Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 301582, 25 pages
doi:10.1155/2009/301582

Research Article

Nonnegativity Preservation under Singular Values Perturbation

Emedin Montaño,[1,2] Mario Salas,[1] and Ricardo L. Soto[1]

[1] *Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280. Antofagasta, Chile*
[2] *Departamento de Matemática y Física, Universidad de Magallanes, Casilla 113-D, Punta Arenas, Chile*

Correspondence should be addressed to Ricardo L. Soto, rsoto@ucn.cl

Received 30 December 2008; Accepted 18 May 2009

Recommended by David Chelidze

We study how singular values and singular vectors of a matrix A change, under matrix perturbations of the form $A+\alpha \mathbf{u}_i\mathbf{v}_i^*$ and $A+\alpha \mathbf{u}_p\mathbf{v}_q^*$, $p \neq q$, where $\alpha \in \mathbb{R}$, A is an $m\times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r = \min\{m,n\}$, and $\mathbf{u}_j, \mathbf{v}_k$, $j = 1,\ldots,m; k = 1,\ldots,n$, are the left and right singular vectors, respectively. In particular we give conditions under which this kind of perturbations preserve nonnegativity and certain matrix structures.

1. Introduction

A *singular value decomposition* of a matrix $A \in \mathbb{C}^{m\times n}$ is a factorization $A = U\Sigma V^*$, where $\Sigma = \operatorname{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_r\} \in \mathbb{R}^{m\times n}$, $r = \min\{m,n\}$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ and both $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary. The diagonal entries of Σ are called *the singular values* of A. The columns $\mathbf{u}_j$ of U are called *left singular vectors* of A and the columns $\mathbf{v}_j$ of V are called *right singular vectors* of A. Every $A \in \mathbb{C}^{m\times n}$ has a *singular value decomposition* $A = U\Sigma V^*$ and the following relations hold: $A\mathbf{v}_j = \sigma_j\mathbf{u}_j, A^*\mathbf{u}_j = \sigma_j\mathbf{v}_j$, and $\mathbf{u}_j^*A\mathbf{v}_j = \sigma_j$. If $A \in \mathbb{R}^{m\times n}$, then U and V may be taken to be real (see [1]).

Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r = \min\{m,n\}$ and left and right singular vectors $\mathbf{u}_j, \mathbf{v}_k$, $j = 1,\ldots,m$; $k = 1,\ldots,n$, respectively. In this paper we study how singular values and singular vectors of A change, under matrix perturbations of the form $A+\alpha\mathbf{u}_i\mathbf{v}_i^*$ and $A+\alpha\mathbf{u}_p\mathbf{v}_q^*$, $p\neq q$, $\alpha\in\mathbb{R}$. Perturbations of the form $A+\alpha\mathbf{u}_i\mathbf{v}_i^*$ were used in [2] to construct nonnegative matrices with prescribed extremal singular values. Both kinds of perturbations are closely related to the *inverse singular value problem* (ISVP), which is the problem of constructing a structured matrix from its singular values. ISVP arises in many areas of application, such as circuit theory, computed tomography, irrigation theory, mass distributions, and so forth (see [3]). The ISVP can be seen as an extension of the *inverse eigenvalue problem* (IEP), which look for necessary and suffcient conditions for the existence of a structured matrix with prescribed spectrum. This problem arises in different applications, see for instance [4]. When the matrix is required to be nonnegative, we have the *nonnegative inverse eigenvalue problem* (NIEP).

In [5, 6] and references therein, in connection with the NIEP, it was used as a perturbation result due to Brauer [7], which shows how to modify one single eigenvalue of a matrix via a rank-one perturbation, without changing any of the remaining eigenvalues. This result was extended by Rado and presented by Perfect [8] to modify r eigenvalues of a matrix of order n, $r \leq n$, via a perturbation of rank-r, without changing any of the $n - r$ remaining eigenvalues. It was also used in conection with NIEP in [8, 9]. Since the eigenvalues and singular values of a matrix are closely related, the perturbation results of this paper, which preserve nonnegativity, may be also important in the NIEP. In particular, for the symmetric case, that is, the construction of a symmetric nonnegative matrix with prescribed spectrum, since the singular values are absolute values of the eigenvalues, similar results are obtained (see [10]). In [2] the following simple *singular value version* of the Rado and Brauer results were given.

Theorem 1.1. *Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$. Let*

$$U = (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p), \quad V = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_p), \quad p \leq r, \tag{1.1}$$

be matrices of order $m \times p$ and $n \times p$, whose columns are the left and right singular vectors, respectively, corresponding to σ_i, $i = 1, \ldots, p$. Let $D = \operatorname{diag}\{d_1, d_2, \ldots, d_p\}$ with $\sigma_i + d_i \geq 0$. Then $A + UDV^*$ has singular values

$$\{\sigma_1 + d_1, \ldots, \sigma_p + d_p, \sigma_{p+1}, \ldots, \sigma_r\}. \tag{1.2}$$

Note that the singular values of $A + \mathrm{UDV}^*$ are not necessarily in nondecreasing order. However we can reorderer them by using an appropriate permutation.

Corollary 1.2. *Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$. Let $\mathbf{u}_i$ and $\mathbf{v}_i$, respectively, the left and right singular vectors corresponding to σ_i, $i = 1, \ldots, r$. Let $\alpha \in \mathbb{R}$ such that $\alpha + \sigma_i \geq 0$, $i = 1, \ldots, r$. Then $A + \alpha \mathbf{u}_i \mathbf{v}_i^*$ has singular values*

$$\sigma_1, \ldots, \sigma_{i-1}, \sigma_i + \alpha, \sigma_{i+1}, \ldots, \sigma_r. \tag{1.3}$$

Remark 1.3. The perturbation given by Theorem 1.1 allow us to have certain control on the spectral condition number of the perturbed matrix. That is, if $\kappa_2(A) = \sigma_1/\sigma_r$, then we may choose $0 < \alpha_1 \leq \sigma_1 - \sigma_2$ and $0 < \alpha_r \leq \sigma_{r-1} - \sigma_r$ in such a way that

$$\kappa_2(A - \alpha_1 \mathbf{u}_1 \mathbf{v}_1^* + \alpha_r \mathbf{u}_r \mathbf{v}_r^*) = \frac{\sigma_1 - \alpha_1}{\sigma_r + \alpha_r} < \kappa_2(A). \tag{1.4}$$

The paper is organized as follows. In Section 2 we consider perturbations of the form $A + \alpha \mathbf{u}_i \mathbf{v}_i^*$, which we will call *simple perturbations*, and give sufficient conditions under which the perturbation preserves nonnegativity. In Section 3 we discuss perturbations of the form

$A + \alpha \mathbf{u}_p \mathbf{v}_q^*$, $p \neq q$, which, because of their different indices, we will call *mixed perturbations*. We also give sufficient conditions in order that mixed perturbations preserve nonnegativity. It is also shown that both, simple and mixed perturbations, preserve doubly stochastic structure. Finally, we show some examples to illustrate the results.

2. Nonnegativity Preservation under Simple Perturbations

Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$. In this section we consider perturbations $A + \alpha \mathbf{u}_i \mathbf{v}_i^T$, which preserve nonnegativity. Note that if A is an $m \times n$ nonnegative matrix, then the left and right singular vectors $\mathbf{u}_1$ and $\mathbf{v}_1$, corresponding to the maximal singular value σ_1, respectively, are nonnegative. Hence, in this case, the matrix $A + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ is nonnegative for all $\alpha > 0$.

Now, let us consider the perturbation $A + \alpha \mathbf{u}_i \mathbf{v}_i^T$, with $i > 1$. Let $\mathbf{u}_s$ and $\mathbf{v}_s$ be the left and right singular vectors corresponding to σ_s, $s > 1$, respectively. Let $\alpha > 0$ and let the entry in position (i, k) of $\mathbf{u}_s \mathbf{v}_s^T$ be negative. That is, $(\mathbf{u}_s \mathbf{v}_s^T)_{ik} < 0$. Then if $A = (a_{ik})$ is nonnegative,

$$a_{ik} + \alpha \left(\mathbf{u}_s \mathbf{v}_s^T \right)_{ik} \geq 0 \quad \text{iff } 0 < \alpha \leq \frac{a_{ik}}{\left| \left(\mathbf{u}_s \mathbf{v}_s^T \right)_{ik} \right|}. \tag{2.1}$$

Thus, to preserve the nonnegativity of A it is enough to choose α in the interval

$$\left(0, \min_{i,k} \frac{a_{ik}}{\left| \left(\mathbf{u}_s \mathbf{v}_s^T \right)_{ik} \right|} \right], \tag{2.2}$$

provided that $a_{ik} > 0$, otherwise $(\mathbf{u}_s \mathbf{v}_s^T)_{ik}$ must be zero. Then from (2.2) and Theorem 1.1 we have the following result.

Lemma 2.1. *Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$. Let α be in the interval*

$$\left(0, \min_{s} \min_{i,k} \frac{a_{ik}}{\left| \left(\mathbf{u}_s \mathbf{v}_s^T \right)_{ik} \right|} \right]. \tag{2.3}$$

Then $A + \alpha \mathbf{u}_s \mathbf{v}_s^T$, $s = 2, \ldots, r$, is nonnegative with singular values

$$\sigma_1, \ldots, \sigma_{s-1}, \sigma_s + \alpha, \sigma_{s+1}, \ldots, \sigma_r. \tag{2.4}$$

Remark 2.2. It is clear that if in Lemma 2.1, α is taken in

$$\left(0, \min_{s} \min_{i,k} \frac{a_{ik}}{\left| \left(\mathbf{u}_s \mathbf{v}_s^T \right)_{ik} \right|} \right), \tag{2.5}$$

then $A + \alpha \mathbf{u}_s \mathbf{v}_s^T$, $2 \leq s \leq r$, is positive with singular values

$$\sigma_1, \ldots, \sigma_{s-1}, \sigma_s + \alpha, \sigma_{s+1}, \ldots, \sigma_r. \tag{2.6}$$

Moreover, for α in intervals in Lemma 2.1 and this remark, the nonnegativity is obtained independently of the chosen singular vectors $\mathbf{u}_s$, $\mathbf{v}_s$.

A more handle interval for α is given by the following lemma.

Lemma 2.3. *Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m,n\}$. Let α be in the interval*

$$\left(0, \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j}\right]. \tag{2.7}$$

Then $A + \alpha \mathbf{u}_s \mathbf{v}_s^T$, $2 \leq s \leq r$, is nonnegative with singular values

$$\sigma_1, \ldots, \sigma_{s-1}, \sigma_s + \alpha, \sigma_{s+1}, \ldots, \sigma_r. \tag{2.8}$$

Proof. From (2.2) and since $|a_{ik}| \leq \sigma_1$, see [1, Corollary 3.1.3], we have

$$\begin{aligned}
\max_{i,k} \left| \left(\mathbf{u}_s \mathbf{v}_s^T \right)_{ik} \right| &= \max_{i,k} \left| \frac{a_{ik}}{\sigma_s} - \sum_{j=1, j \neq s}^{r} \frac{\sigma_j}{\sigma_s} \left(\mathbf{u}_j \mathbf{v}_j^T \right)_{ik} \right| \\
&\leq \frac{1}{\sigma_s} \max_{i,k} |a_{ik}| + \sum_{j=1, j \neq s}^{r} \max_{i,k} \frac{\sigma_j}{\sigma_s} \left| \left(\mathbf{u}_j \mathbf{v}_j^T \right)_{ik} \right| \\
&\leq \frac{1}{\sigma_s} \max_{i,k} |a_{ik}| + \sum_{j=1, j \neq s}^{r} \max_{i,k} \frac{\sigma_j}{\sigma_s} \\
&\leq \frac{\sigma_1}{\sigma_s} + \sum_{j=1, j \neq s}^{r} \frac{\sigma_j}{\sigma_s} \\
&\leq \frac{1}{\sigma_s} \left(2\sigma_1 + \sum_{j=2, j \neq s}^{r} \sigma_j \right) \\
&= \frac{1}{\sigma_s} \left(2\sigma_1 - \sigma_s + \sum_{j=2}^{r} \sigma_j \right) \\
&= 2\frac{\sigma_1}{\sigma_s} - 1 + \sum_{j=2}^{r} \frac{\sigma_j}{\sigma_s} \\
&\leq 2\frac{\sigma_1}{\sigma_r} - 1 + \sum_{j=2}^{r} \frac{\sigma_j}{\sigma_r} \\
&= \frac{\sigma_1}{\sigma_r} + \sum_{j=1}^{r-1} \frac{\sigma_j}{\sigma_r}.
\end{aligned} \tag{2.9}$$

Then

$$\frac{1}{\max_{i,k}\left|\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}\right|} \geq \frac{\sigma_r}{\sigma_1+\sum_{j=1}^{r-1}\sigma_j},$$
$$\frac{\min_{i,k} a_{ik}}{\max_{i,k}\left|\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}\right|} \geq \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1+\sum_{j=1}^{r-1}\sigma_j}. \tag{2.10}$$

Hence we have

$$\left(0, \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1+\sum_{j=1}^{r-1}\sigma_j}\right] \subseteq \left(0, \frac{\min_{i,k} a_{ik}}{\max_{i,k}\left|\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}\right|}\right] \subseteq \left(0, \min_{i,k} \frac{a_{ik}}{\left|\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}\right|}\right],$$
$$\left(0, \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1+\sum_{j=1}^{r-1}\sigma_j}\right] \subseteq \left(0, \min_{s}\min_{i,k} \frac{a_{ik}}{\left|\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}\right|}\right]. \tag{2.11}$$

Then, from Lemma 2.1 the result follows. □

Remark 2.4. For positive $A = (a_{ij})$, and $\alpha \in \mathbb{R}$, we repeat the arguments from Lemmas 2.1 and 2.3 to obtain that if α is in the interval

$$\left[-\frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1+\sum_{j=1}^{r-1}\sigma_j}, \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1+\sum_{j=1}^{r-1}\sigma_j}\right], \tag{2.12}$$

then $A + \alpha\mathbf{u}_s\mathbf{v}_s^T$, $s = 2, \ldots, r$, is nonnegative with singular values

$$\sigma_1, \ldots, \sigma_{s-1}, |\sigma_s + \alpha|, \sigma_{s+1}, \ldots, \sigma_r. \tag{2.13}$$

Now we consider rank-2 perturbations, $A + \mathrm{UDV}^*$, where $U = (\mathbf{u}_s, \mathbf{u}_t)$, $D = \mathrm{diag}\{\alpha_1, \alpha_2\}$, $V = (\mathbf{v}_s, \mathbf{v}_t)$. That is, perturbations of the form $A + \alpha_1\mathbf{u}_s\mathbf{v}_s^T + \alpha_2\mathbf{u}_t\mathbf{v}_t^T$ as in Theorem 1.1. Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$. Then $A + \alpha_1\mathbf{u}_s\mathbf{v}_s^T + \alpha_2\mathbf{u}_t\mathbf{v}_t^T$ will be nonnegative if

$$a_{ik} + \alpha_1\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik} + \alpha_2\left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik} \geq 0, \quad i = 1, \ldots, m; \; j = 1, \ldots, n. \tag{2.14}$$

From the family of straight lines

$$\alpha_2 = -\frac{\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}}{\left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik}}\alpha_1 - \frac{a_{ik}}{\left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik}}, \quad i = 1, \ldots, m; \; j = 1, \ldots, n, \tag{2.15}$$

it follows that they intersect the axes α_1 and α_2 at points

$$\left(-\frac{a_{ik}}{\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}},0\right), \qquad \left(0,\frac{a_{ik}}{\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}}\right), \tag{2.16}$$

where $(\mathbf{u}_s\mathbf{v}_s^T)_{ik} \neq 0$ and $(\mathbf{u}_t\mathbf{v}_t^T)_{ik} \neq 0$, respectively. Let

$$\begin{aligned} E &= \max_{\substack{i,k \\ \left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}>0}} \left(-\frac{a_{ik}}{\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}}\right), \\ F &= \max_{\substack{i,k \\ \left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik}>0}} \left(-\frac{a_{ik}}{\left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik}}\right), \\ G &= \min_{\substack{i,k \\ \left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}<0}} \left(-\frac{a_{ik}}{\left(\mathbf{u}_s\mathbf{v}_s^T\right)_{ik}}\right), \\ H &= \min_{\substack{i,k \\ \left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik}<0}} \left(-\frac{a_{ik}}{\left(\mathbf{u}_t\mathbf{v}_t^T\right)_{ik}}\right). \end{aligned} \tag{2.17}$$

Let

$$\begin{aligned} R_1 &= \left\{(\alpha_1,\alpha_2) : E \leq \alpha_1 \leq 0 \wedge -\frac{F}{E}\alpha_1 + F \leq \alpha_2 \leq -\frac{H}{E}\alpha_1 + H\right\}, \\ R_2 &= \left\{(\alpha_1,\alpha_2) : 0 \leq \alpha_1 \leq G \wedge -\frac{F}{G}\alpha_1 + F \leq \alpha_2 \leq -\frac{H}{G}\alpha_1 + H\right\}. \end{aligned} \tag{2.18}$$

Then $A + \alpha_1\mathbf{u}_s\mathbf{v}_s^T + \alpha_2\mathbf{u}_t\mathbf{v}_t^T$ is nonnegative for $(\alpha_1,\alpha_2) \in R_1 \cup R_2$.

A more handle region for (α_1,α_2) is given by the following lemma.

Lemma 2.5. *Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m,n\}$. Let $P_1 = (E,0)$, $P_2 = (0,F)$, $P_3 = (G,0)$, $P_4 = (0,H)$ be the intersection points in* (2.17)*. Let*

$$(\alpha_1,\alpha_2) \in S = \left\{(x,y) : \left\|(x,y)\right\|_1\right\} \leq \min_{k=1,2,3,4}\{\|P_k\|_1\}, \tag{2.19}$$

where $\|\cdot\|_1$ is the l_1 norm. Then $A + \alpha_1\mathbf{u}_s\mathbf{v}_s^T + \alpha_2\mathbf{u}_t\mathbf{v}_t^T$, $2 \leq s$, $t \leq r$, is nonnegative with singular values $\sigma_1,\ldots,|\sigma_s+\alpha_1|,\ldots,|\sigma_t+\alpha_2|,\ldots,\sigma_r$.

Example 2.6. Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 4 & 9 & 4 \end{pmatrix}, \tag{2.20}$$

with singular values $\sigma_1 = 15.5687298$, $\sigma_2 = 3.9581084$, and $\sigma_3 = 0.9736668$. Let

$$U = (\mathbf{u}_2 \mid \mathbf{u}_3), \qquad V = (\mathbf{v}_2 \mid \mathbf{v}_3), \qquad D = \operatorname{diag}\{\alpha_1, \alpha_2\}, \tag{2.21}$$

where

$$\mathbf{u}_2 = \begin{pmatrix} 0.4219823 \\ 0.5264618 \\ -0.7380847 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 0.8658718 \\ -0.4753192 \\ 0.1560054 \end{pmatrix},$$
$$\mathbf{v}_2 = \begin{pmatrix} 0.0257579 \\ -0.6669921 \\ 0.7446195 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} -0.9106840 \\ 0.2915541 \\ 0.2926616 \end{pmatrix}. \tag{2.22}$$

From (2.17) we compute E, F, G, H. Then, the intersection points are

$$(-12.7300876, 0), \quad (7.1058357, 0), \quad (0, -7.9224079), \quad (0, 1.2681734), \tag{2.23}$$

and $S = \{(x,y) : \|(x,y)\|_1 \leq 1.2681734\}$. Thus, from Lemma 2.5, for $(\alpha_1, \alpha_2) = (-1/2, 1/2)$ we have

$$A + \alpha_1 \mathbf{u}_2 \mathbf{v}_2^T + \alpha_2 \mathbf{u}_3 \mathbf{v}_3^T = \begin{pmatrix} 0.60030 & 2.2670 & 3.9696 \\ 5.2097 & 6.1063 & 7.7344 \\ 3.9385 & 8.7766 & 4.2976 \end{pmatrix}, \tag{2.24}$$

with singular values $\sigma_1, \sigma_2 + \alpha_1, \sigma_3 + \alpha_2$.

Remark 2.7. Let A be an $m \times n$ complex matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$ and singular value decomposition $A = U\Sigma V^*$. In [11] it was defined the concept of energy of A as $\mathcal{E}(A) = \sigma_1 + \sigma_2 + \cdots + \sigma_r$. If A is positive, then as an application of the rank-2 perturbation result, from Lemma 2.5 and Remark 2.4, we may construct, for

$$0 < \alpha \leq \min\left\{ \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j}, \sigma_j \right\}, \tag{2.25}$$

a family of nonnegative matrices $B = A + \alpha \mathbf{u}_i \mathbf{v}_i^T - \alpha \mathbf{u}_j \mathbf{v}_j^T$, with $\mathcal{E}(B) = \mathcal{E}(A)$. Now, suppose A is nonnegative with $\mathcal{E}(A) \leq C$, where C is an upper bound. Then from (1.2), by taking $\alpha = C - \mathcal{E}(A)$ we may construct a family of nonnegative matrices $B = A + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ with $\mathcal{E}(B) = \mathcal{E}(A) + \alpha = C$.

Now, in order to show that simple perturbations preserve doubly stochastic structure we need the following lemma. First we introduce a definition and a notation. An $n \times n$ matrix $A = (a_{ij})$ is said to be *with constant row sums* if $\sum_{j=1}^{n} a_{ij} = \alpha$, $i = 1, 2, \ldots, n$. We denote by CS_α the set of all matrices with constant row sums equal to α.

Lemma 2.8. *Let A be an $n \times n$ irreducible doubly stochastic matrix and let $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x}^T = (x_1, \dots, x_n)$, with $\lambda \neq 1$. Then*

$$S(\mathbf{x}) = x_1 + x_2 + \cdots + x_n = 0. \tag{2.26}$$

Proof. Since A is doubly stochastic, then

$$\begin{aligned} S(A\mathbf{x}) &= \Sigma a_{1j}x_j + \Sigma a_{2j}x_j + \cdots + \Sigma a_{nj}x_j \\ &= x_1\Sigma a_{i1} + x_2\Sigma a_{i2} + \cdots + x_n\Sigma a_{in} \\ &= S(x), \end{aligned} \tag{2.27}$$

and $S(A\mathbf{x}) = S(\lambda\mathbf{x}) = \lambda S(\mathbf{x})$. Then,

$$S(\mathbf{x}) = \lambda S(\mathbf{x}), \tag{2.28}$$

and since $\lambda \neq 1$, $S(\mathbf{x}) = 0$. □

The following result shows that simple perturbations preserve doubly stochastic structure.

Proposition 2.9. *Let A be an $n \times n$ irreducible doubly stochastic matrix. Then,*

$$\begin{aligned} &\text{(i)} \quad \left(A + \alpha_1\mathbf{u}_1\mathbf{v}_1^T\right) \in CS_{1+\alpha_1}, \qquad \left(A + \alpha_1\mathbf{u}_1\mathbf{v}_1^T\right)^T \in CS_{1+\alpha_1}, \\ &\text{(ii)} \quad \left(A + \alpha_i\mathbf{u}_i\mathbf{v}_i^T\right) \in CS_1, \quad \left(A + \alpha_i\mathbf{u}_i\mathbf{v}_i^T\right)^T \in CS_1; \quad i = 2, \dots, n, \\ &\text{(iii)} \quad \left(A + \textstyle\sum_{i=1}^{n}\alpha_i\mathbf{u}_i\mathbf{v}_i^T\right) \in CS_{1+\alpha_1}. \end{aligned} \tag{2.29}$$

Proof. Since $A, A^T \in \mathrm{CS}_1$, then $AA^T \in \mathrm{CS}_1$ and $A^TA \in \mathrm{CS}_1$. Hence, the singular vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are

$$\mathbf{u}_1 = \mathbf{v}_1 = \frac{1}{\sqrt{n}}\mathbf{e} = \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^T. \tag{2.30}$$

Then $\alpha_1\mathbf{u}_1\mathbf{v}_1^T = \alpha_1\mathbf{v}_1\mathbf{u}_1^T = (1/n)\alpha_1\mathbf{e}\mathbf{e}^T$ and $\alpha_1\mathbf{u}_1\mathbf{v}_1^T \in \mathrm{CS}_{\alpha_1}$. Thus (i) holds. From Lemma 2.8, $\alpha_i\mathbf{u}_i\mathbf{v}_i^T\mathbf{e} = \alpha_i(\mathbf{v}_i^T\mathbf{e})\mathbf{u}_i = 0$ and (ii) holds. From (i) and (ii) we have (iii). □

Example 2.10. Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \tag{2.31}$$

which is nonnegative generalized doubly stochastic, that is, A is nonnegative with $A, A^T \in \mathrm{CS}_{10}$, and A has singular values 10, 2.8284, 2, 8284, 2. Let $U\Sigma V^*$ be the singular value decomposition of A. Let $D = \mathrm{diag}\{4, 3, 2, 1\}$. Then

$$B = A + UDV^* = \begin{pmatrix} 6.2596 & 4.8500 & 2.2404 & 0.6501 \\ 1.0822 & 6.0081 & 4.4178 & 2.4919 \\ 2.2404 & 0.6501 & 6.2596 & 4.8500 \\ 4.4178 & 2.4919 & 1.0822 & 6.0081 \end{pmatrix} \tag{2.32}$$

is nonnegative generalized doubly stochastic with singular values 14, 5.8284, 4.8284, 3.

3. Nonnegativity Preservation under Mixed Perturbations

In this section we discuss matrix perturbations of the form $A + \alpha \mathbf{u}_k \mathbf{v}_j^*$, with $k \neq j$, which we will call mixed perturbations, and we study how the singular values and vectors change under this kind of perturbations. We also give sufficient conditions under which mixed perturbations preserve nonnegativity and preserve doubly stochastic structure. Let us start by considering the following particular case: let A be a 4×3 matrix with singular values σ_1, σ_2, σ_3. Let $A = U\Sigma V^*$ with $U = (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4)$ and $V = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3)$. Let $\alpha \geq 0$. That is,

$$A = (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4) \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \mathbf{v}_3^* \end{pmatrix} = \sum_{k=1}^{3} \sigma_k \mathbf{u}_k \mathbf{v}_k^*. \tag{3.1}$$

The matrix $\alpha \mathbf{u}_1 \mathbf{v}_2^*$ has the singular value decomposition:

$$\begin{aligned} \alpha \mathbf{u}_1 \mathbf{v}_2^* &= (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4) \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_2^* \\ \mathbf{v}_1^* \\ \mathbf{v}_3^* \end{pmatrix} \\ &= (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4) \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \mathbf{v}_3^* \end{pmatrix} \\ &= (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4) \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \mathbf{v}_3^* \end{pmatrix}. \end{aligned} \tag{3.2}$$

Then,

$$A + \alpha \mathbf{u}_1 \mathbf{v}_2^* = (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4) \begin{pmatrix} \sigma_1 & \alpha & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \mathbf{v}_3^* \end{pmatrix}. \tag{3.3}$$

Now we compute the singular values of the matrix

$$C = \begin{pmatrix} \sigma_1 & \alpha & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

by computing the eigenvalues of $\widetilde{C}\widetilde{C}^T$, where $\widetilde{C} = \begin{pmatrix} \sigma_1 & \alpha \\ 0 & \sigma_2 \end{pmatrix}$. Since

$$\widetilde{C}\widetilde{C}^T = \begin{pmatrix} \alpha^2 + \sigma_1^2 & \alpha\sigma_2 \\ \alpha\sigma_2 & \sigma_2^2 \end{pmatrix}, \tag{3.5}$$

then

$$\begin{aligned} \operatorname{tr}\left(\widetilde{C}\widetilde{C}^T\right) &= \alpha^2 + \sigma_1^2 + \sigma_2^2 = \lambda_1 + \lambda_2, \\ \det\left(\widetilde{C}\widetilde{C}^T\right) &= \sigma_1^2\sigma_2^2 = \lambda_1\lambda_2 \end{aligned} \tag{3.6}$$

with λ_1, λ_2 being the eigenvalues of $\widetilde{C}\widetilde{C}^*$. Thus, we obtain

$$\begin{aligned} \lambda_1 &= \frac{\alpha^2 + \sigma_1^2 + \sigma_2^2 + \sqrt{\left(\alpha^2 + \sigma_1^2 + \sigma_2^2\right)^2 - 4\sigma_1^2\sigma_2^2}}{2}, \\ \lambda_2 &= \frac{\alpha^2 + \sigma_1^2 + \sigma_2^2 - \sqrt{\left(\alpha^2 + \sigma_1^2 + \sigma_2^2\right)^2 - 4\sigma_1^2\sigma_2^2}}{2}. \end{aligned} \tag{3.7}$$

Hence, the singular values of $A + \alpha \mathbf{u}_1 \mathbf{v}_2^*$ are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sigma_3$.

Now we generalize these results for $m \times n$ matrices.

Theorem 3.1. *Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0, r = \min\{m, n\}$, and with singular value decomposition $A = U\Sigma V^*$, where*

$$\begin{aligned} U = (\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_k \mid \cdots \mid \mathbf{u}_m), \qquad V &= \left(\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_j \mid \cdots \mid \mathbf{v}_n\right), \\ \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \quad k, j &\leq r. \end{aligned} \tag{3.8}$$

Let $\alpha \in \mathbb{R}$. Then $A + \alpha \mathbf{u}_k \mathbf{v}_j^*$ has singular values,

$$\{\sigma_1, \dots \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \dots, \sigma_{j-1}, \tilde{\sigma}_j, \sigma_{j+1}, \dots, \sigma_r\}, \tag{3.9}$$

where

$$\tilde{\sigma}_k = \left(\frac{\alpha^2 + \sigma_k^2 + \sigma_j^2 + \sqrt{\left(\alpha^2 + \sigma_k^2 + \sigma_j^2\right)^2 - 4\sigma_k^2 \sigma_j^2}}{2} \right)^{1/2}, \tag{3.10}$$

$$\tilde{\sigma}_j = \left(\frac{\alpha^2 + \sigma_k^2 + \sigma_j^2 - \sqrt{\left(\alpha^2 + \sigma_k^2 + \sigma_j^2\right)^2 - 4\sigma_k^2 \sigma_j^2}}{2} \right)^{1/2}. \tag{3.11}$$

Proof. Without loss of generality we may assume that $m \geq n$ and $j > k$. The matrix $\alpha \mathbf{u}_k \mathbf{v}_j^*$ has a singular value decomposition

$$\alpha \mathbf{u}_k \mathbf{v}_j^* = (\pm \mathbf{u}_k \mid \cdots \mid \mathbf{u}_{k-1} \mid \mathbf{u}_1 \mid \mathbf{u}_{k+1} \mid \cdots \mid \mathbf{u}_m) \tilde{\tilde{\Sigma}} \begin{pmatrix} \mathbf{v}_j^* \\ \vdots \\ \mathbf{v}_{j-1}^* \\ \mathbf{v}_1^* \\ \mathbf{v}_{j+1}^* \\ \vdots \\ \mathbf{v}_n^* \end{pmatrix}, \tag{3.12}$$

where $\tilde{\tilde{\Sigma}} = \operatorname{diag}\{|\alpha|, 0, \dots, 0\}$. The decomposition in (3.12) can be written as

$$\alpha \mathbf{u}_k \mathbf{v}_j^* = U P_k \tilde{\tilde{\Sigma}} Q_j V^*, \tag{3.13}$$

where P_k and Q_j are $m \times m$ and $n \times n$ permutation matrices of the form

$$P_k = \begin{pmatrix} W_k & 0 \\ 0 & I_{m-k} \end{pmatrix}, \qquad Q_j = \begin{pmatrix} W_j & 0 \\ 0 & I_{n-j} \end{pmatrix}, \tag{3.14}$$

with

$$W_l = \begin{pmatrix} & & 1 \\ & I_{l-2} & \\ 1 & & \end{pmatrix} \quad \text{of order } l. \tag{3.15}$$

Since

$$\tilde{\Sigma} = \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \ddots \end{pmatrix}, \quad \text{then } P_k\tilde{\Sigma}Q_j = \begin{pmatrix} \ddots & & \\ & (\alpha)_{k,j} & \\ & & \ddots \end{pmatrix}, \tag{3.16}$$

where if $\alpha < 0$, we multiply the $\mathbf{u}_k$ vector by minus one. Then

$$A + \alpha \mathbf{u}_k \mathbf{v}_j^* = U\left(\Sigma + P_k\tilde{\Sigma}Q_j\right)V^* = U\begin{pmatrix} \Lambda \\ 0 \end{pmatrix}V^*, \tag{3.17}$$

where

$$\Lambda = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & \cdots & \alpha & \\ & & & \ddots & \vdots & \\ & & & & \sigma_j & \\ & & & & & \\ & & & & & \sigma_n \end{pmatrix}. \tag{3.18}$$

By applying row and column permutations on the matrix $\Sigma + P_k\tilde{\Sigma}Q_j$, it follows from (3.17) and (3.18) that the singular values of $A + \alpha \mathbf{u}_k \mathbf{v}_j^*$ are

$$\left\{\sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_{j-1}, \tilde{\sigma}_j, \sigma_{j+1}, \ldots, \sigma_n\right\}, \tag{3.19}$$

where $\tilde{\sigma}_k$ and $\tilde{\sigma}_j$ are as in (3.10) and (3.11), respectively. □

Observe that in Theorem 3.1, $A + \alpha \mathbf{u}_k \mathbf{v}_j^*$ has singular values

$$\left\{\sigma_1, \ldots \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_{j-1}, \tilde{\sigma}_j, \sigma_{j+1}, \ldots, \sigma_r\right\}, \tag{3.20}$$

if $k, j \leq r$. If $r < k \leq m$, then only σ_j, corresponding to the right singular vector $\mathbf{v}_j$, changes and take the form $\tilde{\sigma}_j = \sqrt{\alpha^2 + \sigma_j^2}$. A straightforward calculation shows that for $\alpha > 0$, $\tilde{\sigma}_k \leq \sigma_k + \alpha$. Observe that

$$\tilde{\sigma}_j = \sqrt{\alpha^2 + \sigma_j^2} \leq \sigma_j + \alpha. \tag{3.21}$$

Example 3.2. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 6 & 1 & 3 \\ 0 & 4 & 1 \\ 1 & 5 & 0 \end{pmatrix}, \tag{3.22}$$

with singular values $\sigma_1 = 7.8207$, $\sigma_2 = 5.6257$, $\sigma_3 = 1.09$. Let

$$\begin{aligned} \mathbf{u}_1 &= \begin{pmatrix} 0.26030 & 0.70386 & 0.39776 & 0.52784 \end{pmatrix}^T, \\ \mathbf{v}_2 &= \begin{pmatrix} 0.63252 & -0.71647 & 0.29425 \end{pmatrix}^T, \end{aligned} \tag{3.23}$$

left and right singular vectors of A, respectively. Let $\alpha = 1$. Then, from (3.10) and (3.11), the matrix

$$A + \alpha \mathbf{u}_1 \mathbf{v}_2^* = \begin{pmatrix} 1.1646 & 1.8135 & 0.076593 \\ 6.4452 & 0.49571 & 3.2071 \\ 0.25159 & 3.715 & 1.117 \\ 1.3339 & 4.6218 & 0.15532 \end{pmatrix} \tag{3.24}$$

has singular values $\tilde{\sigma}_1 = 7.9477$, $\tilde{\sigma}_2 = 5.5358$ and σ_3. By using Theorem 1.1 we have that $A + \mathbf{u}_1\mathbf{v}_1^* + \mathbf{u}_2\mathbf{v}_2^*$ has singular values $\sigma_1 + 1 = 8.8207$, $\sigma_2 + 1 = 6.6257$, and $\sigma_3 = 1.09$.

Different from perturbations of the form $A + \alpha \mathbf{u}_i \mathbf{v}_i^T$, the perturbation of Theorem 3.1 affects not only the singular values σ_k and σ_j, but also the corresponding left and right singular vectors $\mathbf{u}_k$ and $\mathbf{v}_j$. To make this modification clear, we consider again the previous discussion to Theorem 3.1:

$$A + \alpha \mathbf{u}_1 \mathbf{v}_2^* = (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4) \begin{pmatrix} \sigma_1 & \alpha & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \mathbf{v}_3^* \end{pmatrix}. \tag{3.25}$$

Let

$$\widetilde{C} = \begin{pmatrix} \sigma_1 & \alpha \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} \tilde{\sigma}_1 & 0 \\ 0 & \tilde{\sigma}_2 \end{pmatrix} \begin{pmatrix} \upsilon_{11} & \upsilon_{12} \\ \upsilon_{21} & \upsilon_{22} \end{pmatrix}^T, \tag{3.26}$$

the SVD of $\widetilde{C}$ with $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ obtained from (3.10) and (3.11), respectively. The left singular vectors of

$$C = \begin{pmatrix} \sigma_1 & \alpha & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.27}$$

(eigenvectors of CC^T) are

$$\begin{aligned} \widetilde{\mathbf{u}}_1 &= \begin{pmatrix} \omega_{11} & \omega_{21} & 0 & 0 \end{pmatrix}^T, \\ \widetilde{\mathbf{u}}_2 &= \begin{pmatrix} \omega_{12} & \omega_{22} & 0 & 0 \end{pmatrix}^T, \\ \widetilde{\mathbf{u}}_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T = e_3, \\ \widetilde{\mathbf{u}}_4 &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T = e_4, \end{aligned} \tag{3.28}$$

and its corresponding right singular vectors (eigenvectors of $C^T C$) are

$$\begin{aligned} \widetilde{\mathbf{v}}_1 &= \begin{pmatrix} \upsilon_{11} & \upsilon_{21} & 0 \end{pmatrix}^T, \\ \widetilde{\mathbf{v}}_2 &= \begin{pmatrix} \upsilon_{12} & \upsilon_{22} & 0 \end{pmatrix}^T, \\ \widetilde{\mathbf{v}}_3 &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T = e_3. \end{aligned} \tag{3.29}$$

Thus, $A + \alpha \mathbf{u}_1 \mathbf{v}_2^*$ can be written as $A + \alpha \mathbf{u}_1 \mathbf{v}_2^* = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*$, where

$$\begin{aligned} \widetilde{U} &= (\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4)(\widetilde{\mathbf{u}}_1 \mid \widetilde{\mathbf{u}}_2 \mid \widetilde{\mathbf{u}}_3 \mid \widetilde{\mathbf{u}}_4) \\ &= (\omega_{11}\mathbf{u}_1 + \omega_{21}\mathbf{u}_2 \mid \omega_{12}\mathbf{u}_1 + \omega_{22}\mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4), \\ \widetilde{V} &= (\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3)(\widetilde{\mathbf{v}}_1 \mid \widetilde{\mathbf{v}}_2 \mid \widetilde{\mathbf{v}}_3) \\ &= (\upsilon_{11}\mathbf{v}_1 + \upsilon_{21}\mathbf{v}_2 \mid \upsilon_{12}\mathbf{v}_1 + \upsilon_{22}\mathbf{v}_2 \mid \mathbf{v}_3), \\ \widetilde{\Sigma} &= \operatorname{diag}(\widetilde{\sigma}_1, \widetilde{\sigma}_2, \sigma_3). \end{aligned} \tag{3.30}$$

Now, we generalize this result. Without loss generality we assume that $k < j$. From (3.17) and (3.18), by permuting rows and columns of the matrix $\Sigma + P_k\widetilde{\Sigma}Q_j$, we have

$$A + \alpha \mathbf{u}_k \mathbf{v}_j^* = U_1 \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} V_1^* \quad \text{with } \Lambda = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & \alpha & & \\ & & & \sigma_j & & \\ & & & & \ddots & 0 \\ & & & & & \sigma_n \end{pmatrix},$$
$$U_1 = \left(\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_k \mid \mathbf{u}_j \mid \cdots \mid \mathbf{u}_{k+1} \mid \mathbf{u}_{j+1} \mid \cdots \mid \mathbf{u}_m\right),$$
$$V_1 = \left(\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_k \mid \mathbf{v}_j \mid \cdots \mid \mathbf{v}_{k+1} \mid \mathbf{v}_{j+1} \mid \cdots \mid \mathbf{v}_n\right). \tag{3.31}$$

The singular vectors of $\left(\begin{smallmatrix} \Lambda \\ 0 \end{smallmatrix}\right)$ are obtained from the singular vectors of the 2×2 matrix $\left(\begin{smallmatrix} \sigma_k & \alpha \\ 0 & \sigma_j \end{smallmatrix}\right)$. Let

$$\begin{pmatrix} \sigma_k & \alpha \\ 0 & \sigma_j \end{pmatrix} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} \widetilde{\sigma}_k & 0 \\ 0 & \widetilde{\sigma}_j \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}^T. \tag{3.32}$$

Then the unitary matrices of the singular value decomposition of $\left(\begin{smallmatrix} \Lambda \\ 0 \end{smallmatrix}\right)$ are

$$U_2 = \begin{pmatrix} I & & & \\ & \omega_{11} & \omega_{12} & \\ & \omega_{21} & \omega_{22} & \\ & & & I \end{pmatrix} = \left(\mathbf{e}_1 \mid \cdots \mid \mathbf{e}_{k-1} \mid \widetilde{\mathbf{u}}_k \mid \widetilde{\mathbf{u}}_{k+1} \mid \mathbf{e}_{k+2} \mid \cdots \mid \mathbf{e}_m\right),$$
$$V_2 = \begin{pmatrix} I & & & \\ & v_{11} & v_{12} & \\ & v_{21} & v_{22} & \\ & & & I \end{pmatrix} = \left(\mathbf{e}_1 \mid \cdots \mid \mathbf{e}_{k-1} \mid \widetilde{\mathbf{v}}_k \mid \widetilde{\mathbf{v}}_{k+1} \mid \mathbf{e}_{k+2} \mid \cdots \mid \mathbf{e}_n\right), \tag{3.33}$$

of order $m \times m$ and $n \times n$, respectively. Then

$$A + \alpha \mathbf{u}_k \mathbf{v}_j^* = \widetilde{U} \begin{pmatrix} \widetilde{\Lambda} \\ 0 \end{pmatrix} \widetilde{V}^* \quad \text{with } \widetilde{U} = U_1 U_2,\ \widetilde{V} = V_1 V_2,$$
$$\widetilde{\Lambda} = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \widetilde{\sigma}_k & & & \\ & & & \widetilde{\sigma}_j & & \\ & & & & \ddots & \\ & & & & & \sigma_n \end{pmatrix}, \tag{3.34}$$

where $\widetilde{\sigma}_k$ and $\widetilde{\sigma}_j$ are not ordered. Since

$$\widetilde{U} = \left(\mathbf{u}_1 \mid \cdots \mid \omega_{11}\mathbf{u}_k + \omega_{21}\mathbf{u}_j \mid \omega_{12}\mathbf{u}_k + \omega_{22}\mathbf{u}_j \mid \cdots \mid \mathbf{u}_m\right),$$
$$\widetilde{V} = \left(\mathbf{v}_1 \mid \cdots \mid \upsilon_{11}\mathbf{v}_k + \upsilon_{21}\mathbf{v}_j \mid \upsilon_{12}\mathbf{v}_k + \upsilon_{22}\mathbf{v}_j \mid \cdots \mid \mathbf{v}_n\right), \tag{3.35}$$

then the singular vectors corresponding to σ_k, σ_j have been modified. We have prove the following result.

Corollary 3.3. *Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r = \min\{m, n\}$ and with singular value decomposition $A = U\Sigma V^*$, where*

$$U = (\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_k \mid \cdots \mid \mathbf{u}_m), \quad V = \left(\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_j \mid \cdots \mathbf{v}_n\right), \quad k, j \leq r,$$
$$\Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}. \tag{3.36}$$

Let $\alpha \in \mathbb{R}$. Then $A + \alpha \mathbf{u}_k \mathbf{v}_j^*$ has left singular vectors,

$$\widetilde{\mathbf{u}}_i = \mathbf{u}_i, \quad i = 1, \ldots, m,\ i \neq k,\ i \neq j,$$
$$\widetilde{\mathbf{u}}_k = \omega_{11}\mathbf{u}_k + \omega_{21}\mathbf{u}_j,$$
$$\widetilde{\mathbf{u}}_j = \omega_{12}\mathbf{u}_k + \omega_{22}\mathbf{u}_j \tag{3.37}$$

and right singular vectors

$$\widetilde{\mathbf{v}}_i = \mathbf{v}_i, \quad i = 1, \ldots, n,\ i \neq k,\ i \neq j,$$
$$\widetilde{\mathbf{v}}_k = \upsilon_{11}\mathbf{v}_k + \upsilon_{21}\mathbf{v}_j,$$
$$\widetilde{\mathbf{v}}_j = \upsilon_{12}\mathbf{v}_k + \upsilon_{22}\mathbf{v}_j. \tag{3.38}$$

Observe that if 2×2 orthogonal matrices in (3.32) are of the same type

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \tag{3.39}$$

then a straight forward calculation shows that (3.37) and (3.38) become

$$\begin{gathered} \tilde{\mathbf{u}}_i = \mathbf{u}_i, \quad i = 1, \ldots, m, \; i \neq k, \; i \neq j, \\ \tilde{\mathbf{u}}_k = c_1 \mathbf{u}_k - s_1 \mathbf{u}_j, \\ \tilde{\mathbf{u}}_j = s_1 \mathbf{u}_k + c_1 \mathbf{u}_j \end{gathered} \tag{3.40}$$

and

$$\begin{gathered} \tilde{\mathbf{v}}_i = \mathbf{v}_i, \quad i = 1, \ldots, n, \; i \neq k, \; i \neq j, \\ \tilde{\mathbf{v}}_k = c_2 \mathbf{v}_k - s_2 \mathbf{v}_j, \\ \tilde{\mathbf{v}}_j = s_2 \mathbf{v}_k + c_2 \mathbf{v}_j, \end{gathered} \tag{3.41}$$

respectively, while if they are of different type, then (3.38) becomes

$$\begin{gathered} \tilde{\mathbf{v}}_i = \mathbf{v}_i, \quad i = 1, \ldots, m, \; i \neq k, \; i \neq j, \\ \tilde{\mathbf{v}}_k = c_2 \mathbf{v}_k + s_2 \mathbf{v}_j, \\ \tilde{\mathbf{v}}_j = s_2 \mathbf{v}_k - c_2 \mathbf{v}_j. \end{gathered} \tag{3.42}$$

From (3.10) and (3.11) it is clear that $\tilde{\sigma}_k > \tilde{\sigma}_j$ with $k < j$. The following result tells us how the new singular values $\tilde{\sigma}_k$, $\tilde{\sigma}_j$ relate with the previous singular values σ_k, σ_j.

Corollary 3.4. *Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0, r = \min\{m, n\}$ and with singular value decomposition $A = U\Sigma V^*$, where*

$$\begin{gathered} U = (\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_k \mid \cdots \mid \mathbf{u}_m), \quad V = (\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_j \mid \cdots \mathbf{v}_n), \quad k, j \leq r, \\ \Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}. \end{gathered} \tag{3.43}$$

Let $\tilde{\sigma}_k$, $\tilde{\sigma}_j, k < j$, be respectively, as in (3.10) *and* (3.11)*, the singular values of $A + \alpha \mathbf{u}_k \mathbf{v}_j^*$. Then $\tilde{\sigma}_k > \sigma_k \geq \sigma_j > \tilde{\sigma}_j$.*

Proof. Since $4\sigma_j^2\alpha^2 > 0$ we have

$$\begin{aligned}
\left(\sigma_k^2+\alpha^2\right)^2+\sigma_j^4+2\sigma_j^2\left(\sigma_k^2+\alpha^2\right)-4\sigma_k^2\sigma_j^2 &> \left(\sigma_k^2+\alpha^2\right)^2+\sigma_j^4-2\sigma_j^2\left(\sigma_k^2+\alpha^2\right),\\
\left(\sigma_k^2+\sigma_j^2+\alpha^2\right)^2-4\sigma_k^2\sigma_j^2 &> \left(\left(\sigma_k^2+\alpha^2\right)-\sigma_j^2\right)^2,\\
\sigma_k^2+\alpha^2+\sigma_j^2-\sqrt{\left(\sigma_k^2+\sigma_j^2+\alpha^2\right)^2-4\sigma_k^2\sigma_j^2} &< 2\sigma_j^2.
\end{aligned} \tag{3.44}$$

Thus

$$\sigma_j > \left(\frac{1}{2}\left[\sigma_k^2+\alpha^2+\sigma_j^2-\sqrt{\left(\sigma_k^2+\sigma_j^2+\alpha^2\right)^2-4\sigma_k^2\sigma_j^2}\right]\right)^{1/2}. \tag{3.45}$$

In the same way,

$$\begin{aligned}
\left(\sigma_j^2+\alpha^2\right)^2+\sigma_k^4+2\sigma_k^2\left(\sigma_j^2+\alpha^2\right)-4\sigma_k^2\sigma_j^2 &> \left(\sigma_j^2+\alpha^2\right)^2+\sigma_k^4-2\sigma_k^2\left(\sigma_j^2+\alpha^2\right),\\
\left(\sigma_k^2+\sigma_j^2+\alpha^2\right)^2-4\sigma_k^2\sigma_j^2 &> \left(\sigma_k^2-\left(\sigma_j^2+\alpha^2\right)\right)^2,\\
\sigma_k^2+\sigma_j^2+\alpha^2+\sqrt{\left(\sigma_k^2+\sigma_j^2+\alpha^2\right)^2-4\sigma_k^2\sigma_j^2} &> 2\sigma_k^2.
\end{aligned} \tag{3.46}$$

Then

$$\sigma_k < \left(\frac{1}{2}\left[\sigma_k^2+\alpha^2+\sigma_j^2+\sqrt{\left(\sigma_k^2+\sigma_j^2+\alpha^2\right)^2-4\sigma_k^2\sigma_j^2}\right]\right)^{1/2}. \tag{3.47}$$

Therefore,

$$\tilde{\sigma}_k > \sigma_k \geq \sigma_j > \tilde{\sigma}_j. \tag{3.48}$$

Observe that $\sigma_i \in (\tilde{\sigma}_j, \tilde{\sigma}_k)$, $i = k, k+1, \ldots, j$. In particular for $k = 1$ and $j = n$, all singular values of A are in the interval $(\tilde{\sigma}_n, \tilde{\sigma}_1)$. □

Now we extend the mixed perturbation result given by Theorem 3.1 to rank-2 perturbations, that is, perturbations of the form $B = A+\alpha_1\mathbf{u}_{k_1}\mathbf{v}_{j_1}+\alpha_2\mathbf{u}_{k_2}\mathbf{v}_{j_2}$, with α_1, α_2 nonzero real numbers and $\mathbf{u}_{k_i}, \mathbf{v}_{j_i}$, $i = 1, 2$, $k_i < j_i$, $k_1 < j_1 < k_2 < j_2$, being left and right singular vectors of A, respectively. Then as in (3.12)

$$(\pm\mathbf{u}_{k_1}, \pm\mathbf{u}_{k_2}, \dots, \mathbf{u}_1, \mathbf{u}_{k_1+1}, \dots, \mathbf{u}_{k_2-1}, \mathbf{u}_2, \mathbf{u}_{k_2+1}, \dots, \mathbf{u}_m)\widetilde{\Sigma}\begin{pmatrix} \mathbf{v}_{j_1}^* \\ \mathbf{v}_{j_2}^* \\ \vdots \\ \mathbf{v}_1^* \\ \mathbf{v}_{j_1+1}^* \\ \vdots \\ \mathbf{v}_{j_2-1}^* \\ \mathbf{v}_2^* \\ \mathbf{v}_{j_2+1}^* \\ \vdots \\ \mathbf{v}_n^* \end{pmatrix}, \tag{3.49}$$

with $\widetilde{\Sigma} = \operatorname{diag}(|\alpha_1|, |\alpha_2|, 0, \dots, 0)$. It is clear that the matrix in (3.49) can be written as

$$UP_1P_2\widetilde{\Sigma}Q_2Q_1V^*, \tag{3.50}$$

where P_1, P_2, Q_1, Q_2 are permutation matrices of the form

$$\begin{gathered} P_1 = \begin{pmatrix} W_{k_1} & \\ & I_{m-k_1} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & & \\ & W_{k_2} & \\ & & I_{m-(k_2+1)} \end{pmatrix}, \\ Q_1 = \begin{pmatrix} W_{j_1} & \\ & I_{n-j_1} \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & & \\ & W_{j_2} & \\ & & I_{n-(j_2+1)} \end{pmatrix}. \end{gathered} \tag{3.51}$$

From (3.50), the matrix in (3.49) is

$$\begin{pmatrix} (\alpha_1)_{k_1,j_1} & \\ & (\alpha_2)_{k_2,j_2} \end{pmatrix}, \tag{3.52}$$

where if α_1, α_2 are negative, we multiply $\mathbf{u}_{k_1}$, $\mathbf{u}_{k_2}$ by minus one. Thus,

$$B = A + \alpha_1 \mathbf{u}_{k_1} \mathbf{v}_{j_1}^* + \alpha_2 \mathbf{u}_{k_2} \mathbf{v}_{j_2}^* = U \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} V^*, \tag{3.53}$$

where

$$\Lambda = \begin{pmatrix} \sigma_1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & \sigma_{k_1} & \cdots & \alpha_{k_1,j_1} & & & & & & \\ & & & \ddots & \vdots & & & & & & \\ & & & & \sigma_{j_1} & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & \sigma_{k_2} & \cdots & \alpha_{k_2,j_2} & & \\ & & & & & & & \ddots & \vdots & & \\ & & & & & & & & \sigma_{j_2} & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & \sigma_n \end{pmatrix}. \tag{3.54}$$

By permuting rows and columns of Λ it follows that singular values of $\begin{pmatrix} \Lambda \\ 0 \end{pmatrix}$ are

$$\sigma_1, \ldots, \sigma_{k_1-1}, \sigma_{k_1+1}, \ldots, \sigma_{j_1-1}, \sigma_{j_1+1}, \ldots, \sigma_{k_2-1}, \sigma_{k_2+1}, \ldots, \sigma_{j_2-1}, \sigma_{j_2+1}, \ldots, \sigma_n \tag{3.55}$$

together with the singular values of

$$C_1 = \begin{pmatrix} \sigma_{k_1} & \alpha_{k_1,j_1} \\ 0 & \sigma_{j_1} \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \sigma_{k_2} & \alpha_{k_2,j_2} \\ 0 & \sigma_{j_2} \end{pmatrix}. \tag{3.56}$$

Is immediate that the singular values of these matrices are

$$\tilde{\sigma}_{k_1} = \left(\frac{\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2 + \sqrt{\left(\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2\right)^2 - 4\sigma_{k_1}^2 \sigma_{j_1}^2}}{2} \right)^{1/2},$$
$$\tilde{\sigma}_{j_1} = \left(\frac{\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2 - \sqrt{\left(\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2\right)^2 - 4\sigma_{k_1}^2 \sigma_{j_1}^2}}{2} \right)^{1/2},$$
$$\tilde{\sigma}_{k_2} = \left(\frac{\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2 + \sqrt{\left(\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2\right)^2 - 4\sigma_{k_2}^2 \sigma_{j_2}^2}}{2} \right)^{1/2},$$
$$\tilde{\sigma}_{j_2} = \left(\frac{\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2 - \sqrt{\left(\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2\right)^2 - 4\sigma_{k_2}^2 \sigma_{j_2}^2}}{2} \right)^{1/2}. \tag{3.57}$$

The matrix $B = A + \alpha_1 \mathbf{u}_{k_1} \mathbf{v}_{j_1}^* + \alpha_2 \mathbf{u}_{k_2} \mathbf{v}_{j_2}^*$ can be written as

$$B = \tilde{U} \begin{pmatrix} \tilde{\Lambda} \\ 0 \end{pmatrix} \tilde{V}^* \quad \text{with } \tilde{U} = U_1 U_2,\ \tilde{V} = V_1 V_2,$$
$$\tilde{\Lambda} = \begin{pmatrix} \sigma_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \tilde{\sigma}_{k_1} & & & & & & \\ & & & \tilde{\sigma}_{j_1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{\sigma}_{k_2} & & & \\ & & & & & & \tilde{\sigma}_{j_2} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \sigma_n \end{pmatrix}, \tag{3.58}$$

where the $\widetilde{\sigma}$'s are not ordered,

$$\begin{aligned}
U_1 &= \left(\mathbf{u}_1, \mathbf{u}_2 \cdots \mathbf{u}_{k_1}, \mathbf{u}_{j_1} \cdots \mathbf{u}_{k_1+1}, \mathbf{u}_{j_1+1} \cdots \mathbf{u}_{k_2}, \mathbf{u}_{j_2} \cdots \mathbf{u}_{k_2+1}, \mathbf{u}_{j_2+1} \cdots \mathbf{u}_m\right), \\
V_1 &= \left(\mathbf{v}_1, \mathbf{v}_2 \cdots \mathbf{v}_{k_1}, \mathbf{v}_{j_1} \cdots \mathbf{v}_{k_1+1}, \mathbf{v}_{j_1+1} \cdots \mathbf{v}_{k_2}, \mathbf{v}_{j_2} \cdots \mathbf{v}_{k_2+1}, \mathbf{v}_{j_2+1} \cdots \mathbf{v}_n\right), \\
U_2 &= \left(\mathbf{e}_1 \cdots \mathbf{e}_{k_1-1}, \widetilde{\mathbf{u}}_{k_1} \widetilde{\mathbf{u}}_{k_1+1}, \mathbf{e}_{k_1+2} \cdots \mathbf{e}_{k_2-1}, \widetilde{\mathbf{u}}_{k_2}, \widetilde{\mathbf{u}}_{k_2+1}, \mathbf{e}_{k_2+2} \cdots \mathbf{e}_m\right), \\
V_2 &= \left(\mathbf{e}_1 \cdots \mathbf{e}_{k_1-1}, \widetilde{\mathbf{v}}_{k_1} \widetilde{\mathbf{v}}_{k_1+1}, \mathbf{e}_{k_1+2} \cdots \mathbf{e}_{k_2-1}, \widetilde{\mathbf{v}}_{k_2}, \widetilde{\mathbf{v}}_{k_2+1}, \mathbf{e}_{k_2+2} \cdots \mathbf{e}_n\right).
\end{aligned} \tag{3.59}$$

We have proved the following result.

Theorem 3.5. *Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$ and SVD $A = U\Sigma V^*$, where*

$$U = \left(\mathbf{u}_1, \mathbf{u}_2 \cdots \mathbf{u}_{k_1} \cdots \mathbf{u}_{k_2} \cdots \mathbf{u}_m\right), \qquad V = \left(\mathbf{v}_1, \mathbf{v}_2 \cdots \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_2} \cdots \mathbf{v}_n\right), \tag{3.60}$$

$k_i, j_i \leq r$, $k_i \neq j_i$, are $m \times m$ and $n \times n$ unitary matrices, respectively, and $\Sigma = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$. Let α_1, α_2 be real numbers. Then $B = A + \alpha_1 \mathbf{u}_{k_1} \mathbf{v}_{j_1}^* + \alpha_2 \mathbf{u}_{k_2} \mathbf{v}_{j_2}^*$ has singular values

$$\bigcup_{\substack{q=1 \\ q \neq k_i, j_i}}^{r} \{\sigma_q\} \cup \{\widetilde{\sigma}_{k_1}, \widetilde{\sigma}_{j_1}, \widetilde{\sigma}_{k_2}, \widetilde{\sigma}_{j_2}\}, \tag{3.61}$$

$$\widetilde{\sigma}_{k_i} = \left(\frac{\alpha_i^2 + \sigma_{k_i}^2 + \sigma_{j_i}^2 + \sqrt{\left(\alpha_i^2 + \sigma_{k_i}^2 + \sigma_{j_i}^2\right)^2 - 4\sigma_{k_i}^2 \sigma_{j_i}^2}}{2} \right)^{1/2}, \tag{3.62}$$

$$\widetilde{\sigma}_{j_i} = \left(\frac{\alpha_i^2 + \sigma_{k_i}^2 + \sigma_{j_i}^2 - \sqrt{\left(\alpha_i^2 + \sigma_{k_i}^2 + \sigma_{j_i}^2\right)^2 - 4\sigma_{k_i}^2 \sigma_{j_i}^2}}{2} \right)^{1/2} \tag{3.63}$$

$i = 1, 2$. The singular vectors are given by

$$\begin{aligned}
\widetilde{\mathbf{u}}_q = \mathbf{u}_q, \quad q = 1, \ldots, m, \ q \neq k_i, \ q \neq j_i, \ i = 1, 2, \\
\widetilde{\mathbf{u}}_{k_1} &= \omega_{11}^{(1)} \mathbf{u}_{k_1} + \omega_{21}^{(1)} \mathbf{u}_{j_1}, \\
\widetilde{\mathbf{u}}_{j_1} &= \omega_{12}^{(1)} \mathbf{u}_{k_1} + \omega_{22}^{(1)} \mathbf{u}_{j_1}, \\
\widetilde{\mathbf{u}}_{k_2} &= \omega_{11}^{(2)} \mathbf{u}_{k_2} + \omega_{21}^{(2)} \mathbf{u}_{j_2}, \\
\widetilde{\mathbf{u}}_{j_2} &= \omega_{12}^{(2)} \mathbf{u}_{k_2} + \omega_{22}^{(2)} \mathbf{u}_{j_2},
\end{aligned} \tag{3.64}$$

where the coefficients ω's are obtained as before, and

$$\begin{aligned}
\widetilde{\mathbf{v}}_q &= \mathbf{v}_q, \quad q = 1,\dots,n,\ q \neq k_i,\ q \neq j_i,\ i = 1,2,\\
\widetilde{\mathbf{v}}_{k_1} &= \upsilon_{11}^{(1)}\mathbf{v}_{k_1} + \upsilon_{21}^{(1)}\mathbf{v}_{j_1},\\
\widetilde{\mathbf{v}}_{j_1} &= \upsilon_{12}^{(1)}\mathbf{v}_{k_1} + \upsilon_{22}^{(1)}\mathbf{v}_{j_1},\\
\widetilde{\mathbf{v}}_{k_2} &= \upsilon_{11}^{(2)}\mathbf{v}_{k_2} + \upsilon_{21}^{(2)}\mathbf{v}_{j_2},\\
\widetilde{\mathbf{v}}_{j_2} &= \upsilon_{12}^{(2)}\mathbf{v}_{k_2} + \upsilon_{22}^{(2)}\mathbf{v}_{j_2}.
\end{aligned} \tag{3.65}$$

Now, as in Lemma 2.1 in Section 2, we look for a condition to preserve nonnegativity when we deal with mixed perturbations. Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r = \min\{m,n\}$, and with singular value decomposition $A = U\Sigma V^*$, where

$$\begin{gathered}
U = (\mathbf{u}_1 \mid \cdots \mid \mathbf{u}_p \mid \cdots \mid \mathbf{u}_m), \quad V = (\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_q \mid \cdots \mid \mathbf{v}_n), \quad p,q \leq r,\\
\Sigma = \operatorname{diag}\{\sigma_1,\sigma_2,\dots,\sigma_r\}.
\end{gathered} \tag{3.66}$$

Then $A + \alpha\mathbf{u}_p\mathbf{v}_q^T$ is nonnegative for $\alpha > 0$ if $a_{ik} + \alpha(\mathbf{u}_p\mathbf{v}_q^T)_{ik} \geq 0$, $i = 1,\dots,m$, $k = 1,\dots,n$. If $(\mathbf{u}_p\mathbf{v}_q^T)_{ik} < 0$, then

$$a_{ik} + \alpha\left(\mathbf{u}_p\mathbf{v}_q^T\right)_{ik} \geq 0 \quad \text{iff } 0 < \alpha \leq \min_{i,k}\frac{a_{ik}}{\left|\left(\mathbf{u}_p\mathbf{v}_q^T\right)_{ik}\right|}. \tag{3.67}$$

Then we have the following result.

Lemma 3.6. *Let A be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m,n\}$. Let α be in the interval*

$$\left(0, \min_{p,q}\min_{i,k}\frac{a_{ik}}{\left|\left(\mathbf{u}_p\mathbf{v}_q^T\right)_{ik}\right|}\right]. \tag{3.68}$$

Then $A + \alpha\mathbf{u}_p\mathbf{v}_q^T, 1 \leq p \leq m$, $1 \leq q \leq n$, $p \neq q$, is nonnegative with singular values

$$\sigma_1,\dots,\sigma_{p-1},\widetilde{\sigma}_p,\sigma_{p+1},\dots,\sigma_{q-1},\widetilde{\sigma}_q,\sigma_{q+1},\dots,\sigma_r, \tag{3.69}$$

where $\widetilde{\sigma}_p$ and $\widetilde{\sigma}_q$ are defined as in (3.62) *and* (3.63)*, respectively.*

Example 3.7. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ be with singular values $\sigma_1 = 9.5255$, $\sigma_2 = 0.5143$. $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} 0.22985 & -0.88346 & 0.40825 \\ 0.52474 & -0.24078 & -0.81650 \\ 0.81964 & 0.40190 & 0.40825 \end{pmatrix}, \qquad V = \begin{pmatrix} 0.61963 & 0.78489 \\ 0.78489 & -0.61963 \end{pmatrix}. \tag{3.70}$$

Then, from (3.68) we have for $p = 1, 2, 3$; $q = 1, 2$,

$$\min_{p,q} \min_{i,k} \frac{a_{ik}}{\left| \left(\mathbf{u}_p \mathbf{v}_q^T \right)_{ik} \right|} = 1.8268 \tag{3.71}$$

and $\alpha \in (0, 1.8268]$. For $\alpha = 1.8268$, we have

$$\begin{aligned} A + \alpha \mathbf{u}_1 \mathbf{v}_2^T &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + 1.8268 \begin{pmatrix} 0.18041 & -0.14242 \\ 0.41186 & -0.32514 \\ 0.64333 & -0.50787 \end{pmatrix} \\ &= \begin{pmatrix} 1.3296 & 1.7398 \\ 3.7524 & 3.406 \\ 6.1752 & 5.0722 \end{pmatrix} \end{aligned} \tag{3.72}$$

is nonnegative with singular values $\tilde{\sigma}_1 = 9.6996$ and $\tilde{\sigma}_2 = 0.5050$.

To show that mixed perturbations preserve doubly stochastic structure, observe from Lemma 2.8 that $\mathbf{u}_p \mathbf{v}_q^T \mathbf{e} = 0$ for $q = 2, \ldots, n$. Then, if A is an $n \times n$ positive doubly stochastic matrix, we have that $A + \alpha \mathbf{u}_p \mathbf{v}_q^T$ is doubly stochastic.

Acknowledgment

This work was supported by Fondecyt 1085125, Chile.

References

[1] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1991.

[2] E. Montaño, M. Salas, and R. L. Soto, "Nonnegative matrices with prescribed extremal singular values," *Computers & Mathematics with Applications*, vol. 56, no. 1, pp. 30–42, 2008.

[3] A. G. Gramm, *Inverse Problems: Mathematical and Analytical Thechniques with Applications to Engineering*, Springer, New York, NY, USA, 2005.

[4] M. T. Chu and G. H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, NY, USA, 2005.

[5] H. Perfect, "Methods of constructing certain stochastic matrices," *Duke Mathematical Journal*, vol. 20, pp. 395–404, 1953.

[6] R. L. Soto, “Existence and construction of nonnegative matrices with prescribed spectrum,” *Linear Algebra and Its Applications*, vol. 369, pp. 169–184, 2003.
[7] A. Brauer, “Limits for the characteristic roots of a matrix—IV: applications to stochastic matrices,” *Duke Mathematical Journal*, vol. 19, pp. 75–91, 1952.
[8] H. Perfect, “Methods of constructing certain stochastic matrices. II,” *Duke Mathematical Journal*, vol. 22, pp. 305–311, 1955.
[9] R. L. Soto and O. Rojo, “Applications of a Brauer theorem in the nonnegative inverse eigenvalue problem,” *Linear Algebra and Its Applications*, vol. 416, no. 2-3, pp. 844–856, 2006.
[10] R. L. Soto, O. Rojo, J. Moro, and A. Borobia, “Symmetric nonnegative realization of spectra,” *Electronic Journal of Linear Algebra*, vol. 16, pp. 1–18, 2007.
[11] V. Nikiforov, “The energy of graphs and matrices,” *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1472–1475, 2007.