Global Smoothness Preservation and the Variation-Diminishing Property

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In the center of our paper are two counterexamples showing the independence of the
concepts of global smoothness preservation and variation diminution for sequences of
approximation operators. Under certain additional assumptions it is shown that the
variation-diminishing property is the stronger one. It is also demonstrated, however, that
there are positive linear operators giving an optimal pointwise degree of approximation,
and which preserve global smoothness, monotonicity and convexity, but are not variation-
diminishing.

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1. INTRODUCTION

The preservation of global smoothness has recently drawn some interest
in various fields of mathematics. We refer to [4] and the references cited
there for a partial survey.

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In [2] it was shown that for the classical Bernstein operators B_n, given for $f \in C[0, 1]$ and $x \in [0, 1]$ by

$$B_n f(x) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot p_{n,k}(x)$$

with $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ one has

$$\omega_1(B_n f; \delta) \leq \tilde{\omega}_1(f; \delta) \leq 2 \cdot \omega_1(f; \delta), \quad 0 \leq \delta \leq 1. \tag{1}$$

Here ω_1 is the first order modulus, and $\tilde{\omega}_1$ denotes its least concave majorant.

If $\text{Lip}_M \alpha$ are the Lipschitz classes with respect to ω_1, the left inequality of (1) implies

$$B_n(\text{Lip}_M \alpha) \subseteq \text{Lip}_M \alpha, \quad 0 < \alpha \leq 1.$$

This statement was recently supplemented by Zhou [19] who showed that

$$B_n(\text{Lip}_M^* \alpha) \subseteq \text{Lip}_{2M}^* \alpha, \quad 0 < \alpha \leq 2.$$

The symbol $\text{Lip}_M^* \alpha$ stands for the Lipschitz classes with respect to the (classical) second order modulus of smoothness ω_2. Zhou's result was recently modified in an interesting note of Adell and Pérez-Palomares [1].

If inequalities and inclusions of the above type are valid, then (in informal language) one speaks about global smoothness preservation. This notion has not yet been formally defined, nor should it be, in our opinion, at this early stage of the development.

On the other hand, in 1959 it was shown by Schoenberg [17] that the Bernstein operators also have the so-called (strong) variation-diminishing property. To be more specific, let us recall the following definition: Let K be any interval of the real line, and let $f: K \to \mathbb{R}$ be an arbitrary function. For an ordered sequence $x_0 < x_1 < \cdots < x_n$ of points in K, let $S[f(x_k)]$ denote the number of sign changes in the finite sequence of ordinates $f(x_k)$, where zeroes are disregarded. The number of changes of sign of f in the interval K is defined by

$$S_K[f] = \sup S[f(x_k)],$$

where the supremum is taken over all ordered finite sets $\{x_k\}$.

Let I and J be two intervals, let U be a subspace of $C(I)$, and suppose that $L: U \to C(J)$ is a linear operator reproducing constant functions.

The operator L is said to be *strongly variation-diminishing* (as an operator from U into $C(J)$) if

$$S_J[Lf] \leq S_I[f] \quad \text{for all } f \in U.$$

Schoenberg then showed that

$$S_{[0,1]}[B_n f] \leq S_{[0,1]}[f],$$

a result usually referred to as the variation-diminishing property of the Bernstein operators. This concept can be carried over to other approximation operators in a straightforward manner.

Guided by a section in Farin's thesis [6, p. 2–14] ('B_n as a smoothing operator') which essentially contains a global smoothness preservation statement, as well as by the importance of the variation-diminishing property in CAGD (cf., e.g., [7]), the question arose in several recent discussions as to what the relationship between global smoothness preservation and the variation-diminishing property might be.

It is the aim of the present note to show that both properties are independent of each other in the sense that there are positive approximation operators (which, moreover, reproduce constant functions) having one of the two properties, but not the other. Nevertheless, utilizing mild additional assumptions, it will be shown that the strong variation-diminishing property implies the preservation of global smoothness, so that, in this sense, the former is the stronger concept. We will also discuss certain positive linear operators giving optimal degrees of approximation, having good shape-preservation properties, preserving global smoothness, but not being variation-diminishing in the strong sense.

It should be emphasized that preservation of global smoothness and the variation-diminishing property can be investigated for (almost) any individual operator, without having other features in mind. However, both properties appear to be of interest mainly as additional properties of members of sequences of approximation operators. It is for this reason that in the sequel we will focus on such operators, and in particular on the case of sequences of operators providing uniform approximation for any function $f \in C[0, 1]$.

Before we proceed to present our counterexamples we recall the following results from [2]. In their formulation, the notations

$$\mathrm{Lip}[0,1] = \bigcup_{M>0} \mathrm{Lip}_M 1, \qquad |g|_{\mathrm{Lip}} = \sup_{|x-y|>0} \frac{|g(x) - g(y)|}{|x - y|}$$

will be used.

THEOREM 1   *Let* $L: C[0,1] \to C[0,1]$, $L \neq 0$, *be a bounded linear operator. If L maps $C^1[0,1]$ into* $\mathrm{Lip}[0,1]$, *then one has*

$$\omega(Lf; t) \le \|L\| \cdot \tilde{\omega}_1\left(f; \frac{ct}{\|L\|}\right), \quad \text{for all } t \ge 0, \ f \in C[0,1],$$

if and only if

$$|Lg|_{\mathrm{Lip}} \le c \cdot \|g'\| \quad \text{for all } g \in C^1[0,1].$$

2. THE FIRST COUNTEREXAMPLE (AND ITS DEFICIENCIES)

First we show that the variation-diminishing property does not imply global smoothness preservation.

Example   Let $L_n: C[0,1] \to C[0,1]$ be defined by

$$L_n(f; x) := B_n(f(t^2); \sqrt{x}) = \sum_{k=0}^{n} f\left(\frac{k^2}{n^2}\right) \cdot p_{n,k}(\sqrt{x}).$$

We observe that $(L_n)_{n \in \mathbb{N}}$ is a sequence of positive linear approximation operators which reproduce constants. Indeed, the approximation property follows for example from a Korovkin-type argument using the classical test functions (or more easily: from the approximation property of B_n).

The variation-diminishing property of L_n can be verified as follows: Let $0 \le x_0 < \cdots < x_m \le 1$. Then for $\{y_j = \sqrt{x_j}\}_{j=0}^{m}$, $\{a_k = f(k^2/n^2)\}_{k=0}^{n}$, we have

$$S(\{L_n(f; x_j)\}_{j=0}^{m}) = S\left(\left\{\sum_{k=0}^{n} a_k \cdot p_{n,k}(y_j)\right\}_{k=0}^{m}\right) \le S(\{a_k\}_{k=0}^{n}) \le S_{[0,1]}(f).$$

Hence, $S_{[0,1]}(L_n(f)) \le S_{[0,1]}(f)$.

But L_n does not preserve global smoothness. This can be seen by considering the function f with $f(x) = x$. We have

$$L_n(f; x) = B_n(t^2; \sqrt{x}) = x - \frac{x}{n} + \frac{\sqrt{x}}{n}.$$

Furthermore, $\omega_1(f; \delta) = \delta$ and $\omega_1(L_n f; \delta) = \delta(1 - 1/n) + \sqrt{\delta}/n$. Assume now that the L_n preserve global smoothness. It follows that there exists a positive constant c, such that $\delta(1 - 1/n) + \sqrt{\delta}/n \leq c\delta$, for all $\delta \in [0, 1]$, implying $1/n \leq \sqrt{\delta}(c - 1 + 1/n)$, for all $\delta \in [0, 1]$.

For $\delta = 1/n^4$, the latter inequality becomes

$$1 \leq \frac{1}{n}\left(c - 1 + \frac{1}{n}\right),$$

which gives a contradiction.

Remark 2 Note that the sequence of operators $(L_n)_{n \in \mathbb{N}}$ from the last example does not map $C^1[0, 1]$ into $\mathrm{Lip}[0, 1]$. Also the L_n do not reproduce linear functions.

This leads us to following natural questions.

PROBLEM 1 *Suppose $(L_n)_{n \in \mathbb{N}}$ is a sequence of positive linear operators mapping $C^1[0, 1]$ into $\mathrm{Lip}[0, 1]$, and having the variation-diminishing property. Do the L_n then preserve global smoothness?*

PROBLEM 2 *Let $(L_n)_{n \in \mathbb{N}}$ be a sequence of positive linear operators having the variation-diminishing property, and such that $L_n e_i = e_i$, $i = 0, 1$. Is it true that the L_n preserve global smoothness?*

3. PARTIAL ANSWERS TO PROBLEMS 1 AND 2

In the sequel we will give partial answers to the two problems indicated. In particular the relationship between the variation-diminishing property and the preservation of monotonicity and global smoothness will be discussed.

LEMMA 3 *If the operator $L: C[0, 1] \to C[0, 1]$ is variation-diminishing, then it preserves positivity and monotonicity.*

Proof The positivity is a consequence of the reproduction of constants. Furthermore, if f is a monotone function, then for any constant a we have

$$S_{[0,1]}(L(f) - a) \leq S_{[0,1]}(L(f - a)) \leq S_{[0,1]}(f - a).$$

So, since f is a continuous function, one has

$$S_{[0,1]}(f - a) \leq 1.$$

Thus the continuous $L(f)$ changes sign at most once, showing it is also monotone.

In the following we will prove several assertions concerning the relationship between preservation of monotonicity and that of global smoothness. Note that a related result was given by Della Vecchia and Raşa [5].

We denote by

$\mathcal{M} \subset \mathrm{Lip}[0, 1]$ the set of all monotone functions on $[0, 1]$,
$\mathcal{M}^+ \subset \mathcal{M}$ the set of all increasing functions on $[0, 1]$, and
$\mathcal{M}^- \subset \mathcal{M}$ the set of all decreasing functions on $[0, 1]$.

The following two theorems provide a partial solution to Problem 1.

THEOREM 4   *Let* $L : C[0, 1] \to C[0, 1]$ *be a positive linear operator mapping* $C^1[0, 1]$ *into* $\mathrm{Lip}[0, 1]$, *and which reproduces constant functions. If* $L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^+$ *or* $L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^-$, *then*

$$\omega_1(Lf; \delta) \leq \tilde{\omega}_1(f; c\delta),$$

for all $\delta \in [0, 1]$ *and* $f \in C[0, 1]$, *where the best constant c is* $|Le_1|_{\mathrm{Lip}}$.

Proof It suffices to show that $|Lg|_{\mathrm{Lip}} \leq c \cdot \|g'\|$, for all $g \in \mathrm{Lip}[0, 1]$ (see Theorem 1 and observe that $\|L\| = 1$).

Assume first that $L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^+$, and let $g \in C^1[0, 1]$. Then

$$h_1(x) = g(x) + x \cdot \|g'\| \in \mathcal{M}^+,$$
$$h_2(x) = g(x) - x \cdot \|g'\| \in \mathcal{M}^-.$$

Thus Lh_1 is an increasing function and Lh_2 is decreasing. It follows that

$$
\frac{(Lh_1)(x_1) - (Lh_1)(x_2)}{x_1 - x_2}
$$
$$
= \frac{(Lg)(x_1) - (Lg)(x_2)}{x_1 - x_2} + \|g'\| \frac{(Le_1)(x_1) - (Le_1)(x_2)}{x_1 - x_2} \geq 0,
$$
$$
\frac{(Lh_2)(x_1) - (Lh_2)(x_2)}{x_1 - x_2}
$$
$$
= \frac{(Lg)(x_1) - (Lg)(x_2)}{x_1 - x_2} - \|g'\| \frac{(Le_1)(x_1) - (Le_1)(x_2)}{x_1 - x_2} \leq 0,
$$

for all $x_1, x_2 \in [0, 1]$ with $x_1 \neq x_2$.

The latter two inequalities imply $|Lg|_{\text{Lip}} \leq \|g'\| |Le_1|_{\text{Lip}}$. Notice that actually in the proof it is only required that $Le_1 \in \text{Lip}[0,1]$. Notice also that for $g = e_1$ one obtains equality in the latter inequality.

The case $L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^-$ can be treated similarly.

Under the additional assumption that L reproduces linear functions, global smoothness preservation can be characterized as follows:

THEOREM 5 *Let $L: C[0,1] \to C[0,1]$ be a positive linear operator mapping $C^1[0,1]$ into $\text{Lip}[0,1]$, and such that $Le_i = e_i$, $i = 0, 1$. Then, in order to have*

$$
\omega_1(Lf; \delta) \leq \tilde{\omega}_1(f; \delta), \quad \text{for all } \delta \in [0, 1] \text{ and } f \in C[0,1],
$$

it is necessary and sufficient that

$$
L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^+. \tag{2}
$$

Proof If $L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^+$, from Theorem 4 we have

$$
\omega_1(Lf; \delta) \leq \tilde{\omega}_1(f; c\delta).
$$

Because $c = \|(Le_1)'\| = 1$, we have the desired inequality.

In order to prove necessity of (2), we will use the following:

LEMMA 6 (see Lupaş [15 Theorem 1.1]) *Let $J \subset \mathbb{R}$ be a compact interval. We denote by $M(J)$ the linear space of all functions $f: J \to \mathbb{R}$ which are bounded on J, endowed with the norm $\|\cdot\| = \sup_J |\cdot|$. Let now*

$H: M(J_1) \to M(J_2)$ *be a linear operator having the property: there exists a positive number m such that*

$$(He_0)(x) \geq m > 0, \quad x \in J_2.$$

H is a positive operator if and only if the operator $\mathcal{H}: M(J_1) \to M(J_2)$ *defined by* $\mathcal{H}f = (1/H(e_0))H(f), f \in M(J_1)$, *satisfies* $\|\mathcal{H}\| = 1$.

Proof of Theorem 5 (*continued*)   We consider the linear operator $H: C[0,1] \to C[0,1]$ given by $H(g) = (L \circ I)'(g)$, where $I(g)(x) = \int_0^x g(t)\,dt$. Since L preserves global smoothness, it follows $\|H(g)\| \leq \|g\|$, which implies $\|H\| \leq 1$. But $H(e_0) = 1$, so we obtain $\|H\| = 1$.

From Lemma 6 it follows that H is a positive operator. Let now $g \in C^1[0,1]$, $g' \geq 0$. We can write $g(x) = g(0) + \int_0^x g'(t)\,dt$, and thus $(Lg)' = H(g') \geq 0$.

This completes the proof.

Our next result gives a partial answer to Problem 2. In contrast with the previous statements, here it is only assumed that monotone functions are mapped to monotone functions (which is true for any variation-diminishing operator, see Lemma 3).

THEOREM 7   *Let* $L: C[0,1] \to C[0,1]$ *be a positive linear operator mapping* $C^1[0,1]$ *into* Lip[0,1], *and such that* $Le_i = e_i$, $i = 0,1$. *If* $L(\mathcal{M} \cap C[0,1]) \subset \mathcal{M}$, *then*

$$\omega_1(Lf; \delta) \leq \tilde{\omega}_1(f; \delta), \quad \text{for all } \delta \in [0,1] \text{ and } f \in C[0,1].$$

Proof   First note that a positive linear operator L that reproduces linear functions also interpolates at the endpoints. In fact, this follows from the classical result of Mamedov [14] stating that for such operators one has for all $f \in C[0,1]$ and all $x \in [0,1]$ the inequality

$$|L(f; x) - f(x)| \leq 2 \cdot \omega_1(f; L(|\cdot -x|; x)).$$

Putting now $x = 0$ one gets

$$|L(f; 0) - f(0)| \leq 2 \cdot \omega_1(f; L(e_1; 0)) = 2 \cdot \omega_1(f; e_1(0))$$
$$= 2 \cdot \omega_1(f; 0) = 0.$$

The same argument works for $x = 1$.

Now suppose that f is an increasing, non-constant function. We will show that Lf is also increasing. Assume the contrary. Then we have

$$f(1) = (Lf)(1) \leq (Lf)(x) \leq (Lf)(0) = f(0), \quad \text{for all } x \in [0, 1].$$

Hence f is a constant, which gives a contradiction. And so, $L(\mathcal{M}^+ \cap C^1) \subset \mathcal{M}^+$. From Theorem 4 we can conclude that we have global smoothness preservation with constant $c = 1$.

COROLLARY 8   *Any variation-diminishing operator L reproducing linear functions and mapping C^1 into Lip1 preserves global smoothness in the sense that*

$$\omega_1(Lf; \delta) \leq \tilde{\omega}_1(f; \delta) \quad \text{for all } \delta \geq 0, \text{ for all } f \in C[0, 1].$$

4. THE SECOND COUNTEREXAMPLE (AND ITS SHORTCOMINGS)

In this section, among others, we give an example of a sequence of positive linear approximation operators which preserve global smoothness but which are not variation-diminishing. In order to put this counterexample into a more general framework, we will prove the following theorem dealing with modifications of the Bernstein polynomials.

To this end, consider the following Bernstein-type operators: $B_n^* : C[0, 1] \to \Pi_n$, defined by

$$(B_n^* f)(x) = \sum_{k=0}^{n} f(x_{k,n}) p_{n,k}(x), \quad f \in C[0, 1].$$

with $x_{k,n} \in [0, 1]$.

THEOREM 9   *Let $n \in \mathbb{N}$ be fixed. The operator B_n^* has the variation-diminishing property on $C[0, 1]$ if and only if one of the following two conditions holds:*

$$0 \leq x_{0,n} \leq x_{1,n} \leq \cdots \leq x_{n,n} \leq 1, \tag{3}$$

or

$$1 \geq x_{0,n} \geq x_{1,n} \geq \cdots \geq x_{n,n} \geq 0. \tag{4}$$

Proof Since n is fixed, we use during the proof the simpler notation x_k instead of $x_{k,n}$. Assume first that relation (3) is verified. Then $S_{[0,1]}[B_n^* f] \leq S\{f(x_k)\} \leq S_{[0,1]}[f]$. The same is true if condition (4) is satisfied.

Suppose now that B_n^* has the variation-diminishing property on $C[0,1]$. It follows that B_n^* preserves monotonicity (see Lemma 3).

Case 1 Assume that $(B_n^* e_1)' \geq 0$. Then

$$(B_n^* e_1)'(x) = n \sum_{k=0}^{n-1} (x_{k+1} - x_k) p_{n-1,k}(x) \geq 0, \quad x \in [0,1]. \tag{5}$$

Taking $x = 0$ in the latter inequality we obtain $x_1 \geq x_0$, and for $x = 1$ it follows that $x_n \geq x_{n-1}$. Furthermore, integrating (5) we get $x_n \geq x_0$.

We denote

$$p = \min_{i \in \{1,2,\ldots,n-2\}} \{i\colon x_{i+1} < x_i\}.$$

Under the assumption that p exists, we construct the function g_p as follows. Let $x_s \in \{x_0, \ldots, x_n\}$ be such that $x_s < x_p$ and $(x_s, x_p) \cap \{x_0, \ldots, x_n\} = \emptyset$. Then, clearly, $x_{p+1} \leq x_s$.

Consider the function $g_p : [0,1] \to \mathbb{R}$ defined by

$$g_p(x) := \begin{cases} 0, & \text{for } x \in \left[0, \dfrac{x_s + x_p}{2}\right], \\[2mm] \dfrac{2x - x_s - x_p}{x_p - x_s}, & \text{for } x \in \left(\dfrac{x_s + x_p}{2}, x_p\right], \\[2mm] 1, & \text{for } x \in (x_p, 1]. \end{cases}$$

Obviously $g_p \in C[0,1]$ is an increasing function.

We show that $(B_n^* g_p)'(x) \geq 0$, $x \in [0,1]$. Otherwise, since $B_n^* g_p$ is monotone,

$$(B_n^* g_p)(x) \leq (B_n^* g_p)(0) = g_p(x_0) = \begin{cases} 0, & x_0 \leq x_s, \\ 1, & x_0 > x_s. \end{cases}$$

But $(B_n^* g_p)(x) \leq 0$ leads us to a contradiction, since $(B_n^* g_p)(1) = 1$.

The case $(B_n^* g_p)(x) \le 1$ for all x also leads to a contradiction, because

$$1 = g_p(x_n) = (B_n^* g_p)(1) \le (B_n^* g_p)(x) \le 1,$$

which implies $B_n^* g_p = e_0$. But $(B_n^* g_p)(x) \le 1 - p_{n,s}(x) < 1$, $x \in (0,1)$. Thus $(B_n^* g_p)'(x) \ge 0$, $x \in [0,1]$.

On the other hand one has

$$(B_n^* g_p)'(x) = n \sum_{k=0}^{n-1} [g_p(x_{k+1}) - g_p(x_k)] p_{n-1,k}(x)$$

$$\le n \left[-\binom{n-1}{p} x^p (1-x)^{n-1-p} + \sum_{k=p+1}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} \right]$$

$$= nx^p \left[-\binom{n-1}{p} (1-x)^{n-1-p} + \sum_{k=p+1}^{n-1} \binom{n-1}{k} x^{k-p} (1-x)^{n-1-k} \right].$$

But

$$-\binom{n-1}{p} (1-x)^{n-1-p} + \sum_{k=p+1}^{n-1} \binom{n-1}{k} x^{k-p} (1-x)^{n-1-k}$$

is a continuous function taking the value $-\binom{n-1}{p}$ at the point $x = 0$. It follows that there exists an interval of the form $(0, \varepsilon)$, $\varepsilon > 0$ where $(B_n^* g_p)' < 0$, so we obtained a contradiction.

Case 2   The assumption that $(B_n^* e_1)' \le 0$ can be treated similarly.

In the sequel we present a concrete example of a sequence of positive linear approximation operators which preserve global smoothness, but which are not variation-diminishing.

Example   Consider the $(n+1) \times (n+1)$-matrix $A_n = (a_{i,j})_{0 \le i,j \le n}$ with $a_{i,i} = 1$ for $i \ne 1, 2$, $a_{1,2} = a_{2,1} = 1$, and $a_{i,j} = 0$ otherwise.

Define $L_n : C[0,1] \to C[0,1]$ via

$$L_n(f; x) := (p_{n,0}(x), p_{n,1}(x), \ldots, p_{n,n}(x)) \cdot A_n \cdot \begin{pmatrix} f(0/n) \\ f(1/n) \\ \vdots \\ f(n/n) \end{pmatrix}.$$

L_n is positive with $L_n(1) = 1$. From the representation

$$L_n(f; x) = B_n(f; x) - f\left(\frac{1}{n}\right)p_{n,1}(x) - f\left(\frac{2}{n}\right)p_{n,2}(x)$$
$$+ f\left(\frac{2}{n}\right)p_{n,1}(x) + f\left(\frac{1}{n}\right)p_{n,2}(x),$$

we see that

$$\|L_n(f) - f\|_\infty \le \|B_n(f) - f\|_\infty + 2 \cdot \omega_1\left(f; \frac{1}{n}\right) \to 0;$$

thus $L_n(f) \to f$ for $n \to \infty$. Since

$$(L_n f)'(x) = n \cdot \left\{ p_{n-1,0}(x) \cdot \left[f\left(\frac{2}{n}\right) - f\left(\frac{0}{n}\right) \right] + p_{n-1,1}(x) \cdot \left[f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right) \right] \right.$$
$$+ p_{n-1,2}(x) \cdot \left[f\left(\frac{3}{n}\right) - f\left(\frac{1}{n}\right) \right]$$
$$\left. + \sum_{k=3}^{n-1} p_{n-1,k}(x) \cdot \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] \right\},$$

we obtain for $g \in C^1[0, 1]$ that

$$\|(L_n g)'\|_\infty \le 2 \cdot \|g'\|_\infty.$$

Hence L_n satisfies a global smoothness preservation property.

But the nodes of L_n neither satisfy condition (3) nor (4) from Theorem 9, so L_n cannot be variation-diminishing.

Remark 10 Elementary computation yields $L_n(e_1; x) = x + (\sqrt{x}(1 - \sqrt{x})/n$, thus the operators from our second counterexample do not reproduce linear functions. Note also that they preserve global smoothness with a constant $c = 2$ only.

In a natural fashion, this leads to

PROBLEM 3   *Let (L_n) be a sequence of positive linear approximation operators having the following properties*:
  (i) $L_n e_i = e_i$, $i = 0, 1$;
 (ii) L_n *maps* $C^1[0, 1]$ *into* $C^1[0, 1]$;
(iii) $\|(L_n g)'\| \le 1 \cdot \|g'\|$, *for all* $g \in C^1[0, 1]$.

Under these conditions, do the L_n have the variation-diminishing property?

A negative answer to Problem 3 will be given in the following section.

5. A NEGATIVE ANSWER TO PROBLEM 3

In the following we will carry out further investigations concerning the shape preservation potential of operators introduced by Gavrea in 1996 (see [8]). The paper mentioned provided the first solution to a problem in approximation theory which had been open for many years, namely to find positive linear polynomial operators providing a DeVore–Gopengauz inequality. An explicit statement of the problem can be found in [12, Problem #1]. Further details will be given below.

As indicated in the title of this section, the outcome will be (essentially) negative. As can be seen from Theorems 13 and 16 below, the operators H_{m+2} possess most of the shape preservation properties relevant to CAGD. Nonetheless, they turn out not to be variation-diminishing for (at least) large degrees.

Let $D_n^{\langle \alpha \rangle}$ be the generalized Durrmeyer operator

$$(D_n^{\langle \alpha \rangle} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{\langle p_{n,k}, f \rangle_\alpha}{\langle p_{n,k}, 1 \rangle_\alpha}, \quad f \in C[0,1], \quad x \in [0,1],$$

where

$$\langle f, g \rangle_\alpha = \int_0^1 f(x) g(x) \, dw(x, \alpha),$$

$$dw(t, \alpha) = \frac{t^\alpha (1 - t)^\alpha}{B(\alpha + 1, \alpha + 1)}, \quad \alpha > -1.$$

THEOREM 11 (Lupaş [16, Lemma 4.2]) *For $f \in C^s[0,1]$, one has*

$$(D_n^{\langle \alpha \rangle} f)^{(s)}(x) = \frac{(-1)^s (-n)_s}{(n + 2\alpha + 2)_s} (D_{n-s}^{\langle \alpha+s \rangle} f^{(s)})(x), \tag{6}$$

where $(a)_0 = 1$, $(a)_s = a(a+1) \cdots (a+s-1)$ is the Pochhammer symbol.

Lupaş [16] considered sequences of operators of the form

$$(L_n^{\langle \alpha \rangle} f)(x) = \sum_{k=0}^{n} \frac{(\alpha + 1)_k}{(2\alpha + 2)_k} a_{k,n} (D_k^{\langle \alpha \rangle} f)(x), \tag{7}$$

where $a_{k,n}$ are real numbers, $a_{n,n} \neq 0$.

The sequence of polynomials (p_n) defined by $p_n(x) = \sum_{k=0}^{n} a_{k,n} x^k$ is called generator sequence of the operators $(L_n^{\langle \alpha \rangle})$.

Lupaş [16, Theorem 5.2] showed that $(L_n^{\langle\alpha\rangle})_{n\in\mathbb{N}}$ is a sequence of positive linear operators, provided $p_n(x) \geq 0$, $x \in [0, 1]$. Furthermore, if one also has $\langle p_n, 1 \rangle_\alpha = 1$, then $L_n^{\langle\alpha\rangle} e_0 = e_0$.

In [8] Gavrea introduced the sequence of operators $(L_m)_{m\in\mathbb{N}}$, with $L_m : C[0, 1] \to \Pi_m$ given by

$$(L_m f)(x) = f(0)(1 - x)^m + x^m f(1)$$
$$+ (m - 1) \sum_{k=1}^{m-1} p_{m,k}(x) \int_0^1 p_{m-2,k-1}(t) f(t) \, \mathrm{d}t.$$

Now take a polynomial $P_m \in \Pi_m$, $P_m(x) = \sum_{k=0}^m a_k x^k$, also called generator below, which satisfies the following conditions:

$$P_m(x) \geq 0, \quad x \in [0, 1],$$
$$\int_0^1 P_m(x) \, \mathrm{d}x = 1, \tag{8}$$
$$P'_m(x) \geq 0, \quad x \in [0, 1].$$

In [8] Gavrea provided the essential idea to construct the operators $H_{m+2} : C[0, 1] \to \Pi_{m+2}$, given by

$$(H_{m+2} f)(x) = \sum_{k=0}^m \frac{a_k}{k+1} (L_{k+2} f)(x).$$

While Gavrea's original approximants are in Π_{2m+1} only, it was shown in [9,10] that their degree can be reduced to $m + 2$ by using a slightly modified construction. The operators H_{m+2} are linear and positive, they reproduce linear functions, and they satisfy the following DeVore–Gopengauz inequality:

$$|(H_{m+2} f)(x) - f(x)| \leq c\omega_2\left(f; \frac{\sqrt{x(1 - x)}}{m}\right). \tag{9}$$

The following result was communicated to us by Jia-ding Cao (Fudan University, Shanghai).

THEOREM 12   *Let f be an absolutely continuous function on* $[0, 1]$. *Then*

$$(L_{n+1} f)'(x) = (D_n f')(x). \tag{10}$$

Proof   It is well-known that

$$p'_{n+1,i}(x) = (n+1)[p_{n,i-1}(x) - p_{n,i}(x)], \quad 1 \le i \le n.$$

So we obtain

$$
\begin{aligned}
(L_{n+1}f)'(x) &= n\sum_{i=1}^{n}(n+1)[p_{n,i-1}(x) - p_{n,i}(x)]\int_0^1 p_{n-1,i-1}(t)f(t)\,\mathrm{d}t \\
&\quad - (n+1)f(0)(1-x)^n + (n+1)f(1)x^n \\
&= n(n+1)\sum_{i=0}^{n-1}p_{n,i}(x)\int_0^1 p_{n-1,i}(t)f(t)\,\mathrm{d}t \\
&\quad - n(n+1)\sum_{i=1}^{n}p_{n,i}(x)\int_0^1 p_{n-1,i-1}(t)f(t)\,\mathrm{d}t \\
&\quad - (n+1)f(0)(1-x)^n + (n+1)f(1)x^n \\
&= n(n+1)\sum_{i=1}^{n-1}p_{n,i}(x)\int_0^1 [p_{n-1,i}(t) - p_{n-1,i-1}(t)]f(t)\,\mathrm{d}t \\
&\quad + n(n+1)(1-x)^n\int_0^1 f(t)(1-t)^{n-1}\,\mathrm{d}t - n(n+1)x^n \\
&\quad \times \int_0^1 f(t)t^{n-1}\,\mathrm{d}t - (n+1)f(0)(1-x)^n + (n+1)f(1)x^n \\
&= -(n+1)\sum_{i=1}^{n-1}p_{n,i}(x)\int_0^1 f(t)p'_{n,i}(t)\,\mathrm{d}t \\
&\quad + n(n+1)(1-x)^n\int_0^1 f(t)(1-t)^{n-1}\,\mathrm{d}t - n(n+1)x^n \\
&\quad \times \int_0^1 f(t)t^{n-1}\,\mathrm{d}t - (n+1)f(0)(1-x)^n + (n+1)f(1)x^n \\
&= (n+1)\sum_{i=1}^{n-1}p_{n,i}(x)\int_0^1 f'(t)p_{n,i}(t)\,\mathrm{d}t \\
&\quad - (n+1)f(t)(1-t)^n\Big|_0^1(1-x)^n + (n+1)p_{n,0}(x) \\
&\quad \times \int_0^1 f'(t)p_{n,0}(t)\,\mathrm{d}t - (n+1)f(t)t^n\Big|_0^1 x^n + (n+1)p_{n,n}(x) \\
&\quad \times \int_0^1 f'(t)p_{n,n}(t)\,\mathrm{d}t - (n+1)f(0)(1-x)^n + (n+1)f(1)x^n \\
&= (D_n f')(x).
\end{aligned}
$$

THEOREM 13   *The operator H_{m+2} preserves the monotonicity and the convexity of the function f.*

Proof It suffices to prove the theorem for f absolutely continuous, with $f' \geq 0$. From Theorem 12 it follows

$$(H_{m+2}f)'(x) = \sum_{k=0}^{m} \frac{a_k}{k+1} (D_{k+1}f')(x) = \sum_{k=0}^{m} a_k \frac{k+2}{k+1} \frac{1}{k+2} (D_{k+1}f')(x)$$

$$= \sum_{k=1}^{m+1} a_{k-1} \frac{k+1}{k(k+1)} (D_k f')(x).$$

By Lupaş' observation, in order to prove the theorem it is enough to show that the polynomial

$$q_{m+1}(x) := \sum_{k=1}^{m+1} a_{k-1} \frac{k+1}{k} x^k$$

is positive on [0, 1]. But

$$q_{m+1}(x) = \sum_{k=0}^{m} a_k \frac{k+2}{k+1} x^{k+1} = x P_m(x) + \int_0^x P_m(t)\, dt,$$

where P_m is the positive generator polynomial, and hence $q_{m+1}(x) \geq 0$ for $x \in [0, 1]$.

From Theorem 12 we also obtain for f with f' absolutely continuous, that

$$(H_{m+2}f)''(x) = \sum_{k=0}^{m} \frac{a_k}{k+1} \frac{k+1}{k+3} (D_k^{\langle 1 \rangle} f'')(x)$$

$$= \sum_{k=0}^{m} \frac{a_k}{k+3} \frac{(4)_k (2)_k}{(2)_k (4)_k} (D_k^{\langle 1 \rangle} f'')(x).$$

For finishing the proof it suffices to show that

$$q_m(x) := \sum_{k=0}^{m} \frac{a_k}{k+3} \frac{(4)_k}{(2)_k} x^k$$

is positive on $[0, 1]$. We have

$$q_m(x) = \sum_{k=0}^{m} a_k \frac{4 \cdot 5 \cdots (k+3)}{(k+3) \cdot 2 \cdot 3 \cdots (k+1)} x^k$$

$$= \frac{1}{6} \sum_{k=0}^{m} a_k(k+2)x^k = \frac{1}{6} x P_m'(x) + \frac{1}{3} P_m(x) \geq 0, \quad \text{for } x \in [0, 1].$$

Thus the theorem is proved.

We recall here that a function f is called *i-convex*, $i \geq 1$, if $f \in C[a, b]$ and all ith forward differences

$$\Delta_h^i f(t) := \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} f(t + kh), \quad 0 \leq h \leq (b-a)/i, \quad t \in [a, b - ih]$$

are non-negative. Also, the function f is said to be *0-convex* if it is non-negative on $[a, b]$. Furthermore, an operator $L: C[0, 1] \to C[0, 1]$ preserves the convexity of order k, if for every function f convex of order k, one has Lf convex of order k.

THEOREM 14   Let $P_m(x) = \sum_{k=0}^{m} a_k x^k$ be the generator polynomial for H_{m+2}, thus satisfying conditions (8). If P_m is convex up to the order r, then H_{m+2} preserves the convexity up to the order $r+1$.

Proof   It suffices to prove the theorem for the case $f \in C^{r+1}[0, 1]$.
Let $2 \leq s \leq r+1$ be a fixed natural number. Then one has

$$(H_{m+2}f)^{(s)}(x) = \sum_{k=0}^{m} \frac{a_k}{k+1} (D_{k+1} f')^{(s-1)}(x)$$

$$= \sum_{k=s-2}^{m} \frac{a_k}{k+1} \cdot \frac{(-1)^{s-1}(-k-1)_{s-1}}{(k+3)_{s-1}} \left(D_{k-s+2}^{\langle s-1 \rangle} f^{(s)} \right)(x)$$

$$= \sum_{k=s-2}^{m} a_k \frac{(s-1)! \binom{k+1}{s-1}}{(k+1)(k+3) \cdots (k+s+1)} \left(D_{k-s+2}^{\langle s-1 \rangle} f^{(s)} \right)(x).$$

In order to show that $(H_{m+2}f)^{(s)} \geq 0$, provided $f^{(s)} \geq 0$, it suffices to prove that the polynomial

$$q_{m-s+2}(x) = \sum_{k=s-2}^{m} a_k \frac{(s-1)! \binom{k+1}{s-1}}{(k+1)(k+3)\cdots(k+s+1)} \frac{(2s)_{k-s+2}}{(s)_{k-s+2}} x^{k-s+2}$$

is positive on $[0, 1]$. We have

$$q_{m-s+2}(x)$$
$$= \sum_{k=s-2}^{m} a_k x^{k-s+2} \frac{k(k-1)\cdots(k-s+3)}{(k+3)\cdots(k+s+1)} \cdot \frac{2s(2s+1)\cdots(k+s+1)}{s(s+1)\cdots(k+1)}$$
$$= \sum_{k=s-2}^{m} a_k x^{k-s+2} \frac{k(k-1)\cdots(k-s+3)(k+2)(s-1)!}{(2s-1)!}$$
$$= \frac{(s-1)!}{(2s-1)!} \sum_{k=s-2}^{m} a_k x^{k-s+2} k(k-1)\cdots(k-s+3)(k-s+2+s)$$
$$= \frac{(s-1)!}{(2s-1)!} \left(x \sum_{k=s-1}^{m} a_k x^{k-s+1} k(k-1)\cdots(k-s+2) \right.$$
$$\left. + s \sum_{k=s-2}^{m} a_k x^{k-s+2} k(k-1)\cdots(k-s+3) \right)$$
$$= \frac{(s-1)!}{(2s-1)!} \left(x P_m^{(s-1)}(x) + s P_m^{(s-2)}(x) \right) \geq 0.$$

LEMMA 15   *Let $L: C[0, 1] \to \Pi_n$ be a positive linear operator having the variation-diminishing property and which interpolates at one of the endpoints. If degree $Le_i = i$, $i = 0, 1, \ldots, n$, then L preserves the convexity of orders $0, 1, \ldots, n$.*

Proof   In [11] we showed that an operator satisfying the conditions of the lemma transforms a function convex of order k into a function convex of order k, or into one concave of order k. In the sequel we will show that such an operator transforms convex functions of order k into convex functions of order k.

Let $f \in C[0, 1]$ be a function convex of order k, $0 \leq k \leq n$ fixed. From [11, Theorem 7], it follows that for any choice of points $x, x_1, \ldots, x_k$

the divided difference $[x, x_1, x_2, \ldots, x_k; Lf]$ has constant sign. Let $x_1, x_2, \ldots, x_k$ be fixed and distinct points in the range $(0, 1)$.

Since $Le_i, i = 0, 1, \ldots, k - 1$, form a basis in Π_{k-1}, there exist constants $c_i, i = 0, 1, \ldots, k - 1$, such that the Lagrange interpolator of Lf can be written as

$$L_{k-1}(x_1, x_2, \ldots, x_k; Lf) = \sum_{i=0}^{k-1} c_i Le_i.$$

We have

$$(Lf)(x) - \sum_{i=0}^{k-1} c_i (Le_i)(x) = L\left(f - \sum_{i=0}^{k-1} c_i e_i\right)(x)$$
$$= (x - x_1) \cdots (x - x_k) \cdot [x, x_1, x_2, \ldots, x_k; Lf]. \qquad (11)$$

(See, e.g. [13, p. 248].)

Put $g := f - \sum_{i=0}^{k-1} c_i e_i$. Then $L(g; x) = (x - x_1) \cdots (x - x_k) \cdot [x, x_1, x_2, \ldots, x_k; Lf]$ is a polynomial of degree not greater than n, the polynomial factor $[x, x_1, x_2, \ldots, x_k; Lf]$ of which has constant sign, i.e., it is ≥ 0 or ≤ 0. This implies that Lg changes its sign in the k points $x_1, \ldots, x_k$ and nowhere else. Since L is variation-diminishing, it follows that g changes its sign in at least k points.

Let $y_1, y_2, \ldots, y_k$ be distinct points where g changes its sign. Then

$$\sum_{i=0}^{k-1} c_i e_i = L_{k-1}(y_1, y_2, \ldots, y_k; f).$$

Thus we have

$$g(t) = f(t) - L_{k-1}(y_1, \ldots, y_k; f)(t)$$
$$= (t - y_1) \cdots (t - y_k) \cdot [t, y_1, y_2, \ldots, y_k; f], \qquad (12)$$

which again follows from the error representation for Lagrange interpolation. Hence

$$(Lg)(x) = L((e_1 - y_1) \cdots (e_1 - y_k) \cdot [e_1, y_1, y_2, \ldots, y_k; f])(x), \quad x \in [0, 1].$$

Assuming further that L interpolates each function at the endpoint 1, from the latter relation one obtains

$$(Lg)(1) = L((e_1 - y_1) \cdots (e_1 - y_k) \cdot [e_1, y_1, y_2, \ldots, y_k; f]) (1)$$
$$= (1 - y_1) \cdots (1 - y_k) \cdot [1, y_1, y_2, \ldots, y_k; f].$$

On the other hand, one has (see (11))

$$(Lg)(1) = (1 - x_1) \cdots (1 - x_k) \cdot [1, x_1, x_2, \ldots, x_k; Lf].$$

Since $[1, y_1, y_2, \ldots, y_k; f] \geq 0$, we obtain that $[1, x_1, x_2, \ldots, x_k; Lf] \geq 0$.

Thus the divided differences of f and of Lf agree in sign everywhere, and the proof is complete for the case in which L interpolates at 1.

The case $(Lf)(0) = f(0)$ can be treated similarly.

The following considerations provide a solution to Problem 3.

THEOREM 16 *Let H_{m+2}, $m \in \mathbb{N}$, be given as above. Then the following statements hold:*

(i) *H_{m+2} preserves global smoothness.*
(ii) *$H_{m+2}f \in \Pi_{m+2}$, for all $f \in C[0, 1]$.*
(iii) *$H_{m+2} e_i = e_i$, for $i = 0, 1$.*
(iv) *For $m \geq M_0$, H_{m+2} does not have the variation-diminishing property.*

Proof (i) Our earlier observations (and Theorem 13, in particular) show that H_{m+2} is an operator satisfying the conditions of Theorem 5. Thus we have

$$\omega_1(H_{m+2}f; \delta) \leq \tilde{\omega}_1(f; \delta), \quad \text{for all } \delta \in [0, 1] \text{ and } f \in C[0, 1].$$

(ii) This was mentioned above already as an immediate consequence of the definition of H_{m+2}.

(iii) The two inequalities in question are an immediate consequence of inequality (9).

(iv) Suppose that H_{m+2} has the variation-diminishing property. Under this assumption, we will show that degree $H_{m+2}e_i = i$, $i = 0, 1, \ldots, m + 2$. It suffices to show that H'_{m+2} transforms polynomials of degree i in polynomials of degree $i - 1$. From the proof of Theorem 13 we have

$$(H_{m+2}f)'(x) = \sum_{k=1}^{m+1} a_{k-1} \frac{k+1}{k(k+1)} (D_k f')(x).$$

Hence $(H_{m+2}f)' = L_{m+1}^{(0)}f'$, where $L_{m+1}^{(0)}$ are the operators considered by Lupaş [16] and having the generator polynomial

$$q_{m+1}(x) = \sum_{k=1}^{m+1} a_{k-1}\frac{k+1}{k}x^k,$$

since $\int_0^1 q_{m+1}(x)\,dx = 1$. It follows from [16, Lemma 5.2] that we can write

$$(H_{m+2}f)'(x) = \sum_{k=0}^{m+1} \gamma_k^{(0)}\rho_{k,m+1}\left\langle f',\varphi_k^{(0)}\right\rangle\varphi_k^{(0)}(x),$$

where $\varphi_k^{(0)}$, $k=0,1,2,\ldots,m+1$ are the Legendre polynomials of degree k related to the interval $[0,1]$ and such that $\varphi_k^{(0)}(1) = 1$, and

$$\gamma_k^{(0)} = \frac{1}{\int_0^1 [\varphi_k^{(0)}(x)]^2\,dx},$$

$$\rho_{k,m+1} = \int_0^1 q_{m+1}(x)\varphi_k^{(0)}(x)\,dx, \quad k=0,1,2,\ldots,m+1.$$

Hence we have

$$(H_{m+2}f)(x) = \sum_{k=0}^{m+1} \gamma_k^{(0)}\rho_{k,m+1}\left\langle f',\varphi_k^{(0)}\right\rangle\int_0^x \varphi_k^{(0)}(t)\,dt + f(0).$$

The latter representation ensures that $H_{m+2}\Pi_k \subseteq \Pi_k, k=0,1,\ldots,m+2$. We will show in the sequel that degree $H_{m+2}\,e_i$ is exactly i, $i=0,1,\ldots,m+2$. This is equivalent to showing that $\rho_{k,m+1}\neq 0$, $k=0,1,\ldots,m+1$.

We have $\rho_{0,m+1} = 1$, $\rho_{1,m+1} > 0$, $\rho_{2,m+1} > 0$. Since

$$\rho_{m+1,m+1} = \int_0^1 q_{m+1}(x)\varphi_{m+1}^{(0)}(x)\,dx$$

$$= \frac{a_m}{m+1}\cdot(m+2)\int_0^1 x^{m+1}\varphi_{m+1}^{(0)}(x)\,dx,$$

and $a_m \neq 0$, it follows that $\rho_{m+1,m+1}\neq 0$.

We assume there exists $3 \leq i \leq m$, such that $\rho_{i,m+1} = 0$. Then we also show that $\rho_{i+1,m+1} = 0$.

To that end, consider the polynomial $P(x) = \int_0^x \varphi_{i+1}^{(0)}(t) \, dt - \int_0^x \varphi_i^{(0)}(t) \, dt \in \Pi_{i+2}$. One has $P(0) = P(1) = 0$ and $P'(1) = 0$. Hence P changes its sign in at most $i - 1$ points in the interval $(0, 1)$.

On the other hand, one has

$$(H_{m+2}P)(x) = \rho_{i+1,m+1} \int_0^x \varphi_{i+1}^{(0)}(t) \, dt.$$

But it is well known that

$$\varphi_{i+1}^{(0)}(x) = \frac{(-1)^{i+1}}{(i+1)!} \left[x^{i+1}(1-x)^{i+1} \right]^{(i+1)}.$$

So we have

$$(H_{m+2}P)(x) = \rho_{i+1,m+1} \frac{(-1)^{i+1}}{(i+1)!} \left[x^{i+1}(1-x)^{i+1} \right]^{(i)}$$

$$= -\rho_{i+1,m+1} x(1-x) \cdot J_i^{(1,1)}(x),$$

where

$$J_i^{(1,1)}(x) = \frac{(-1)^i}{x(1-x) \cdot (i+1)!} \left[x^{i+1}(1-x)^{i+1} \right]^{(i)}$$

is the Jacobi polynomial of degree i, relative to the interval $[0, 1]$. It is known that $J_i^{(1,1)}$ has i distinct roots in the range $(0, 1)$.

Now, if $\rho_{i+1,m+1} \neq 0$, then $H_{m+2}P$ would change sign i times in the interval $(0, 1)$, which contradicts the variation-diminishing property of H_{m+2}, so that $\rho_{i+1,m+1} = 0$. Now the fact that $\rho_{i,m+1} = 0$ implies $\rho_{i+1,m+1} = 0$, and hence $\rho_{m+1,m+1} = 0$, which is a contradiction. It follows that degree $H_{m+2}e_i = i$, $i = 0, 1, \ldots, m+2$.

Thus H_{m+2} satisfies all the conditions of [11, Theorem 7]. From Lemma 15 it follows that H_{m+2} preserves the convexity of order i, for $i = 0, 1, \ldots, m+2$.

This latter conclusion, together with (ii) and (iii) shows that H_{m+2} satisfies the conditions in a statement of Berens and DeVore [3, p. 214].

From this we conclude that

$$\frac{x(1-x)}{m+2} = B_{m+2}((\cdot - x)^2; x) \leq H_{m+2}((\cdot - x)^2; x)$$

$$\leq c \cdot \frac{x(1-x)}{m^2}, \quad \text{for all } x \in [0, 1],$$

where the last inequality is again following from (9). Since the constant c is independent of m, this cannot be true for $m \geq M_0$. This yields a contradiction to our assumption that H_{m+2} has the variation-diminishing property.

Acknowledgements

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