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Weighted Uniform Approximation on the Semiaxis by Rational Operators

B. DELLA VECCHIA [a], G. MASTROIANNI [b] and J. SZABADOS [c,*]

[a] *Dipartimento di Matematica, Università degli Studi di Roma "La Sapienza", Piazzale Aldo Moro 2, 00185 Roma, Italy;* [b] *Dipartimento di Matematica, Università della Basilicata, Via Nazario Sauro 85, 85100 Potenza, Italy;* [c] *Mathematical Institute of the Hungarian Academy of Sciences, H-1364 Budapest, P.O.B. 127, Hungary*

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We construct rational operators for the weighted uniform approximation on the semiaxis. Direct and converse results are shown which are not possible for polynomials.

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1 INTRODUCTION

Given a matrix of distinct nodes $X = \{x_i\}_{i=0}^n \subseteq [0, +\infty)$, consider the Balázs–Shepard operator relative to the matrix X defined by

$$S_n(X;f;x) := S_n(f;x) = \frac{\sum_{k=0}^n |x - x_k|^{-s} f(x_k)}{\sum_{k=0}^n |x - x_k|^{-s}}, \quad s > 2, \tag{1}$$

for $f \in C([0, +\infty))$.

We recall that for s an even integer, S_n is a positive rational operator of interpolatory type, of interest in approximation theory and in many applications (see, e.g., [1,2,4–7,9–12]).

* Corresponding author. E-mail: szabados@math-inst.hu. Research of this author supported by Hungarian National Foundation for Scientific Research, Grant No. T017425.

Recently the authors in [7] investigated the approximation behaviour of the operator (1), when f is a continuous and bounded function on the semiaxis and they proved direct and converse results.

In this paper we want to investigate the more general weighted case when the function f may be unbounded on the semiaxis.

In [11] Mastroianni and Szabados studied the weighted convergence of S_n operator with exponential-type weights in a similar situation (the real line) and they obtained only direct results for particular meshes.

Here first we show that, unlike for polynomials, for Shepard operators the weighted convergence with exponential weights is not guaranteed in general (Proposition 2.1). Therefore here we consider weights of rational type, i.e., functions having an algebraic growth rate at $+\infty$. For such functions we give weighted uniform approximation estimates by the operators S_n, involving a weighted modulus of smoothness related to the distribution mesh (Theorems 2.1 and 2.2). We also give converse results (Theorem 2.2). Our results are based on new weighted Markov–Bernstein inequalities for S_n (Lemmas 3.2 and 3.3). We also show that our results are sharp in some sense (see remarks to Theorems 2.1 and 2.2 and Proposition 3.1). Finally we remark that similar results are not possible for polynomials.

2 MAIN RESULTS

Letting

$$w(x) = (1+x)^{-\beta}, \quad \beta > 0 \tag{2}$$

we consider functions f continuous on $[0,+\infty)$ such that

$$w(x)|f(x)| \downarrow 0, \quad x \to +\infty. \tag{3}$$

Then we construct the Balázs–Shepard operator S_n given in (1) on the nodes

$$x_k = \frac{k^\gamma}{n^{\gamma/2}}, \quad k = 0, \ldots, n, \;\; \gamma \geq 1. \tag{4}$$

Here we want to study the weighted uniform convergence of $S_n(f)$ to f, with weight w given by (2), i.e., the convergence behaviour of $w(x)|f(x) - S_n(f;x)|$, for $x \geq 0$.

First we remark that it is not restrictive to consider weights of type (2) for weighted approximation by S_n based on the nodes (4), since for exponential-type weights the convergence is not guaranteed in general.

Indeed, putting $\|wf\| = \sup_{x\geq 0} w(x)|f(x)|$, we have

PROPOSITION 2.1 *Let S_n be the operator given by* (1). *If* $w(x) = \exp(-x)$, $f(x) = \exp(\sqrt{x})$ *and* $x_k = k/\sqrt{n}$, $k = 0, \dots, n$, *then*

$$\limsup_{n\to+\infty} \|wS_n(f)\| = +\infty. \tag{5}$$

We also remark that we can get weighted convergence of S_n with exponential weights, provided that we modify slightly the mesh like in [11], but in such case the study of direct and converse results is rather complicated (cf. [11]).

In the following C denotes a positive constant which may assume different values in different formulas. Moreover if ν and μ are two quantities depending on some parameters, we write $\nu \sim \mu$ if $|\nu/\mu|^{\pm 1} \leq C$, with C independent of the parameters.

Let

$$\omega^\phi(f;t)_w = \sup_{0\leq h\leq t} \|w\Delta_{h\phi} f\| \tag{6}$$

be the weighted modulus of smoothness of f with step function

$$\phi(x) = x^{1-1/\gamma}, \quad \gamma \geq 1 \tag{7}$$

(cf. [8]). Then we can give the following weighted uniform error estimate.

THEOREM 2.1 *Let S_n be the operator defined by* (1) *and* (4). *If f satisfies* (3), *then for* $s \geq (\beta\gamma+4)/(\gamma+1)$

$$\|w[f - S_n(f)]\| \leq C\left[\omega^\phi(f;1/\sqrt{n})_w + \frac{1}{n^{(\beta\gamma+1)/2}} \int_0^{n^{\gamma/2}} \frac{|f(t)|}{\phi(t)}\,\mathrm{d}t\right]. \tag{8}$$

Remarks From Theorem 2.1 we deduce the weighted uniform convergence of the Shepard operator based on the nodes given by (4), if $s \geq (\beta\gamma+4)/(\gamma+1)$.

From (8) it follows that our error estimates are strongly influenced by the mesh distribution (see the presence of the function ϕ and the exponent γ on the right-hand side of (8)).

Concerning the sharpness of Theorem 2.1, we remark that the appearance of the integral on the right-hand side of (8) is necessary. Indeed let $f_0(x)=1+x$ and $w(x)=(1+x)^{-1-\varepsilon/\gamma}$, $0<\varepsilon<1$. Then Theorem 2.1 gives

$$\begin{aligned}\|w[f_0 - S_n(f_0)]\| &\le C\left\{\frac{1}{\sqrt{n}} + \frac{n^{\gamma/2+1/2}}{n^{\gamma/2+\varepsilon/2+1/2}}\right\}\\ &\le C\left\{\frac{1}{\sqrt{n}} + \frac{1}{n^{\varepsilon/2}}\right\} \le \frac{C}{n^{\varepsilon/2}}.\end{aligned} \tag{9}$$

Now we show that

$$\sup_{x\ge 0} w(x)|f_0(x) - S_n(f_0;x)| \ge \frac{C}{n^{\varepsilon/2}},$$

i.e., (9) is sharp. In fact if $x=2n^{\gamma/2}$, then

$$\begin{aligned}&w(2n^{\gamma/2})|f_0(2n^{\gamma/2}) - S_n(f_0;2n^{\gamma/2})|\\ &= (1+2n^{\gamma/2})^{-1-\varepsilon/\gamma}\frac{\sum_{k=0}^{n}\left(2n^{\gamma/2} - (k^\gamma/n^{\gamma/2})\right)^{-s+1}}{\sum_{k=0}^{n}(2n^{\gamma/2} - (k^\gamma/n^{\gamma/2}))^{-s}}\\ &\ge Cn^{(-\gamma+\varepsilon)/2}n^{\gamma/2}\frac{\sum_{k=0}^{n}\left(2n^{\gamma/2} - (k^\gamma/n^{\gamma/2})\right)^{-s}}{\sum_{k=0}^{n}(2n^{\gamma/2} - (k^\gamma/n^{\gamma/2}))^{-s}} \ge Cn^{-\varepsilon/2}.\end{aligned}$$

Moreover if we assume something more on the integral on the right-hand side of (8), then we can state a direct and converse result.

Indeed, putting

$$K^{\phi}(f;t)_w = \inf_{\substack{h\in C([0,+\infty))\\ \|wh'\phi\|<+\infty}}\{\|w(f-h)\| + t\|w\phi h'\|\}, \tag{10}$$

the weighted K-functional of f with step function ϕ, we have

THEOREM 2.2 *Let*

$$s \ge (\beta\gamma+4)/(\gamma+1). \tag{11}$$

If f satisfies (3) *and*

$$\int_0^x \frac{|f(t)|}{\phi(t)}\,\mathrm{d}t = O(w^{-1}(x)), \quad \forall x > 0, \tag{12}$$

then

$$\|w[f - S_n(f)]\| \le C\omega^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_w \sim CK^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_w. \tag{13}$$

Moreover if

$$s \ge (\gamma(\beta+1)+3)/(\gamma+1) \tag{14}$$

and

$$\int_0^x \frac{|f(t)|}{\phi(t)} = O(w^{\alpha/(\beta\gamma)-1}(x)), \quad \forall x > 0,\ 0 < \alpha < 1, \tag{15}$$

then

$$\|w[f - S_n(f)]\| = O(n^{-\alpha/2}) \Leftrightarrow \omega^{\phi}(f; t)_w = O(t^{\alpha}). \tag{16}$$

Remarks First we remark that the above results are not possible for polynomials.

Note that from Theorem 2.1, if (12) holds true, the first term dominates on the right-hand side of (8), i.e., we have (13).

Furthermore from (13) it follows that, if f satisfies (12) and $w(x)|f(x)| = u(x)v(x)$, where $u(x)$ is a good function on $[0, +\infty)$ (for example $u \in C^2[0, +\infty)$) and $v(x) \le Cx^{1/\gamma}$, $x \le a$, $a > 0$, then the error is not greater that $Cn^{-1/2}$. We remark that such result is not possible for polynomials.

Moreover (13) cannot be improved because of (16).

In a sense, the equivalence relation (16) characterizes the class of functions satisfying (15) and having a given behaviour near 0 and on $[0, +\infty)$ by the order of approximation by the operator S_n.

Finally from the proof of (13) we can also get the following pointwise weighted error estimate for $x \in [x_0, x_n]$:

$$w(x)|f(x) - S_n(f; x)| \le C\omega\left(f; \frac{\phi(x)}{\sqrt{n}}\right)_w.$$

3 PROOF OF THE MAIN RESULTS

Proof of Proposition 2.1 Assume $n=m(m+1)$, with m a positive integer and let $x=1$. Then denote by x_j, $0\leq j\leq n$, a closest knot to x. Consequently

$$|1-x_j|=\left|\frac{\sqrt{n}-j}{\sqrt{n}}\right|\geq C\frac{\sqrt{m(m+1)}-m}{\sqrt{n}}\geq\frac{C}{\sqrt{n}}, \tag{17}$$

with $C>0$ an absolute constant.

Since

$$\sum_{k=0}^{n}\frac{1}{|1-x_k|^s}<\frac{n+1}{|1-x_j|^s},$$

we deduce

$$\begin{aligned} w(1)|S_n(f;1)| &= \exp(-1)\frac{\sum_{k=0}^{n}|1-(k/\sqrt{n})|^{-s}\exp(\sqrt{k/\sqrt{n}})}{\sum_{k=0}^{n}|1-(k/\sqrt{n})|^{-s}} \\ &\geq \frac{\exp(-1)\exp(n^{1/4})}{(\sqrt{n}-1)^s}\frac{|1-(j/\sqrt{n})|^s}{n+1}. \end{aligned}$$

Hence by (17) it follows that

$$w(1)|S_n(f;1)|\geq C\frac{\exp(n^{1/4})}{(\sqrt{n}-1)^s}\frac{1}{(\sqrt{n})^s n},$$

which is unbounded for $n\to+\infty$, that is we have the assertion.

The following lemma will be useful in the sequel. It establishes the boundedness of Shepard operator in the weighted norm.

LEMMA 3.1 *If*

$$s\geq(\beta\gamma+2)/(\gamma+1), \tag{18}$$

then for every function f defined on $[0,+\infty)$ *we have*

$$\|wS_n(f)\|\leq C\|wf\|\left\|wS_n\left(\frac{1}{w}\right)\right\|\leq C\|wf\| \tag{19}$$

with C a positive constant independent of f and n.

Note that Lemma 3.1 does not need the Assumption (3).

Proof Because of the interpolatory property of S_n at x_k, $k=0,\dots,n$, we may assume $x \neq x_k$, $k=0,\dots,n$. We distinguish two cases.

Case 1 $x \geq x_n$.
Then $w(x) \leq w(x_k)$, for every $k=0,\dots,n$, therefore

$$w(x)\frac{\sum_{k=0}^{n}|x-x_k|^{-s}|f(x_k)|}{\sum_{k=0}^{n}|x-x_k|^{-s}} \leq \|wf\|w(x)\frac{\sum_{k=0}^{n}|x-x_k|^{-s}w(x_k)^{-1}}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$
$$\leq \|wf\|\frac{\sum_{k=0}^{n}|x-x_k|^{-s}}{\sum_{k=0}^{n}|x-x_k|^{-s}} \leq \|wf\|.$$

Case 2 $x_0 < x < x_n$.
Let $x_j = j^\gamma/n^{\gamma/2}$, $0 \leq j \leq n$, denote the closest knot to x.
Then

$$\sum_{k=0}^{n}\frac{1}{|x-x_k|^s} \geq \frac{1}{|x-x_j|^s} \sim \frac{n^{s\gamma/2}}{[(j+1)^\gamma - j^\gamma]^s} \sim \frac{n^{s\gamma/2}}{j^{(\gamma-1)s}}. \tag{20}$$

Therefore from (20)

$$w(x)|S_n(f;x)| \leq \|wf\|w(x)\frac{\sum_{k=0}^{n}|x-x_k|^{-s}w(x_k)^{-1}}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$
$$\leq C\|wf\|\frac{j^{(\gamma-1)^s}}{n^{s\gamma/2}}\frac{n^{\beta\gamma/2}}{(n^{\gamma/2}+j^\gamma)^\beta}\sum_{\substack{k=0\\k\neq j}}^{n}\frac{n^{s\gamma/2}}{|j^\gamma-k^\gamma|^s}\frac{(n^{\gamma/2}+k^\gamma)^\beta}{n^{\beta\gamma/2}}$$
$$\leq C\|wf\|\frac{j^{(\gamma-1)s}}{(n^{\gamma/2}+j^\gamma)^\beta}\sum_{\substack{k=0\\k\neq j}}^{n}\frac{(n^{\gamma/2}+k^\gamma)^\beta}{|j-k|^s(j^{s(\gamma-1)}+k^{s(\gamma-1)})}.$$

Now we distinguish four cases.

Case a $0 \leq k \leq 2j$.
Then

$$\frac{j^{(\gamma-1)s}}{(n^{\gamma/2}+j^\gamma)^\beta}\sum_{\substack{k=0\\k\neq j}}^{2j}\frac{(n^{\gamma/2}+k^\gamma)^\beta}{|j-k|^s(j^{s(\gamma-1)}+k^{s(\gamma-1)})} \leq C\sum_{\substack{k=0\\k\neq j}}^{2j}\frac{1}{|j-k|^s} = O(1).$$

Case b $j \le \sqrt{n}$ and $2j \le k \le 2\sqrt{n}$.
Then

$$\begin{aligned}
&\frac{j^{(\gamma-1)s}}{(n^{\gamma/2}+j^{\gamma})^{\beta}}\sum_{k=2j}^{2\sqrt{n}}\frac{(n^{\gamma/2}+k^{\gamma})^{\beta}}{|j-k|^{s}(j^{s(\gamma-1)}+k^{s(\gamma-1)})}\\
&\le Cj^{(\gamma-1)s}\sum_{k=2j}^{2\sqrt{n}}\frac{1}{k^{s\gamma}}\le C\frac{j^{(\gamma-1)s}}{j^{s\gamma-1}}\\
&\le \frac{C}{j^{s-1}}=O(1).
\end{aligned}$$

Case c $j \le \sqrt{n}$ and $2\sqrt{n} \le k \le n$.
Then

$$\begin{aligned}
T:=&\frac{j^{(\gamma-1)s}}{(n^{\gamma/2}+j^{\gamma})^{\beta}}\sum_{k=2\sqrt{n}}^{n}\frac{(n^{\gamma/2}+k^{\gamma})^{\beta}}{(k-j)^{s}(j^{s(\gamma-1)}+k^{s(\gamma-1)})}\\
&\le C\frac{j^{(\gamma-1)s}}{n^{\beta\gamma/2}}\sum_{k=2\sqrt{n}}^{n}\frac{1}{k^{s\gamma-\beta\gamma}}\le Cn^{(s-\beta)\gamma/2-s/2}\sum_{k=2\sqrt{n}}^{n}\frac{1}{k^{(s-\beta)\gamma}}.
\end{aligned}$$

Now, if $s \ge \beta$, then

$$T \le Cn^{(s-\beta)\gamma/2-s/2}\frac{n}{n^{(s-\beta)\gamma/2}} = n^{1-s/2} = O(1).$$

On the other hand, if $s < \beta$, it follows that

$$\begin{aligned}
T&\le Cn^{(s-\beta)\gamma/2-s/2}\sum_{k=2\sqrt{n}}^{n}k^{\beta\gamma-s\gamma}\\
&\le Cn^{(s-\beta)\gamma/2-s/2}n^{(\beta-s)\gamma+1}=Cn^{\beta\gamma/2-s\gamma/2+1-s/2}=O(1),
\end{aligned}$$

if (18) holds true.

Case d $j \ge \sqrt{n}$ and $2j \le k \le n$.
Then

$$\begin{aligned}
U:=&\frac{j^{(\gamma-1)s}}{(n^{\gamma/2}+j^{\gamma})^{\beta}}\sum_{k=2j}^{n}\frac{(n^{\gamma/2}+k^{\gamma})^{\beta}}{(k-j)^{s}(j^{s(\gamma-1)}+k^{s(\gamma-1)})}\\
&\le C\frac{j^{(\gamma-1)s}}{j^{\gamma\beta}}\sum_{k=2j}^{n}\frac{k^{\gamma\beta}}{(k-j)^{s}k^{s(\gamma-1)}}\le Cj^{(\gamma-1)s-\gamma\beta}\sum_{k=2j}^{n}\frac{1}{k^{(s-\beta)\gamma}}.
\end{aligned}$$

Now, if $s \geq \beta$, then

$$U \leq Cj^{(\gamma-1)s-\gamma\beta}\frac{n}{j^{(s-\beta)\gamma}} \leq C\frac{n}{j^s} \leq C\frac{n}{(\sqrt{n})^s} = O(1).$$

On the other hand, if $s < \beta$, it follows that

$$\begin{aligned} U &\leq Cj^{(\gamma-1)s-\gamma\beta}\frac{n}{n^{(s-\beta)\gamma}} \leq n^{(\gamma-1)s/2-\gamma\beta/2}\frac{n}{n^{(s-\beta)\gamma}} \\ &= Cn^{-s\gamma/2-s/2+\gamma\beta/2+1} = O(1), \end{aligned}$$

if (18) holds true. So finally we get

$$\|wS_n(f)\| \leq C\|wf\|,$$

that is the assertion.

We remark that condition (18) in Lemma 3.1 is sharp in some sense. Indeed we have the following:

PROPOSITION 3.1 *Let x_k, $k=0,\dots,n$, be the knots given by* (4). *If $s < (\beta\gamma+2)/(\gamma+1)$, then*

$$\limsup_{n\to+\infty}\left\|wS_n\left(\frac{1}{w}\right)\right\| = +\infty. \tag{21}$$

Proof Let $n = m(m+1)$ and put $x = 1$. Then, if x_j denotes a closest knot to 1, then, from (17) it follows that

$$\begin{aligned} |1-x_j| &= \frac{n^{\gamma/2}-j^\gamma}{n^{\gamma/2}} \geq C\frac{|m^{\gamma/2}(m+1)^{\gamma/2}-j^\gamma|}{n^{\gamma/2}} \\ &\geq C\frac{m^{\gamma/2}m^{\gamma/2-1}}{n^{\gamma/2}} = C\frac{m^{\gamma-1}}{n^{\gamma/2}} \geq \frac{C}{\sqrt{n}}. \end{aligned} \tag{22}$$

Now, recalling that $a^\gamma - b^\gamma > C(a-b)a^{\gamma-1}$, if $a > b$ and $\gamma \geq 1$, from (22) we get

$$\begin{aligned} \sum_{k=0}^{n}\frac{1}{|1-x_k|^s} &\leq \sum_{\substack{k=0\\k\neq j}}^{n}\frac{n^{\gamma s/2}}{|n^{\gamma/2}-k^\gamma|^s} + Cn^{s/2} \\ &\leq Cn^{s/2} + Cn^{\gamma s/2}\sum_{\substack{k=0\\k\neq j}}^{\sqrt{n}}\frac{1}{[\sqrt{n}-k]^s(\sqrt{n})^{(\gamma-1)s}} \end{aligned}$$

$$+ Cn^{\gamma s/2} \sum_{\sqrt{n}+1}^{n} \frac{1}{[k-\sqrt{n}]^s} \frac{1}{k^{(\gamma-1)s}}$$

$$\leq Cn^{s/2} + Cn^{\gamma s/2} \sum_{k \neq j} \frac{1}{|\sqrt{n}-k|^s (\sqrt{n})^{(\gamma-1)s}} \leq Cn^{s/2}.$$

Therefore

$$\frac{1}{\sum_{k=0}^{n} |1-x_k|^{-s}} \geq \frac{C}{n^{s/2}}. \tag{23}$$

Thus

$$V := \frac{\sum_{k=0}^{n} [1/(|1-x_k|^s w(x_k))]}{\sum_{k=0}^{n} |1-x_k|^{-s}} \geq \frac{C}{n^{s/2}} \sum_{k=n/2}^{n} \frac{n^{\gamma s/2}(n^{\gamma/2}+k^{\gamma})^{\beta}}{(n^{\gamma/2}-k^{\gamma})^s n^{\gamma\beta/2}}$$

$$\geq Cn^{\gamma s/2 - s/2 - \beta\gamma/2} \sum_{k=n/2}^{n} \frac{k^{\gamma\beta}}{(k-\sqrt{n})^s k^{(\gamma-1)s}}$$

$$\geq Cn^{\gamma s/2 - s/2 - \beta\gamma/2} \sum_{k=n/2}^{n} \frac{1}{k^{(s-\beta)\gamma}}$$

$$\geq Cn^{\gamma s/2 - s/2 - \beta\gamma/2} \frac{n}{n^{(s-\beta)\gamma}} = Cn^{\beta\gamma/2 + 1 - s/2 - s\gamma/2}$$

which is unbounded, if $s < (\beta\gamma+2)/(\gamma+1)$.

3.1 Proof of Theorem 2.1

Obviously we assume $x \neq x_k$, $k = 0, \ldots, n$. We recall that [7]

$$|x - x_j| \leq C\frac{\phi(x)}{n} \tag{24}$$

$$|x - x_k| \geq C\phi(x)\frac{|j-k|}{n}, \quad j \neq k \tag{25}$$

$$\left[\sum_{k=1}^{n} |x - x_k|^{-s}\right]^{-1} \leq |x - x_j|^s, \tag{26}$$

with x_j, $0 \leq j \leq n$, a closest knot to x and $\phi(x)$ given by (7).

We distinguish two cases.

Case 1 $x_0 < x < x_n$. Then from (1)

$$w(x)|f(x) - S_n(f;x)| \leq w(x)\frac{\sum_{k=0}^n |x - x_k|^{-s}|f(x) - f(x_k)|}{\sum_{k=0}^n |x - x_k|^{-s}}$$

$$= w(x)\frac{\sum_{k=0}^n |x - x_k|^{-s}|g(\theta) - g(k/\sqrt{n})|}{\sum_{k=0}^n |x - x_k|^{-s}},$$

with $g(\theta) = f(\theta^\gamma)$ and $x = \theta^\gamma$.

Hence if $g'\bar{w}$ is bounded on $[0, n]$, with $\bar{w}(\theta) = w(\theta^\gamma) = w(x)$, letting $\|\ \|_{[a,b]}$ be the usual supremum norm on $[a, b]$, we get

$$w(x)|f(x) - S_n(f;x)| \leq \|g'\bar{w}\|_{[0,n]} w(x)\frac{\sum_{k=0}^n |x - x_k|^{-s}\left|\int_\theta^{k/\sqrt{n}} \bar{w}(t)^{-1}\,\mathrm{d}t\right|}{\sum_{k=0}^n |x - x_k|^{-s}}$$

$$\leq C\|f'\phi w\|_{[x_0,x_n]}\frac{w(x)\sum_{k=0}^n |x - x_k|^{-s}}{\sum_{k=0}^n |x - x_k|^{-s}}$$

$$\times\left|\int_\theta^{k/\sqrt{n}} \bar{w}(t)^{-1}\,\mathrm{d}t\right|. \tag{27}$$

Hence it remains to estimate

$$\frac{w(x)\sum_{k=0}^n |x - x_k|^{-s}}{\sum_{k=0}^n |x - x_k|^{-s}}\left|\int_\theta^{k/\sqrt{n}} \bar{w}(t)^{-1}\,\mathrm{d}t\right|.$$

If $k/\sqrt{n} \leq \theta$, then $\bar{w}(t) \geq \bar{w}(\theta)$, for all $t \in [\theta, k/\sqrt{n}]$, therefore

$$\frac{w(x)\sum_{x_k \leq x} |x - x_k|^{-s}\left|\int_{k/\sqrt{n}}^\theta w(t)^{-1}\,\mathrm{d}t\right|}{\sum_{k=0}^n |x - x_k|^{-s}} \leq \frac{\sum_{x_k \leq x} |x - x_k|^{-s}(\theta - k/\sqrt{n})}{\sum_{k=0}^n |x - x_k|^{-s}}$$

and working as usual (see, e.g., [4–7,12]) by (24)–(26)

$$\frac{w(x)\sum_{x_k \leq x} |x - x_k|^{-s}\left|\int_{k/\sqrt{n}}^\theta \bar{w}(t)^{-1}\,\mathrm{d}t\right|}{\sum_{k=0}^n |x - x_k|^{-s}} \leq \frac{C}{\sqrt{n}}. \tag{28}$$

If $\theta \leq k/\sqrt{n}$, then

$$\frac{w(x)\sum_{x_k \geq x}|x-x_k|^{-s}\left|\int_\theta^{k/\sqrt{n}}\bar{w}(t)^{-1}\,\mathrm{d}t\right|}{\sum_{k=0}^n|x-x_k|^{-s}} \leq \frac{w(x)\sum_{x_k\geq x}|x-x_k|^{-s}w(x_k)^{-1}(k/\sqrt{n}-\theta)}{\sum_{k=0}^n|x-x_k|^{-s}}.$$

Now, proceeding case for case as in Lemma 3.1, we get from (11)

$$\frac{w(x)\sum_{x_k \geq x}|x-x_k|^{-s}\left|\int_\theta^{k/\sqrt{n}}\bar{w}(t)^{-1}\,\mathrm{d}t\right|}{\sum_{k=0}^n|x-x_k|^{-s}} \leq \frac{C}{\sqrt{n}}. \tag{29}$$

Therefore from (27)–(29), if $x \in [x_0, x_n]$

$$w(x)|f(x)-S_n(f;x)| \leq \frac{C\|f'\phi w\|_{[x_0,x_n]}}{\sqrt{n}}. \tag{30}$$

Hence for every h such that $wh'\phi$ is bounded on $[x_0, x_n]$, we obtain

$$\begin{aligned}\|w[f-S_n(f)]\|_{[x_0,x_n]} \leq \|w[f-h]\|_{[x_0,x_n]} &+ \|w[h-S_n(h)]\|_{[x_0,x_n]} \\ &+ \|wS_n(f-h)\|_{[x_0,x_n]}.\end{aligned} \tag{31}$$

By (30)

$$\|w[h-S_n(h)]\|_{[x_0,x_n]} \leq \frac{C\|h'\phi w\|_{[x_0,x_n]}}{\sqrt{n}}. \tag{32}$$

On the other hand, if $x \in [x_0, x_n]$, from Lemma 3.1 we get

$$\|wS_n(f-h)\|_{[x_0,x_n]} \leq C\|w[f-h]\|_{[x_0,x_n]}. \tag{33}$$

Hence from (31)–(33)

$$\|w[f-S_n(f)]\|_{[x_0,x_n]} \leq C\|w[f-h]\|_{[x_0,x_n]} + C\frac{\|w\phi h'\|_{[x_0,x_n]}}{\sqrt{n}}$$

and taking the infimum on h

$$\|w[f - S_n(f)]\|_{[x_0,x_n]} \leq CK^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_{w,[x_0,x_n]} \leq CK^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_{w} \tag{34}$$

where $K^{\phi}(f; 1/\sqrt{n})_{w,[x_0,x_n]}$ is the weighted K-functional of f relative to the interval $[x_0, x_n]$. Since [8]

$$K^{\phi}(f; t)_w \sim \omega^{\phi}(f; t)_w,$$

from (34) we deduce

$$\|w[f - S_n(f)]\|_{[x_0,x_n]} \leq C\omega^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_w.$$

Case 2 $\quad x > x_n$.

From the monotonicity of the limit (3), it follows that

$$\begin{aligned} w(x)|f(x) - S_n(f; x)| &\leq w(x)|f(x)| + w(x)|S_n(f; x)| \\ &\leq w(n^{\gamma/2})|f(n^{\gamma/2})| + w(x)\left\{\sum_{x_k > x/2} + \sum_{x_k < x/2}\right\} \\ &\quad \times \frac{|x - x_k|^{-s}|f(x_k)|}{\sum_{k=0}^{n} |x - x_k|^{-s}} \\ &:= w(n^{\gamma/2})|f(n^{\gamma/2})| + \Sigma_1 + \Sigma_2. \end{aligned} \tag{35}$$

Now by (3) and Lemma 3.1

$$\begin{aligned} \Sigma_1 &\leq w(x) \frac{\sum_{x_k > x/2} [|x - x_k|^{-s} w(x_k)|f(x_k)|/w(x_k)]}{\sum_{k=0}^{n} |x - x_k|^{-s}} \\ &\leq w\left(\frac{x}{2}\right)\left|f\left(\frac{x}{2}\right)\right| w(x) \frac{\sum_{x_k > x/2} [|x - x_k|^{-s}/w(x_k)]}{\sum_{k=0}^{n} |x - x_k|^{-s}} \\ &\leq w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right| w(x) \frac{\sum_{x_k > x/2} |x - x_k|^{-s} w(x_k)^{-1}}{\sum_{k=0}^{n} |x - x_k|^{-s}} \\ &\leq Cw\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|. \end{aligned} \tag{36}$$

On the other hand, since

$$\sum_{k=0}^{n} |x - x_k|^{-s} \geq \frac{n}{x^s}, \tag{37}$$

we have

$$\begin{aligned}\Sigma_2 &\leq Cw(x)\frac{x^s}{n}\frac{1}{x^s}\sum_{x_k<x/2}|f(x_k)| \\ &\leq \frac{C}{n}w(x)\sum_{x_k<x/2}|f(x_k)|\frac{\Delta_k}{x_{k-1}^{1-1/\gamma}}\sqrt{n}\end{aligned}$$

with $\Delta_k = x_k - x_{k-1} \geq Cx_{k-1}^{1-1/\gamma}/\sqrt{n}$ by (25).

Hence

$$\begin{aligned}\Sigma_2 &\leq \frac{C}{\sqrt{n}}w(x)\sum_{x_k<x/2}\frac{|f(x_k)|\Delta_k}{x_{k-1}^{1-1/\gamma}} \\ &\leq \frac{C}{\sqrt{n}}\frac{1}{n^{\gamma\beta/2}}\int_0^{n^{\gamma/2}}\frac{|f(t)|}{t^{1-1/\gamma}}\,\mathrm{d}t \\ &\leq \frac{C}{n^{\gamma\beta/2+1/2}}\int_0^{n^{\gamma/2}}\frac{|f(t)|}{\phi(t)}\,\mathrm{d}t. \end{aligned}\tag{38}$$

From the monotonicity of (3) we get

$$\begin{aligned}\frac{1}{n^{\gamma\beta/2+1/2}}\int_0^{n^{\gamma/2}}\frac{|f(t)|}{\phi(t)}\,\mathrm{d}t &\geq \frac{1}{n^{\gamma\beta/2+1/2}}\int_{n^{\gamma/2}/4}^{n^{\gamma/2}/2}\frac{|f(t)|}{\phi(t)}\,\mathrm{d}t \\ &= \frac{1}{n^{\gamma\beta/2+1/2}}\int_{n^{\gamma/2}/4}^{n^{\gamma/2}/2}\frac{w(t)|f(t)|}{w(t)\phi(t)}\,\mathrm{d}t \\ &\geq \frac{C}{n^{\gamma\beta/2+1/2}}w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right| \\ &\quad\times\int_{n^{\gamma/2}/4}^{n^{\gamma/2}/2}\frac{1}{w(t)\phi(t)}\,\mathrm{d}t \\ &\geq Cw\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|.\end{aligned}$$

From (35), (36), (38) and the last inequality we obtain

$$w(x)|f(x) - S_n(f;x)| \le \frac{C}{n^{\gamma\beta/2+1/2}} \int_0^{n^{\gamma/2}} \frac{|f(t)|}{\phi(t)}\,\mathrm{d}t$$

and hence the assertion.

The following lemmas are useful to prove Theorem 2.2. In particular they are interesting in themselves because they establish some weighted Markov–Bernstein type inequalities for the Shepard operator (Lemmas 3.2 and 3.3) (cf. [7] and [4]) for analogous results for the operator S_n in the unweighted case).

LEMMA 3.2 *Let* (18) *hold true. Then*

$$\|w\phi S_n'(f)\| \le C\sqrt{n}\|wf\|,$$

with ϕ given by (7) *and C independent of f and n.*

Proof Since $S_n'(f;x_k) = 0, k = 0,\dots,n$, we assume $x \ne x_k, k = 0,\dots,n$. We distinguish two cases.

Case 1 $x < x_n$.

Working as in [4] we get

$$\begin{aligned}
w(x)\phi(x)|S_n'(f;x)| \le & Cw(x)\phi(x)\Bigg[\sum_{k\ne j} \frac{|x-x_k|^{-s-1}|f(x_k)|}{\left(\sum_{k=1}^n |x-x_k|^{-s}\right)^2} \sum_{k\ne j} |x-x_k|^{-s} \\
& + \frac{\sum_{k\ne j}|x-x_k|^{-s}|f(x_k)|\sum_{k\ne j}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^n |x-x_k|^{-s}\right)^2} \\
& + \frac{|x-x_j|^{-s-1}|f(x_j)|}{\left(\sum_{k=0}^n |x-x_k|^{-s}\right)^2} \sum_{k\ne j} |x-x_k|^{-s} \\
& + \frac{|x-x_j|^{-s}|f(x_j)|}{\left(\sum_{k=0}^n |x-x_k|^{-s}\right)^2} \sum_{k\ne j} |x-x_k|^{-s-1} \\
& + \sum_{k\ne j} \frac{|x-x_k|^{-s}|f(x_k)|\,|x-x_j|^{-s-1}}{\left(\sum_{k=0}^n |x-x_k|^{-s}\right)^2} \\
& + \sum_{k\ne j} \frac{|x-x_k|^{-s-1}|f(x_k)|\,|x-x_j|^{-s}}{\left(\sum_{k=0}^n |x-x_k|^{-s}\right)^2}\Bigg] \\
:= & A_1 + A_2 + A_3 + A_4 + A_5 + A_6,
\end{aligned}$$

where again x_j, $0 \le j \le n$, denotes a closest knot to x.

Since $w(x) \sim w(x_j)$, easily we deduce from (24)–(26)

$$A_3 \le C\|wf\|\phi(x)|x - x_j|^{s-1} \sum_{k \ne j} |x - x_k|^{-s}$$
$$\le C\|wf\| \frac{\phi^s(x)}{(\sqrt{n})^{s-1}} \sum_{k \ne j} \frac{n^{s/2}}{\phi^s(x)|k - j|^s} \le C\|wf\|\sqrt{n}$$

and

$$A_4 \le C\phi(x)\|wf\| \frac{\sqrt{n}}{\phi(x)} \frac{\sum_{k \ne j} |x - x_k|^{-s}}{\sum_{k=0}^{n} |x - x_k|^{-s}}$$
$$\le C\|wf\|\sqrt{n}.$$

By Lemma 3.1 and (24)–(26), if (18) holds true

$$A_2 \le C\phi(x)\|wf\|w(x) \frac{\sum_{k \ne j} |x - x_k|^{-s} w^{-1}(x_k)}{\sum_{k=0}^{n} |x - x_k|^{-s}} \frac{\sqrt{n}}{\phi(x)} \frac{\sum_{k \ne j} |x - x_k|^{-s}}{\sum_{k=0}^{n} |x - x_k|^{-s}}$$
$$\le C\|wf\|\sqrt{n},$$

$$A_1 \le Cw(x)\phi(x)\|wf\| \frac{\sum_{k \ne j} |x - x_k|^{-s} w^{-1}(x_k)}{\sum_{k=0}^{n} |x - x_k|^{-s}} \frac{\sqrt{n}}{\phi(x)}$$
$$\le C\|wf\|\sqrt{n},$$

and

$$A_6 \le Cw(x)\phi(x)\|wf\| \frac{\sum_{k \ne j} |x - x_k|^{-s} w^{-1}(x_k)}{\sum_{k=0}^{n} |x - x_k|^{-s}} \frac{\sqrt{n}}{\phi(x)}$$
$$\le C\|wf\|\sqrt{n}.$$

Moreover working as in Lemma 3.1

$$A_5 \le w(x)\phi(x)\|wf\| \sum_{k \ne j} |x - x_k|^{-s} w(x_k)^{-1} |x - x_j|^{s-1}$$
$$\le \|wf\|\phi(x)w(x) \frac{\sum_{k \ne j} |x - x_k|^{-s} w(x_k)^{-1}}{\sum_{k \ne j} |x - x_k|^{-s}} \sum_{k \ne j} |x - x_k|^{-s} |x - x_j|^{s-1}$$
$$\le C\|wf\|\phi(x) \sum_{k \ne j} \frac{n^{s/2}}{|k - j|^s \phi^s(x)} \frac{\phi(x)^{s-1}}{(\sqrt{n})^{s-1}}$$
$$\le C\|wf\|\sqrt{n}.$$

Case 2 $x > x_n$.

Now $w(x) < w(x_k)$, $k = 0, \dots, n$, hence $w(x)|f(x_k)| \leq \|wf\|$ and working as in [7] we get

$$w(x)\phi(x)|S_n'(f;x)| \leq C\|wf\|\sqrt{n}.$$

LEMMA 3.3 *Let* (14) *hold true. If f satisfies* (3), *then*

$$\|w\phi S_n'(f)\| \leq C\left\{\|w\phi f'\| + \frac{1}{n^{\beta\gamma/2}}\int_0^{n^{\gamma/2}} \frac{|f(t)|}{\phi(t)}\,\mathrm{d}t\right\},$$

with ϕ given by (7) *and C independent of f and n.*

Remark Note that if (15) holds true, then Lemma 3.3 gives

$$\|w\phi S_n'(f)\| \leq C\left\{\|w\phi f'\| + n^{-\alpha/2}\right\}.$$

Proof We assume $x \neq x_k$, $k = 0, \dots, n$. Since

$$S_n'(f;x) = \sum_{k=0}^{n} A_k'(x)[f(x_k) - f(x)] = \sum_{k=0}^{n} A_k'(x)\int_x^{x_k} f'(t)\,\mathrm{d}t$$

with

$$A_k(x) = \frac{|x - x_k|^{-s}}{\sum_{k=0}^{n}|x - x_k|^{-s}},$$

it follows

$$w(x)\phi(x)|S_n'(f;x)| \leq \sum_{k=0}^{n}|A_k'(x)|w(x)\phi(x)\left|\int_x^{x_k}\frac{1}{w(t)\phi(t)}\,\mathrm{d}t\right|\|w\phi f'\|.$$

Now we distinguish three cases.

Case 1 $x_k < x < x_n$.

Then $w(t) \geq w(x)$, $\forall t \in [x_k, x]$ and

$$\begin{aligned}\Sigma_1 &:= \sum_{x_k<x}(w\phi)(x)|A_k'(x)|\left|\int_{x_k}^{x}\frac{1}{w(t)\phi(t)}\,\mathrm{d}t\right| \\ &\leq \sum_{x_k<x}|A_k'(x)|\phi(x)\left|\int_{x_k}^{x}\frac{1}{\phi(t)}\,\mathrm{d}t\right|. \qquad (39)\end{aligned}$$

Now since (see, e.g., [4] or [7])

$$\phi(x)\left|\int_{x_k}^{x}\frac{1}{\phi(t)}\,\mathrm{d}t\right| \le C|x-x_k| \tag{40}$$

from (39) we obtain

$$\Sigma_1 \le C\sum_{x_k<x}|A_k'(x)|\,|x-x_k|.$$

Then working as in [4,7] we get

$$\sum_{x_k<x}|A_k'(x)|\,|x-x_k| \le C$$

and consequently

$$\Sigma_1 \le C.$$

Case 2 $x_k > x$.

Then $w(t) \ge w(x_k)$, therefore

$$\begin{aligned}\Sigma_2 &:= \sum_{x_k>x}|A_k'(x)|w(x)\phi(x)\left|\int_x^{x_k}\frac{1}{w(t)\phi(t)}\,\mathrm{d}t\right|\\ &\le \sum_{x_k>x}|A_k'(x)|\frac{w(x)}{w(x_k)}\phi(x)\left|\int_x^{x_k}\frac{1}{\phi(t)}\,\mathrm{d}t\right|\end{aligned}$$

and by (40)

$$\Sigma_2 \le \sum_{x_k>x}|A_k'(x)|\frac{w(x)}{w(x_k)}|x-x_k|. \tag{41}$$

Now since $w(x) \sim w(x_j)$, with x_j the closest knot to x, then

$$\begin{aligned}&w(x)\sum_{x_k>x}|A_k'(x)|w(x_k)^{-1}|x-x_k|\\ &\le Cw(x)\left[\frac{\sum_{k\ne j}|x-x_k|^{-s}w(x_k)^{-1}\sum_{k\ne j}|x-x_k|^{-s}}{\left(\sum_{k=0}^{n}|x-x_k|^{-s}\right)^2}\right.\\ &\qquad+\frac{\sum_{k\ne j}|x-x_k|^{-s+1}w(x_k)^{-1}\sum_{k\ne j}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^{n}|x-x_k|^{-s}\right)^2}\\ &\qquad+\frac{w(x_j)^{-1}|x-x_j|^{-s}\sum_{k\ne j}|x-x_k|^{-s}}{\left(\sum_{k=0}^{n}|x-x_k|^{-s}\right)^2}\end{aligned}$$

$$+\frac{w(x_j)^{-1}|x-x_j|^{-s+1}\sum_{k\neq j}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}$$
$$+\frac{\sum_{k\neq j}w(x_k)^{-1}|x-x_k|^{-s+1}|x-x_j|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}$$
$$\left.+\frac{\sum_{k\neq j}w(x_k)^{-1}|x-x_k|^{-s}|x-x_j|^{-s}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}\right]$$
$$:=B_1+B_2+B_3+B_4+B_5+B_6.$$

By Lemma 3.1, $B_1\le C$, $B_6\le C$ and since $w(x_j)\sim w(x)$ then $B_3\le C$. Moreover

$$\begin{aligned}B_4&\le C|x-x_j|^{s+1}\sum_{k\neq j}|x-x_k|^{-s-1}\\&\le C\frac{\phi(x)^{s+1}}{(\sqrt{n})^{s+1}}\sum_{k\neq j}\frac{(\sqrt{n})^{s+1}}{\phi(x)^{s+1}|k-j|^{s+1}}\le C.\end{aligned}$$

On the other hand by Lemma 3.1, if (14) holds true,

$$\begin{aligned}B_5&\le w(x)\frac{\sum_{k\neq j}|x-x_k|^{-s+1}w(x_k)^{-1}}{\sum_{k\neq j}|x-x_k|^{-s+1}}\frac{\sum_{k\neq j}|x-x_k|^{-s+1}|x-x_j|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}\\&\le C\frac{\sum_{k\neq j}|x-x_k|^{-s+1}|x-x_j|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}\end{aligned}$$

and working as in [4,7]

$$B_5\le C.$$

Finally by Lemma 3.1, working as in [4,7]

$$\begin{aligned}B_2&\le w(x)\frac{\sum_{k\neq j}|x-x_k|^{-s+1}w(x_k)^{-1}}{\sum_{k\neq j}|x-x_k|^{-s+1}}\frac{\sum_{k\neq j}|x-x_k|^{-s+1}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}\\&\le C.\end{aligned}$$

Case 3 $x > x_n$.

Here

$$\begin{aligned}
&w(x)\phi(x)|S_n'(f;x)| \\
&\quad \le Cw(x)\phi(x)\left[\frac{\sum_{k\neq j}|x-x_k|^{-s-1}|f(x_k)|\sum_{k\neq n}|x-x_k|^{-s}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}\right. \\
&\qquad + \frac{\sum_{k\neq j}|x-x_k|^{-s}|f(x_k)|\sum_{k\neq n}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^{-2}} \\
&\qquad + \frac{|x-x_n|^{-s-1}|f(x_n)|}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^2}\sum_{k\neq n}|x-x_k|^{-s} \\
&\qquad + \frac{|x-x_n|^{-s}|f(x_n)|\sum_{k\neq n}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^{-2}} \\
&\qquad + \frac{\sum_{k\neq n}|x-x_k|^{-s}|f(x_k)||x-x_n|^{-s-1}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^{-2}} \\
&\qquad \left. + \frac{\sum_{k\neq n}|x-x_k|^{-s-1}|f(x_k)||x-x_n|^{-s}}{\left(\sum_{k=0}^n|x-x_k|^{-s}\right)^{-2}}\right] \\
&\quad := E_1+E_2+E_3+E_4+E_5+E_6.
\end{aligned}$$

Now

$$\begin{aligned}
E_1 &\le w(x)\phi(x)\left\{\sum_{x_k>x/2}+\sum_{x_k<x/2}\right\}\frac{(|x-x_k|^{-s-1}/w(x_k))w(x_k)|f(x_k)|}{\sum_{k=0}^n|x-x_k|^{-s}} \\
&:= \Sigma_1+\Sigma_2.
\end{aligned}$$

Since $|x-x_k| > C\phi(x)/\sqrt{n}$, $k\neq n$, by the monotonicity of the limit (3) and Lemma 3.1 it follows that

$$\begin{aligned}
\Sigma_1 &\le Cw(x)w\left(\frac{x}{2}\right)\left|f\left(\frac{x}{2}\right)\right|\frac{\phi(x)}{\phi(x)}\sqrt{n}\frac{\sum_{x_k>x/2}|x-x_k|^{-s}w(x_k)^{-1}}{\sum_{k=0}^n|x-x_k|^{-s}} \\
&\le C\sqrt{n}w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|.
\end{aligned}$$

On the other hand, by (37), working as in the proof of Theorem 2.1, Case 2

$$\begin{aligned}
\Sigma_2 &\le Cw(x)\phi(x)\frac{x^s}{n}\frac{1}{x^{s+1}}\sum_{x_k<x/2}|f(x_k)| \\
&\le \frac{C}{n^{\beta\gamma/2}nx^{1/\gamma}}\sum_{x_k<x/2}|f(x_k)|\frac{\Delta_k}{\Delta_k} \\
&\le \frac{C}{n^{\beta\gamma/2}n\sqrt{n}}\sum_{x_k<x/2}|f(x_k)|\frac{\Delta_k}{\phi(x_k)}\sqrt{n} \\
&\le \frac{C}{n^{\beta\gamma/2}n}\int_0^{n^{\gamma/2}}\frac{|f(t)|}{\phi(t)}\,\mathrm{d}t.
\end{aligned}$$

Moreover

$$E_6 \le \frac{w(x)\phi(x)\sum_{k\ne n}|x-x_k|^{-s-1}|f(x_k)|}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$

and we can work as in the estimate of E_1.

On the other hand, since $w(x)\le w(x_n)$, then

$$E_3 \le w(n^{\gamma/2})|f(n^{\gamma/2})|\frac{|x-x_n|^{-s-1}\sum_{k\ne n}|x-x_k|^{-s}}{\left(\sum_{k=0}^{n}|x-x_k|^{-s}\right)^2}$$

and working as in [7]

$$E_3 \le C\sqrt{n}w(n^{\gamma/2})|f(n^{\gamma/2})|.$$

Moreover

$$\begin{aligned}
E_2 &\le w(x)\phi(x)\frac{\left\{\sum_{x_k>x/2}+\sum_{x_k<x/2}\right\}|x-x_k|^{-s}|f(x_k)|\sum_{k=0}^{n-1}|x-x_k|^{-s-1}}{\left(\sum_{k=0}^{n}|x-x_k|^{-s}\right)^2} \\
&:= \Sigma' + \Sigma''.
\end{aligned}$$

Now by Lemma 3.1

$$\begin{aligned}
\Sigma' \le{}& \phi(x)w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|w(x)\frac{\sum_{x_k>x/2}|x-x_k|^{-s}w(x_k)^{-1}}{\sum_{k=0}^{n}|x-x_k|^{-s}} \\
&\times\frac{\sum_{k=0}^{n-1}|x-x_k|^{-s-1}}{\sum_{k=0}^{n}|x-x_k|^{-s}}
\end{aligned}$$

$$\leq C\phi(x)w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|\frac{\sqrt{n}}{\phi(x)}\frac{\sum_{k=0}^{n-1}|x-x_k|^{-s}}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$
$$\leq C\sqrt{n}w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|.$$

On the other hand

$$\Sigma'' \leq C\frac{w(x)\phi(x)}{x^s}\frac{x^s}{n}\sum_{x_k<x/2}|f(x_k)|\frac{\sum_{k=0}^{n-1}|x-x_k|^{-s-1}}{\sum_{k=0}^{n}|x-x_k|^{-s}}$$
$$\leq C\frac{\phi(x)}{n^{\beta\gamma/2}n}\sum_{x_k<x/2}|f(x_k)|\frac{\sqrt{n}}{\phi(x)}\Delta_k$$
$$\leq \frac{C}{n^{\beta\gamma/2}\sqrt{n}}\sum_{x_k<x/2}|f(x_k)|\Delta_k,$$

and working as in the estimate of Σ_2, we get

$$\Sigma'' \leq \frac{C}{n^{\beta\gamma/2}\sqrt{n}}\int_0^{n^{\gamma/2}}\frac{|f(t)|}{\phi(t)}\,\mathrm{d}t.$$

Moreover, working as in the proof of Lemma 3.2

$$E_4 \leq w(n^{\gamma/2})|f(n^{\gamma/2})|\phi(x)|x-x_n|^s\sum_{k=0}^{n-1}|x-x_k|^{-s-1}$$
$$\leq C\sqrt{n}w(n^{\gamma/2})|f(n^{\gamma/2})|.$$

Finally

$$E_5 \leq w(x)\phi(x)|x-x_n|^{s-1}\sum_{k=0}^{n-1}|x-x_k|^{-s}|f(x_k)|$$
$$= w(x)\phi(x)|x-x_n|^{s-1}\left\{\sum_{x_k<x/2}+\sum_{x_k>x/2}\right\}|x-x_k|^{-s}|f(x_k)|$$

and working as above

$$E_5 \leq C\left\{\sqrt{n}w\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right| + \frac{1}{n^{\beta\gamma/2}}\int_0^{n^{\gamma/2}}\frac{\varepsilon_f^{\infty}(t)}{\phi(t)}\,\mathrm{d}t\right\}.$$

Now from the monotonicity of the limit (3),

$$\begin{aligned}\frac{1}{n^{\beta\gamma/2}}\int_0^{n^{\gamma/2}}\frac{|f(t)|}{\phi(t)}\,dt &\geq \frac{1}{n^{\beta\gamma/2}}\int_{n^{\gamma/2}/4}^{n^{\gamma/2}/2}\frac{w(t)|f(t)|}{w(t)\phi(t)}\,dt\\ &\geq \frac{w\left(n^{\gamma/2}/2\right)\left|f\left(n^{\gamma/2}/2\right)\right|}{n^{\beta\gamma/2}}\int_{n^{\gamma/2}/4}^{n^{\gamma/2}/2}\frac{1}{w(t)\phi(t)}\,dt\\ &\geq Cw\left(\frac{n^{\gamma/2}}{2}\right)\left|f\left(\frac{n^{\gamma/2}}{2}\right)\right|\sqrt{n}\end{aligned}$$

we get the assertion.

3.2 Proof of Theorem 2.2

From Theorem 2.1, under the Assumptions (11) and (12), it follows that

$$\begin{aligned}\|w[f-S_n(f)]\| &\leq C\left\{\omega^\phi\left(f;\frac{1}{\sqrt{n}}\right)_w+\frac{1}{\sqrt{n}}\right\}\\ &\leq C\omega^\phi\left(f;\frac{1}{\sqrt{n}}\right)_w \sim CK^\phi\left(f;\frac{1}{\sqrt{n}}\right)_w\end{aligned}$$

that is (13).

Moreover if (15) holds true and $\omega^\phi(f;t)_w \leq Ct^\alpha$, then

$$\|w[f-S_n(f)]\| \leq Cn^{-\alpha/2}.$$

On the other hand from the definition of $K^\phi(f)_w$, we obtain for $h\in C([0,+\infty))$, $\|wh'\phi\|<+\infty$ and h satisfying (3) and (15)

$$\begin{aligned}K^\phi\left(f;\frac{1}{\sqrt{n}}\right)_w &\leq \|w[f-S_k(f)]\|+\frac{1}{\sqrt{n}}\|w\phi S_k'(f)\|\\ &\leq \|w[f-S_k(f)]\|+\frac{1}{\sqrt{n}}\{\|w\phi S_k'(f-h)\|+\|w\phi S_k'(h)\|\}.\end{aligned}$$

Now, by using Lemma 3.2 and the remark to Lemma 3.3, if (14) and (15) hold true, we get

$$\begin{aligned}K^\phi\left(f;\frac{1}{\sqrt{n}}\right)_w &\leq \|w[f-S_k(f)]\|\\ &\quad + C\frac{\sqrt{k}}{\sqrt{n}}\left\{\|w[f-h]\|+\frac{1}{\sqrt{k}}\|w\phi h'\|\right\}+\frac{C}{\sqrt{n}}k^{-\alpha/2}.\end{aligned}$$

Since it is easy to see that

$$\inf_{\substack{h\in C([0,+\infty))\\ \|wh'\phi\|<+\infty\\ h \text{ satisfying (3) and (15)}}} \left\{ \|w[f-h]\| + \frac{1}{\sqrt{n}}\|w\phi h'\| \right\} \leq CK^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_w$$

we deduce

$$K^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_w \leq \|w[f-S_k(f)]\| + C\frac{\sqrt{k}}{\sqrt{n}}K^{\phi}\left(f; \frac{1}{\sqrt{k}}\right)_w + \frac{C}{\sqrt{n}}k^{-\alpha/2}.$$

Now if $\|w[f-S_n(f)]\| = O(n^{-\alpha/2})$, $0<\alpha<1$, then

$$K^{\phi}\left(f; \frac{1}{\sqrt{n}}\right)_w \leq Ck^{-\alpha/2} + C\frac{\sqrt{k}}{\sqrt{n}}K^{\phi}\left(f; \frac{1}{\sqrt{k}}\right)_w.$$

and from a well-known lemma by Berens and Lorentz (see, e.g., [3] or Lemma 9.34, p. 699 in [10]), (16) follows.

References

[1] G. Allasia, A class of interpolating positive linear operators, in: S.P. Singh (Ed.) *Approx. Theory*, *Wavelets and Applications* (1995), Dordrecht, Kluwer, pp. 1–36.

[2] R.E. Barnhill, R.P. Dube and F.F. Little, Properties of Shepard's surfaces, *Rocky Mountains J. Math.* **13** (1983), 365–382.

[3] G.G. Berens and G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972), 693–708.

[4] B. Della Vecchia, Direct and converse results by rational operators, *Constr. Approx.* **12** (1996), 271–285.

[5] B. Della Vecchia and G. Mastroianni, Pointwise simultaneous approximation by rational operators, *Jour. Approx. Th.* **65** (1991), 140–150.

[6] B. Della Vecchia, G. Mastroianni and J. Szabados, Balázs–Shepard operators on infinite intervals, *Annales Univ. Sci. Budapest., Sectio Comput.* **16** (1996), 93–102.

[7] B. Della Vecchia, G. Mastroianni and J. Szabados, Uniform approximation on the semiaxis by rational operators, to appear in *East Journal on Approximations* (1997).

[8] Z. Ditzian and V. Totik, Moduli of smoothness, *Springer Ser. Comput. Math.* **9** (1987).

[9] T. Foley, Scattered data interpolation and approximation with error bounds, *Comput. Aided Geom. Design* **3** (1986), 163–177.

[10] F.F. Little, Convex combination surfaces, in: R. Barhill and W. Boehm (Eds.), *Surfaces in Computer Aided Geometric Design* (1983), North-Holland, Amsterdam.

[11] G. Mastroianni and J. Szabados, Balázs–Shepard operators on infinite intervals II, *J. Approx. Theory* **90** (1997), 1–8.

[12] J. Szabados, Direct and converse approximation theorems for the Shepard operator, *J. Approx. Th. and its Appl.* **7** (1991), 63–76.