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An Elementary Proof for One-dimensionality of Travelling Waves in Cylinders

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Let ω be a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary, $u_+, u_- \in \mathbb{R}$, $a > 0$, and let $u \in W^{2,n}_{\mathrm{loc}}((-a,a) \times \omega) \cap C^1([-a,a] \times \bar{\omega})$ satisfy $-\Delta u + c(x_1)u_{x_1} = f(x_1, u)$ and $u_{x_1} \geq 0$ in $(-a,a) \times \omega$, $u = u_\pm$ on $\{\pm a\} \times \omega$ and $\partial u/\partial\nu = 0$ on $(-a,a) \times \partial\omega$, where c is bounded and nonincreasing and f is continuous and nondecreasing in x_1. We prove that u is a function of x_1 only. The same result is shown for a related problem in the infinite cylinder $\mathbb{R} \times \omega$. The proofs are based on a rearrangement inequality.

Keywords: Boundary value problem; Monotonicity of the solution; One-dimensionality of the solution; Rearrangement; Travelling wave

AMS Subject Classification: 35B05, 35B50, 35B99, 35J25

1. INTRODUCTION

This paper is concerned with the question of *one-dimensionality* of solutions of certain boundary value problems with Neumann boundary conditions in cylindrical domains. More precisely, we consider problems of the following kind:

Let $\Omega = \mathbb{R} \times \omega$ be an infinite cylinder, where ω is a bounded domain in $\mathbb{R}^{n-1}$. For $x \in \Omega$ we write $x = (x_1, x')$ where $x' \in \omega$. Suppose that u

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satisfies

$$
\begin{aligned}
-\Delta u + c(x)u_{x_1} &= f(x,u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \mathbb{R}\times\partial\omega, \\
u(\pm\infty,\cdot) &= u_\pm,
\end{aligned}
\tag{1}
$$

where the functions c, f are continuous and $u_\pm \in \mathbb{R}$, $u_+ > u_-$. The solutions of (1) can be seen as *travelling waves* of a corresponding semilinear parabolic equation. They arise in various applications, e.g. in combustion and some problems of biology. We note that for one space dimension there is much more literature for problems of this kind. This corresponds to $c = c(x_1)$, $f = f(x_1, u)$ and $u = u(x_1)$. The problem (1) then reduces to

$$
\begin{aligned}
-u'' + c(x)u' &= f(x,u) \quad \text{on } \mathbb{R}, \\
u(\pm\infty) &= u_\pm.
\end{aligned}
\tag{2}
$$

Here a general reference is the book of Fife [8]. Problem (1) has been extensively studied by Berestycki and Nirenberg [4]. They obtained various existence criteria, and it turned out that the solutions of (1) behave qualitatively similar as in the one-dimensional case. In particular, if $c = c(x') > 0$ and if $f = f(u)$ is smooth and satisfies some further conditions near $u_\pm$, then u is unique, $u_{x_1} > 0$ and u tends exponentially fast to $u_\pm$ as $x_1 \to \pm\infty$. We mention that the proof of monotonicity and uniqueness u is based on the so-called *sliding method.* This device turned out to be a very powerful tool to show qualitative properties of solutions of some boundary value problems in cylindrical domains (see [1–4]).

It is natural to raise the following question: Suppose that the functions c and f in (1) are independent of x' and $u_{x_1} > 0$. Is it then true that u is independent of x', too?

Let us first emphasize that if c is constant and if f is smooth and $f = f(u)$, then the uniqueness result of [4] immediately yields the desired answer. On the other hand, if f is not smooth, then we cannot apply the sliding method, and the solutions of (1) might be not unique. (In fact, it is easy to construct counterexamples with "flat zones", see Remark 3). Nevertheless the answer to the above question is positive in some relevant cases, even if $f \notin C^1$.

We consider solutions of (1) and also of some related problem in a finite cylinder $(-a, a) \times \omega$ with the decaying conditions replaced by

Dirichlet boundary conditions on $\{\pm a\} \times \omega$. We assume that c is decreasing and that f is increasing in x_1 and continuous. In case of the problem in the infinite cylinder we impose some further conditions on $f(x_1, u)$ near $u = u_\pm$ which ensure exponential decay of u and ∇u at infinity. Exploiting some appropriate transformation of variables and a simple rearrangement inequality we prove the one-dimensionality of the solutions (Theorems 1 and 2). The following Lemma 1 shows that the method is not restricted to continuous nonlinearities f.

We conclude with a simple comparison result for solutions u of some related boundary value problem which satisfy the stronger condition $u_{x_1} > 0$ (Lemma 2).

Remark 1 (1) Our work is also motivated by a paper of Carbou [7]. The author studied the following minimum problem for the Ginzburg–Landau functional:

$$J(u) \equiv \int_\Omega \left(|\nabla u|^2 + \frac{1}{2}(u^2 - 1)^2 \right) \mathrm{d}x \rightarrow \text{Min}!, \quad u \in K, \tag{3}$$

where $K := \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\colon \lim_{x_1 \to \pm\infty} u(x) = \pm 1\}$. This is related to the solutions of $-\Delta u = u - u^3$ in Ω. Using a rearrangement inequality he proved that (3) admits only the trivial solution $u(x) = \tanh(x_1/\sqrt{2})$ (see also Remark 2).

(2) Recently I studied the Cauchy problem for a convolution model of phase transitions in a cylinder in [6]. In particular, I proved the monotonicity and one-dimensionality of travelling and stationary waves by using the sliding method.

(3) In a forthcoming paper we will study the problem whether the solutions of (1) are monotonous in direction x_1. The main tool will be some kind of *continuous rearrangement*. Note that recently a similar construction has been investigated by the author in [5].

2. RESULTS AND PROOFS

By $\mathcal{L}^k$ we denote k-dimensional Lebesgue measure $(1 \le k \le n)$. Let ω a bounded domain in $\mathbb{R}^{n-1}$ with C^1-boundary, and let

$$\Omega = \mathbb{R} \times \omega, \quad \Omega_a = (-a, a) \times \omega,$$

where $a>0$. For points $x\in\Omega$ we write $x=(x_1,x')$, where $x_1\in\mathbb{R}$, $x'\in\omega$, and similarly, $\xi=(\xi_1,\xi')$. Furthermore, we write $\nabla=(\partial/\partial_{x_1},\nabla')$ for the gradient, where $\nabla'=(\partial/\partial_{x_2},\dots,\partial/\partial_{x_n})$. Our first result is

THEOREM 1 *Let $c\in L^\infty((-a,a))$ and nonincreasing, and let $f=f(x_1,u)$ be continuous on $(-a,a)\times\mathbb{R}$ and nondecreasing in x_1. Furthermore, let $u\in W^{2,n}_{\mathrm{loc}}(\Omega_a)\cap C^1(\overline{\Omega_a})$ satisfy:*

$$-\Delta u+c(x_1)u_{x_1}=f(x_1,u),\quad u_{x_1}\ge 0 \text{ in } \Omega_a, \tag{4}$$

$$u=u_\pm \quad \text{on } \{\pm a\}\times\omega, \tag{5}$$

$$\frac{\partial u}{\partial\nu}=0 \quad \text{on } (-a,a)\times\partial\omega \quad (\nu\text{: exterior normal}). \tag{6}$$

Then u is independent of x'.

Proof By introducing a new variable,

$$z=\varphi(x_1):=\int_0^{x_1}\exp\left\{-\int_0^t c(s)\,\mathrm{d}s\right\}\mathrm{d}t, \tag{7}$$

and by setting $\tilde f(z,\cdot):=f(x_1,\cdot)$ and $v^\varepsilon(z,x'):=v(z,x')+\varepsilon z$, where

$$v(z,x'):=u(x_1,x') \tag{8}$$

and $\varepsilon>0$, problem (4)–(6) can be rewritten as

$$-\Delta' v^\varepsilon-\frac{\partial}{\partial z}(g^2(z)v_z^\varepsilon)=\tilde f(z,v^\varepsilon-\varepsilon z)-2\varepsilon g(z)g'(z), \tag{9}$$

$$\begin{aligned} &v_z^\varepsilon\ge\varepsilon \quad \text{in } (a_-,a_+)\times\omega,\\ &v^\varepsilon=u_\pm+\varepsilon a_\pm \quad \text{on } \{a_\pm\}\times\omega, \end{aligned} \tag{10}$$

$$\frac{\partial v^\varepsilon}{\partial\nu}=0 \quad \text{on } (a_-,a_+)\times\partial\omega, \tag{11}$$

where $\Delta'=\sum_{i=2}^n(\partial^2/\partial x_i^2)$, $a_\pm=\varphi(\pm a)$,

$$g(z)=\exp\left\{-\int_0^{\psi(z)}c(t)\,\mathrm{d}t\right\}, \tag{12}$$

and $x_1=\psi(z)$ is the inverse function of φ. Note that $g(z)$ is convex since $c(x_1)$ is nonincreasing.

Let $z=y^\varepsilon(v,x')$ be the inverse function of v^ε with respect to z, and let $U^\varepsilon := (u_- + \varepsilon a_-,\ u_+ + \varepsilon a_+)\times\omega$. We have $y^\varepsilon \in W^{2,n}_{\mathrm{loc}}(U^\varepsilon)\cap C^1(\overline{U^\varepsilon})$, and we compute the derivatives of y^ε as:

$$
\begin{aligned}
v^\varepsilon_z &= \frac{1}{y^\varepsilon_v}, \quad v^\varepsilon_{x_i} = -\frac{y^\varepsilon_{x_i}}{y^\varepsilon_v},\\
v^\varepsilon_{zz} &= -\frac{y^\varepsilon_{vv}}{(y^\varepsilon_v)^3}, \quad v^\varepsilon_{zx_i} = -\frac{y^\varepsilon_{vx_i}}{(y^\varepsilon_v)^2}+\frac{y^\varepsilon_{x_i}y^\varepsilon_{vv}}{(y^\varepsilon_v)^3},\\
v^\varepsilon_{x_ix_j} &= -\frac{y^\varepsilon_{x_ix_j}}{y^\varepsilon_v}+\frac{y^\varepsilon_{x_iv}y^\varepsilon_{x_j}}{(y^\varepsilon_v)^2}+\frac{y^\varepsilon_{x_jv}y^\varepsilon_{x_i}}{(y^\varepsilon_v)^2}-\frac{y^\varepsilon_{x_i}y^\varepsilon_{x_j}y^\varepsilon_{vv}}{(y^\varepsilon_v)^3},\\
&\quad (i,j=2,\dots,n).
\end{aligned}
\tag{13}
$$

Then from (9)–(11) and (13) we obtain:

$$
\begin{aligned}
&\nabla'\left(\frac{\nabla' y^\varepsilon}{y^\varepsilon_v}\right)-\frac{\partial}{\partial v}\left(\frac{|\nabla' y^\varepsilon|^2+g^2(y^\varepsilon)}{2(y^\varepsilon_v)^2}\right)-\frac{g(y^\varepsilon)g'(y^\varepsilon)}{y^\varepsilon_v}\\
&=\tilde f(y^\varepsilon, v-\varepsilon y^\varepsilon)-2\varepsilon g(y^\varepsilon)g'(y^\varepsilon),\\
&\quad 0<y^\varepsilon_v\le 1/\varepsilon \quad \text{in } U^\varepsilon,\\
&\quad y^\varepsilon = u_\pm+\varepsilon a_\pm \quad \text{on } \{u_\pm+\varepsilon a_\pm\}\times\omega,\\
&\quad \nabla' y^\varepsilon = 0 \quad \text{on } (u_-+\varepsilon a_-, u_++\varepsilon a_+)\times\partial\omega.
\end{aligned}
\tag{14}
$$

Let Y^ε the following average of y^ε:

$$
Y^\varepsilon(v,x')\equiv Y^\varepsilon(v) := \frac{\int_\omega y^\varepsilon(v,\xi')\mathrm{d}\xi'}{\mathcal{L}^{n-1}(\omega)}, \quad ((v,x')\in\overline{U^\varepsilon}). \tag{15}
$$

Note that Y^ε is independent of x' and $Y^\varepsilon_v \le 1/\varepsilon$. Then (14) yields the identity

$$
\begin{aligned}
&\iint_{U^\varepsilon}\left(-\frac{\nabla' y^\varepsilon}{y^\varepsilon_v}\nabla'(y^\varepsilon - Y^\varepsilon)+\frac{|\nabla' y^\varepsilon|^2+g^2(y^\varepsilon)}{2(y^\varepsilon_v)^2}(y^\varepsilon_v - Y^\varepsilon_v)\right)\mathrm{d}x'\,\mathrm{d}v\\
&\quad-\iint_{U^\varepsilon}\frac{g(y^\varepsilon)g'(y^\varepsilon)}{y^\varepsilon_v}(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v\\
&\quad=\iint_{U^\varepsilon}\left(\tilde f(y^\varepsilon, v-\varepsilon y^\varepsilon)-2\varepsilon g(y^\varepsilon)g'(y^\varepsilon)\right)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v.
\end{aligned}
\tag{16}
$$

Furthermore, since the function

$$f(\xi_1,\ldots,\xi_{n+1}) := \frac{\sum_{i=2}^n \xi_i^2 + g^2(\xi_1)}{\xi_{n+1}}, \quad ((\xi_1,\ldots,\xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+),$$

is convex, we have that

$$\begin{aligned} &\frac{1}{2}\left(\frac{\sum_{i=2}^n \eta_i^2 + g^2(\eta_1)}{\eta_{n+1}} - \frac{\sum_{i=2}^n \xi_i^2 + g^2(\xi_1)}{\xi_{n+1}}\right) \\ &\quad \geq \frac{\sum_{i=2}^n \xi_i(\eta_i - \xi_i)}{\xi_{n+1}} - \frac{\sum_{i=2}^n \xi_i^2 + g^2(\xi_1)}{2\xi_{n+1}^2}(\eta_{n+1} - \xi_{n+1}) \\ &\quad + \frac{g(\xi_1)g'(\xi_1)}{\xi_{n+1}}(\eta_1 - \xi_1), \\ &\quad ((\xi_1,\ldots,\xi_{n+1}),(\eta_1,\ldots,\eta_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+). \end{aligned} \tag{17}$$

Choosing $\xi_1 = y^\varepsilon$, $\eta_1 = Y^\varepsilon$, $\xi_i = y^\varepsilon_{x_i}$, $\eta_i = y^\varepsilon_{x_i}$ $(i = 2,\ldots,n)$, $\xi_{n+1} = y^\varepsilon_v$ and $\eta_{n+1} = Y^\varepsilon_v$ in (17), and since $\nabla' Y \equiv 0$, we derive from (16)

$$\begin{aligned} &\frac{1}{2}\iint_{U^\varepsilon}\left(\frac{g^2(Y^\varepsilon)}{Y^\varepsilon_v} - \frac{|\nabla' y^\varepsilon|^2 + g^2(y^\varepsilon)}{y^\varepsilon_v}\right) \mathrm{d}x'\,\mathrm{d}v \\ &\quad \geq \iint_{U^\varepsilon}\left(\tilde{f}(y^\varepsilon, v - \varepsilon y^\varepsilon) - 2\varepsilon g(y^\varepsilon)g'(y^\varepsilon)\right)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v. \end{aligned} \tag{18}$$

Next we claim the following rearrangement inequality:

$$\iint_{U^\varepsilon} \frac{g^2(y^\varepsilon)}{y^\varepsilon_v}\,\mathrm{d}x'\,\mathrm{d}v \geq \iint_{U^\varepsilon} \frac{g^2(Y^\varepsilon)}{Y^\varepsilon_v}\,\mathrm{d}x'\,\mathrm{d}v. \tag{19}$$

To show (19), it is sufficient to prove that

$$\int_\omega \frac{g^2(y^\varepsilon)}{y^\varepsilon_v}\,\mathrm{d}x' \geq \mathcal{L}^{n-1}(\omega)\frac{g^2(Y^\varepsilon)}{Y^\varepsilon_v} \quad \forall v \in [u_- + \varepsilon a_-, u_+ + \varepsilon a_+]. \tag{20}$$

By Jensen's inequality we have

$$g(Y^\varepsilon) \leq \frac{\int_\omega g(y^\varepsilon)\,\mathrm{d}x'}{\mathcal{L}^{n-1}(\omega)}. \tag{21}$$

Furthermore, the Cauchy–Schwarz inequality yields

$$\int_\omega \frac{g^2(y^\varepsilon)}{y_v^\varepsilon}\,\mathrm{d}x' \int_\omega y_v^\varepsilon\,\mathrm{d}x' \geq \left(\int_\omega g(y^\varepsilon)\,\mathrm{d}x'\right)^2. \tag{22}$$

Then (20) follows from (21) and (22). Since $\nabla' v^\varepsilon = \nabla' v$, we obtain from (18) and (19):

$$-\frac{1}{2}\int_{a_-}^{a_+}\int_\omega |\nabla' v|^2\,\mathrm{d}x'\,\mathrm{d}z$$
$$\geq \iint\limits_{U^\varepsilon} \left(\tilde{f}(y^\varepsilon, v - \varepsilon y^\varepsilon) - 2\varepsilon g(y^\varepsilon) g'(y^\varepsilon)\right)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v. \tag{23}$$

Since $f(x_1, u)$ is nondecreasing in u, we have

$$\left(\tilde{f}(y^\varepsilon, v - \varepsilon y^\varepsilon) - \tilde{f}(Y^\varepsilon, v - \varepsilon y^\varepsilon)\right)(y^\varepsilon - Y^\varepsilon) \geq 0. \tag{24}$$

Furthermore,

$$\iint\limits_{U^\varepsilon} \tilde{f}(Y^\varepsilon, v)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v = 0. \tag{25}$$

Now (23)–(25) yield

$$-\frac{1}{2}\int_{a_-}^{a_+}\int_\omega |\nabla' v|^2\,\mathrm{d}x'\,\mathrm{d}z$$
$$\geq \iint\limits_{U^\varepsilon} \left(\tilde{f}(Y^\varepsilon, v - \varepsilon y^\varepsilon) - 2\varepsilon g(y^\varepsilon) g'(y^\varepsilon)\right)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v$$
$$= \iint\limits_{U^\varepsilon} \left(\tilde{f}(Y^\varepsilon, v - \varepsilon y^\varepsilon) - \tilde{f}(Y^\varepsilon, v) - 2\varepsilon g(y^\varepsilon) g'(y^\varepsilon)\right)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v. \tag{26}$$

Since the last integral in (26) tends to zero as $\varepsilon \to 0$, we obtain that $\nabla' v \equiv 0$ in $(a_-, a_+) \times \omega$. This proves the assertion.

Remark 2 (1) The averaging transformation $y^\varepsilon \to Y^\varepsilon$ defined by (15) is related to a very simple type of rearrangement: Let $(v^\varepsilon)^*(z) \equiv (v^\varepsilon)^*(z, x')$,

$((z,x')\in(a_-,a_+)\times\omega)$, the inverse of Y^ε. Then $(v^\varepsilon)^*\in C^1([a_-,a_+]\times\bar\omega)$ and $\partial(v^\varepsilon)^*/\partial z\geq\varepsilon$. Furthermore, it is easy to see that v^ε and $(v^\varepsilon)^*$ are *equimeasurable*, i.e. we have

$$\mathcal{L}^n(\{c_1\leq v^\varepsilon\leq c_2\})=\mathcal{L}^n(\{c_1\leq(v^\varepsilon)^*\leq c_2\}),\\ \text{if } u_-+\varepsilon a_-\leq c_1\leq c_2\leq u_++\varepsilon a_+. \tag{27}$$

Finally, inequality (19) can be rewritten as

$$\int_{a_-}^{a_+}\int_\omega g^2(z)\left(\frac{\partial v^\varepsilon}{\partial z}\right)^2\mathrm{d}x'\,\mathrm{d}z\geq\int_{a_-}^{a_+}\int_\omega g^2(z)\left(\frac{\partial(v^\varepsilon)^*}{\partial z}\right)^2\mathrm{d}x'\,\mathrm{d}z. \tag{28}$$

(2) The above averaging rearrangement can be generalized for functions which are not increasing in x_1, and one could prove then several inequalities which are similar to (28) (see [7]). But since we actually need only the simple inequality (19) in our proofs, we will not go in detail here.

Next our aim is to apply the method to a related problem in the infinite cylinder Ω with (7) replaced by decaying conditions at infinity. To this we add some further conditions on f near $u_\pm$ which ensure exponential asymptotic behavior of u and ∇u. Note that the assumption $c\geq 0$ in Theorem 2 is not essential, since, if $c<0$, then by setting $w(x_1,x'):=-u(x_1,x')$ we arrive at an analogous problem for w, with c replaced by $-c$.

THEOREM 2 *Let u_+, u_-, $c\in\mathbb{R}$, $u_+>u_-$ and $c\geq 0$, and let $f\in C(\mathbb{R}\times[u_-,u_+])\cap C^{1,\alpha}(\mathbb{R}\times([u_-,u_-+\delta]\cup[u_+-\delta,u_+]))$ for some $\delta>0$ and $\alpha\in(0,1)$. Furthermore, let $f(x_1,u)$ nondecreasing in x_1, and let*

$$f(x_1,u_\pm)=0,\ f_u(x_1,u_\pm)=-b_\pm\quad\forall x_1\in\mathbb{R}, \tag{29}$$

for some $b_+,b_-\in\mathbb{R}$, satisfying

$$b_-\geq-\frac{c^2}{4},\quad b_+>0 \tag{30}$$

and

$$b_->0\quad \textit{if } c=0. \tag{31}$$

Finally, let $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C^1(\bar{\Omega})$ *satisfy:*

$$-\Delta u + cu_{x_1} = f(x_1, u), \quad u_{x_1} \geq 0 \text{ in } \Omega, \tag{32}$$

$$\lim_{x_1 \to \pm\infty} u(x_1, x') = u_\pm \quad \forall x' \in \bar{\omega}, \tag{33}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \quad (\nu\text{: exterior normal}). \tag{34}$$

Then u is independent of x'.

Proof Let

$$\lambda_\pm = \frac{c}{2} \mp \sqrt{\frac{c^2}{4} + b_\pm}. \tag{35}$$

Note that $\lambda_- > c$ and $0 > \lambda_+$. We choose $\lambda'_\pm, \lambda''_\pm \in \mathbb{R}$, such that

$$\lambda'_- > \lambda_- > \lambda''_- > c/2 \quad \text{and} \quad 0 > \lambda''_+ > \lambda_+ > \lambda'_+. \tag{36}$$

Then, in view of the results of ([4, Section 4]), we find numbers $c_1, c_2, R > 0$, such that

$$c_1 e^{\lambda'_- x_1} \leq u(x_1, x') - u_- \leq c_2 e^{\lambda''_- x_1}, \tag{37}$$

$$c_1 e^{\lambda'_- x_1} \leq u_{x_1}(x_1, x') \leq c_2 e^{\lambda''_- x_1}, \tag{38}$$

$$|\nabla u(x_1, x')| \leq c_2 e^{\lambda''_- x_1}, \quad \text{if } x_1 \leq -R, \tag{39}$$

$$c_1 e^{\lambda'_+ x_1} \leq u_+ - u(x_1, x') \leq c_2 e^{\lambda''_+ x_1}, \tag{40}$$

$$c_1 e^{\lambda'_+ x_1} \leq u_{x_1}(x_1, x') \leq c_2 e^{\lambda''_+ x_1}, \tag{41}$$

$$|\nabla u(x_1, x')| \leq c_2 e^{\lambda''_+ x_1}, \quad \text{if } x_1 \geq R. \tag{42}$$

We choose $h \in C^\infty(\mathbb{R})$ with $h(t) = -1$ for $t \leq -R$, $h(t) = 1$ for $t \geq R$ and $h'(t) > 0$ for $t \in (-R, R)$, and we set $u^\varepsilon(x) := u(x) + \varepsilon h(x_1)$, where $\varepsilon \in (0, 1)$. Then the functions u^ε satisfy asymptotical conditions analogous to (37)–(42), with $u_\pm$ replaced by $u_\pm \pm\varepsilon$, respectively, and we have

$$-\Delta u^\varepsilon + cu^\varepsilon_{x_1} = f(x_1, u^\varepsilon - \varepsilon h(x_1)) - \varepsilon h''(x_1) + \varepsilon c h'(x_1), \; u^\varepsilon_{x_1} > 0 \text{ in } \Omega. \tag{43}$$

Let $z, v, g(z)$ and $\tilde{f}$ be defined as in the previous proof.

First suppose that $c > 0$. Then

$$z = \frac{1 - e^{-cx_1}}{c} \tag{44}$$

and

$$g(z) = 1 - cz. \tag{45}$$

Setting $v^\varepsilon(z, x') := u^\varepsilon(x_1, x')$, let y^ε and Y^ε be defined as in the previous proof, and let $V^\varepsilon := (u_- - \varepsilon, u_+ + \varepsilon) \times \omega$. Then we obtain from (43) and (13):

$$\begin{aligned}
&\nabla'\left(\frac{\nabla' y^\varepsilon}{y_v^\varepsilon}\right) - \frac{\partial}{\partial v}\left(\frac{|\nabla' y^\varepsilon|^2 + (1 - cy^\varepsilon)^2}{2(y_v^\varepsilon)^2}\right) + \frac{c(1 - cy^\varepsilon)}{y_v^\varepsilon} \\
&\quad = \tilde{f}(y^\varepsilon, v - \varepsilon h(-(1/c)\log(1 - cy^\varepsilon))) - \varepsilon h''(-(1/c)\log(1 - cy^\varepsilon)) \\
&\qquad + \varepsilon c h'(-(1/c)\log(1 - cy^\varepsilon)), \quad 0 < y_v^\varepsilon < +\infty \text{ in } V^\varepsilon,
\end{aligned} \tag{46}$$

$$\lim_{v \to u_\pm \pm \varepsilon} y^\varepsilon(v, x') = \pm\infty \quad \forall x' \in \bar{\omega}, \tag{47}$$

$$\nabla' y^\varepsilon = 0 \quad \text{on } (u_- - \varepsilon, u_+ + \varepsilon) \times \partial\omega. \tag{48}$$

We choose $\delta' \in (0, \delta)$, such that

$$\{x:\ u_- + \delta' < u(x) < u_+ - \delta'\} \supset [-R, +R] \times \omega. \tag{49}$$

Multiplying (46) with $(y^\varepsilon - Y^\varepsilon)$ and then integrating over

$$V_k^\varepsilon := (u_- - \varepsilon + (1/k), u_+ + \varepsilon - (1/k)) \times \omega, \quad (k \in \mathbb{N}),$$

we obtain:

$$\begin{aligned}
&I_1'^{k,\varepsilon} + I_1''^{k,\varepsilon} + I_2^{k,\varepsilon} + I_3^{k,\varepsilon} \\
&\quad \equiv \iint\limits_{V_k^\varepsilon} \left(-\frac{\nabla' y^\varepsilon}{y_v^\varepsilon}\nabla'(y^\varepsilon - Y^\varepsilon) + \frac{|\nabla' y^\varepsilon|^2 + (1 - cy^\varepsilon)^2}{2(y_v^\varepsilon)^2}(y_v^\varepsilon - Y_v^\varepsilon)\right) dx'\, dv \\
&\qquad + \iint\limits_{V_k^\varepsilon} \frac{c(1 - cy^\varepsilon)}{y_v^\varepsilon}(y^\varepsilon - Y^\varepsilon)\, dx'\, dv \\
&\qquad - \int_\omega \frac{|\nabla' y^\varepsilon|^2 + (1 - cy^\varepsilon)^2}{2(y_v^\varepsilon)^2}(y^\varepsilon - Y^\varepsilon)\, dx' \bigg|_{v = u_+ + \varepsilon - (1/k)}
\end{aligned}$$

$$+ \int_\omega \frac{|\nabla' y^\varepsilon|^2 + (1 - cy^\varepsilon)^2}{2(y_v^\varepsilon)^2} (y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x' \bigg|_{v=u_- - \varepsilon + (1/k)}$$

$$= \iint_{V_k^\varepsilon} \tilde{f}(y^\varepsilon, v - \varepsilon h(-(1/c)\log(1 - cy^\varepsilon)))(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v$$

$$- \iint_{V_k^\varepsilon} \varepsilon h''(-(1/c)\log(1 - cy^\varepsilon))(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v$$

$$+ \iint_{V_k^\varepsilon} \varepsilon c h'(-(1/c)\log(1 - cy^\varepsilon))(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v$$

$$\equiv J_1^{k,\varepsilon} + J_2^{\prime k,\varepsilon} + J_2^{\prime\prime k,\varepsilon}. \tag{50}$$

Then we derive analogously as in the previous proof:

$$I_1^{\prime k,\varepsilon} + I_2^{\prime\prime k,\varepsilon} \le \frac{1}{2} \iint_{V_k^\varepsilon} \left(\frac{(1 - cY^\varepsilon)^2}{Y_v^\varepsilon} - \frac{|\nabla' y^\varepsilon|^2 + (1 - cy^\varepsilon)^2}{y_v^\varepsilon} \right) \mathrm{d}x'\,\mathrm{d}v$$

$$\le -\frac{1}{2} \iint_{\{u_- - \varepsilon + (1/k) < v^\varepsilon < u_+ + \varepsilon - (1/k)\}} |\nabla' v^\varepsilon|^2\,\mathrm{d}x'\,\mathrm{d}z. \tag{51}$$

Since $\nabla' v^\varepsilon = \nabla' v$, we can pass to the limit in (51) to obtain

$$\limsup_{\varepsilon \to 0} (I_1^{\prime k,\varepsilon} + I_2^{\prime\prime k,\varepsilon}) \le -\frac{1}{2} \iint_{\{u_- + (1/k) < v < u_+ - (1/k)\}} |\nabla' v|^2\,\mathrm{d}x'\,\mathrm{d}z. \tag{52}$$

Furthermore, we infer as before:

$$J_1^{k,\varepsilon} \ge \iint_{V_k^\varepsilon} \Big(\tilde{f}(Y^\varepsilon, v - \varepsilon h(-(1/c)\log(1 - cy^\varepsilon))) - \tilde{f}(Y^\varepsilon, v) \Big) \times (y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v. \tag{53}$$

The functions y^ε, Y^ε are uniformly bounded in V_k^ε for any $k \in \mathbb{N}$, by the estimates (37) and (40). Hence we derive from (50) and (53):

$$\liminf_{\varepsilon \to 0} (J_1^{k,\varepsilon} + J_2^{\prime k,\varepsilon} + J_2^{\prime\prime k,\varepsilon}) \ge 0. \tag{54}$$

In view of (41) and (49) we have $u_{x_1} > 0$ outside of $[-R, R] \times \omega$. Let $z = y(v, x')$ the inverse function of v with respect to z for $|x_1| > R$, and let Y be the corresponding average:

$$Y(v, x') \equiv Y(v) := \frac{\int_\omega y(v, \xi')\mathrm{d}\xi'}{\mathcal{L}^{n-1}(\omega)},$$
$$((v, x') \in (u_-, u_- + \delta') \cup (u_+ - \delta', u_+) \times \omega). \tag{55}$$

Note that in view of (49) we also have

$$y^\varepsilon(v - \varepsilon, \cdot) = y(v, \cdot) \qquad \text{for } v \le u_- + \delta' \text{ and} \tag{56}$$

$$y^\varepsilon(v + \varepsilon, \cdot) = y(v, \cdot) \quad \text{for } v \ge u_+ - \delta'. \tag{57}$$

Let $(1/k) < \delta'$. Then (50), (52), (54)–(57) yield:

$$\begin{aligned} -\frac{1}{2} \iint_{\{u_- + (1/k) < v < u_+ - (1/k)\}} & |\nabla' v|^2 \, \mathrm{d}x' \, \mathrm{d}z \\ & \ge - \int_\omega \frac{|\nabla' y|^2 + (1 - cy)^2}{2(y_v)^2} (y - Y) \, \mathrm{d}x' \bigg|_{v = u_+ - (1/k)} \\ & \quad + \int_\omega \frac{|\nabla' y|^2 + (1 - cy)^2}{2(y_v)^2} (y - Y) \, \mathrm{d}x' \bigg|_{v = u_- + (1/k)} \\ & \equiv I_2^k + I_3^k. \end{aligned} \tag{58}$$

Recall that

$$\frac{|\nabla' y|^2 + (1 - cy)^2}{(y_v)^2} = |\nabla u|^2.$$

Using (37) and (39) this gives

$$\frac{|\nabla' y|^2 + (1 - cy)^2}{(y_v)^2} \bigg|_{v = u_- + (1/k)} \le c_2^2 (c_1 k)^{-2\lambda''_- / \lambda'_-}. \tag{59}$$

Furthermore, we obtain from (37):

$$|y(u_- + (1/k), x')|, \, |Y(u_- + (1/k))| \le \frac{(c_1 k)^{c/\lambda'_-}}{c} \quad \forall x' \in \omega. \tag{60}$$

Then (59) and (60) yield in view of (36):

$$\lim_{k\to\infty} I_2^k = 0. \tag{61}$$

Similarly we obtain, using (40), (42) and (59):

$$\left.\frac{|\nabla' y|^2 + (1-cy)^2}{(y_v)^2}\right|_{v=u_+-(1/k)} \leq c_2^2(c_1k)^{-2\lambda_+''/\lambda_+'}$$

and

$$|y(u_+ - (1/k), x')|, |Y(u_+ - (1/k))| \leq 1 \quad \forall x' \in \omega, \;\text{ if } kc_1 \geq 1.$$

It follows that

$$\lim_{k\to\infty} I_3^k = 0. \tag{62}$$

Now from (58), (61) and (62) we infer that $\nabla' v \equiv 0$.

The proof is analogous – and even more simple – in the case $c=0$, since then $z=x_1$. The details are left to the reader.

The method of proof also applies to discontinuous nonlinearities f: Let H be the (multivalued) Heaviside function

$$H(t) := \begin{cases} 0 & \text{if } t < 0, \\ [0,1] & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{63}$$

LEMMA 1 **(1)** *Let* $d_i \in \mathbb{R}$ *and* $u_i \in (u_-, u_+)$, $(i=1,\dots,m)$. *Then the conclusions of Theorem 1 hold if the equation in* (4) *is replaced by*

$$-\Delta u + c(x_1)u_{x_1} - f(x_1,u) \in \sum_{i=1}^{m} \mathrm{d}_i H(u-u_i). \tag{64}$$

(2) *Let* d_i, u_i, $(i=1,\dots,m)$, *as in* **(1)** *and let* $g \in C^{1,\alpha}([u_-,u_+])$ *with* $g(u_-)=0$, $g(u_+) = \sum_{i=1}^m \mathrm{d}_i$, $g'(u_\pm)=0$ *and* $\alpha \in (0,1)$. *Then the conclusions of Theorem 2 hold if the equation in* (32) *is replaced by*

$$-\Delta u + cu_{x_1} - f(x_1,u) + g(u) \in \sum_{i=1}^{m} \mathrm{d}_i H(u-u_i). \tag{65}$$

Proof We consider the situation of Theorem 1. Let $-\Delta u+c(x_1)u_{x_1}-f(x,u)=:h(x)$. Then, proceeding as in the proof of Theorem 1, we arrive at

$$\begin{aligned}
&-\frac{1}{2}\int_{a_-}^{a_+}\int_{\omega}|\nabla' v|^2\,\mathrm{d}x'\,\mathrm{d}z \\
&\geq \iint\limits_{U^\varepsilon}\Big(\tilde{f}(Y^\varepsilon, v-\varepsilon y^\varepsilon)-\tilde{f}(Y^\varepsilon, v)-2\varepsilon g(y^\varepsilon)g'(y^\varepsilon)\Big)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v \\
&\quad+\iint\limits_{U^\varepsilon}\tilde{h}(y^\varepsilon, x')(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v \\
&\equiv I_1^\varepsilon + I_2^\varepsilon,
\end{aligned}\tag{66}$$

where $\tilde{h}(z,\cdot):=h(x_1,\cdot)$ and z is given by (7). Recall that

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = 0. \tag{67}$$

Setting

$$H_\varepsilon(t):=\begin{cases}1 & t>0,\\ 1+(t/\sqrt{\varepsilon}) & t\in[-\sqrt{\varepsilon},0],\\ 0 & t<-\sqrt{\varepsilon},\end{cases}$$

we have

$$\begin{aligned}
I_2^\varepsilon &= \iint\limits_{U^\varepsilon}\Big(\tilde{h}(y^\varepsilon, x')-\sum_{i=1}^{m}H_\varepsilon(v-\varepsilon y^\varepsilon-u_i)\Big)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v \\
&\quad+\sum_{i=1}^{m}\iint\limits_{U^\varepsilon}\Big(H_\varepsilon(v-\varepsilon y^\varepsilon-u_i)-H_\varepsilon(v-u_i)\Big)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v \\
&\quad+\sum_{i=1}^{m}\iint\limits_{U^\varepsilon}H_\varepsilon(v-u_i)(y^\varepsilon - Y^\varepsilon)\,\mathrm{d}x'\,\mathrm{d}v \\
&\equiv T_1^\varepsilon + T_2^\varepsilon + T_3^\varepsilon.
\end{aligned}\tag{68}$$

Since H_ε is continuous, we have

$$T_3^\varepsilon = 0 \quad \forall\varepsilon>0. \tag{69}$$

Furthermore,

$$|T_2^\varepsilon| \le \iint\limits_{G^\varepsilon} \frac{1}{\sqrt{\varepsilon}}\varepsilon|y^\varepsilon||y^\varepsilon - Y^\varepsilon|\,\mathrm{d}x'\,\mathrm{d}v \longrightarrow 0 \quad \text{as } \varepsilon \to 0. \tag{70}$$

Finally, the function $\tilde{h}(y^\varepsilon, x')$ differs from $\sum_{i=1}^m d_i H_\varepsilon(v - \varepsilon y^\varepsilon - u_i)$ at most on the set

$$\bigcup_{i=1}^m \{(v,x')\colon\ u_i - \sqrt{\varepsilon} < v - \varepsilon y^\varepsilon(v,x') < u_i\} =: M_\varepsilon.$$

Since $|y^\varepsilon| < c$ for some $c > 0$, independently from ε, we have that

$$M_\varepsilon \subset \bigcup_{i=1}^m \{u_i - \sqrt{\varepsilon} - \varepsilon c < v < u_i + \varepsilon c\} =: N_\varepsilon,$$

and since

$$\lim_{\varepsilon\to 0} \mathcal{L}^n(N_\varepsilon) = 0,$$

this yields

$$\lim_{\varepsilon\to 0} T_1^\varepsilon = 0. \tag{71}$$

Now (68)–(71) gives

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = 0. \tag{72}$$

The assertion then follows from (66), (67) and (72).

The proof carries over with obvious changes to the case of the problem in the infinite cylinder Ω.

Remark 3 We cannot expect uniqueness for solutions of the problems in Theorems 1 and 2 since there are easy counterexamples like the following one:

Let $n = 1$, $a = 2$,

$$u(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 + (x^3 - 1)^3 & \text{if } x \in (0,1), \\ 1 & \text{if } x \ge 1, \end{cases}$$

and $u^t(\cdot) := u(\cdot - t)$ where $|t| \leq 1$. Then both u and u^t are weak solutions of (4) and (5) with $c \equiv 0$, $u_- = 0$, $u_+ = 1$ and

$$f = f(u) = 18(1-u)^{1/3}\left(1-(1-u)^{1/3}\right)^{1/3}\left(3-4(1-u)^{1/3}\right),$$
$$(u \in [0,1]).$$

On the other hand, it is often easy to prove the uniqueness of solutions in similar problems, if we impose the stronger condition $u_{x_1} > 0$. This was kindly pointed out to me by Dolbeaut. For instance, there holds the following comparison result:

LEMMA 2 *Let $D \subset \Omega$ be a bounded domain which is convex in x_1-direction, let Σ be an open portion of $\partial D \cap (\mathbb{R} \times \partial\omega)$. Furthermore, let $c \in L^\infty(D)$ and nonincreasing in x_1, and let $f = f(x,u)$ be continuous in $\bar{D} \times \mathbb{R}$ and nondecreasing in x_1. Finally, let $u^i \in W^{2,\infty}(D) \cap C^1(\bar{D})$ satisfy*

$$-\Delta u^i + c(x)\frac{\partial u^i}{\partial x_1} = f(x,u^i) \ \ in\ D, \qquad \frac{\partial u^i}{\partial x_1} > 0 \ \ in\ \bar{D}, \tag{73}$$

$$\frac{\partial u^i}{\partial \nu} = 0 \ \ on\ \Sigma \quad (\nu\text{: exterior normal}), (i = 1,2), \tag{74}$$

and

$$u^1 \leq u^2 \quad on\ \partial D \backslash \Sigma. \tag{75}$$

Then $u^1 \leq u^2$ in D.

Proof Let $x_1 = y^i(v, x')$ the inverse functions of u^i with respect to x_1, let U_i, S_i the image of D and Σ, respectively, under the mapping $(x_1, x') \mapsto (u^i(x_1,x'), x')$ $(i = 1,2)$, and $U = U_1 \cap U_2$, $S = S_1 \cap S_2$. Then $y^i \in W^{2,\infty}(U) \cap C^1(\bar{U})$,

$$\nabla'\left(\frac{\nabla' y^i}{y^i_v}\right) - \frac{\partial}{\partial v}\left(\frac{|\nabla' y^i|^2 + 1}{2(y^i_v)^2}\right) + \frac{c(y^i, x')}{y^i_v} = f(y^i, x', v) \quad \text{in } U,$$
$$\nabla' y^i = 0 \quad \text{on } S, \ \ (i = 1,2),$$
$$y^1 \geq y^2 \quad \text{on } \partial U \backslash S.$$

Setting $w := y^2 - y^1$ we compute from this:

$$\begin{aligned}
&\frac{\Delta' w}{y^1_v} - 2\sum_{i=2}^{n} \frac{y^1_{x_i}}{(y^1_v)^2} w_{x_i v} + \frac{|\nabla' y^1|^2 + 1}{(y^1_v)^3} w_{vv} + \sum_{i=2}^{n} a_i w_{x_i} + a w_v \\
&\quad = \frac{c(y^1, x') - c(y^2, x')}{y^1_v} + f(y^2, x', v) - f(y^1, x', v) \quad \text{in } U, \\
&\quad \nabla' w = 0 \text{ on } S, \quad w \le 0 \text{ on } \partial U \backslash S,
\end{aligned} \tag{76}$$

where $a, a_i \in L^\infty(U)$ $(i = 2, \ldots, n)$. Since the Equation (76) is uniformly elliptic and the right-hand side is nonpositive by the assumptions, the maximum principle yields $w \le 0$ in G. This proves the lemma.

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