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Research Article

Generalized Variational Principle for Long Water-Wave Equation by He's Semi-Inverse Method

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Variational principles for nonlinear partial differential equations have come to play an important role in mathematics and physics. However, it is well known that not every nonlinear partial differential equation admits a variational formula. In this paper, He's semi-inverse method is used to construct a family of variational principles for the long water-wave problem.

1. Introduction

In this paper we apply He's semi-inverse method [1–12] to establish a family of variational formulations for the following higher-order long water-wave equations:

$$u_t - u_x u - v_x + \alpha u_{xx} = 0, \tag{1.1}$$

$$v_t - (uv)_x - \alpha v_{xx} = 0. \tag{1.2}$$

When $\alpha = 1/2$, equations (1.1) and (1.2) were investigated in [13], but the generalized variational approach for the discussed problem has not been dealt with.

2. Variational Formulation

We rewrite (1.1) and (1.2) in conservation forms:

$$u_t + \left(-\frac{1}{2}u^2 - v + \alpha u_x\right)_x = 0, \tag{2.1}$$

$$v_t + (-uv - \alpha v_x)_x = 0. \tag{2.2}$$

According to (1.1) or (2.1) we can introduce a special function Ψ defined as

$$\Psi_t = -\frac{1}{2}u^2 - v + \alpha u_x, \tag{2.3}$$

$$\Psi_x = -u. \tag{2.4}$$

Similarly from (1.2) or (2.2) we can introduce another special function Φ defined as

$$\begin{aligned} \Phi_t &= -uv - \alpha v_x, \\ \Phi_x &= -v. \end{aligned} \tag{2.5}$$

Our aim in this paper is to establish some variational formulations whose stationary conditions satisfy (1.1), (2.5), or (1.2), (2.3), and (2.4). To this end, we will apply He's semi-inverse method to construct a trial functional:

$$J(u, v, \Psi) = \iint L\, dx\, dt, \tag{2.6}$$

where L is a trial Lagrangian defined as

$$L = v\Psi_t + (-uv - \alpha v_x)\Psi_x + F(u, v), \tag{2.7}$$

where F is an unknown function of u, v and/or their derivatives. The advantage of the above trial Lagrangian is that the stationary condition with respect to Ψ is one of the governing equations (2.2) or (1.2).

Calculating the above functional equation (2.6) stationary with respect to u and v, we obtain the followimg Euler-Lagrange equations:

$$-v\Psi_x + \frac{\delta F}{\delta u} = 0, \tag{2.8}$$

$$\Psi_t - u\Psi_x + \alpha\Psi_{xx} + \frac{\delta F}{\delta v} = 0, \tag{2.9}$$

where $\delta F/\delta u$ is called He's variational derivative [14–17] with respect to u, which was first sugested by He in [2], defined as

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \cdots. \tag{2.10}$$

We search for such an F so that (2.8) is equivalent to (2.3), and (2.9) is equivalent to (2.4). So in view of (2.3) and (2.4), we set

$$\begin{aligned} \frac{\delta F}{\delta u} &= v\Psi_x = -uv, \\ \frac{\delta F}{\delta v} &= -\Psi_t + u\Psi_x - \alpha\Psi_{xx} = -\frac{1}{2}u^2 + v, \end{aligned} \tag{2.11}$$

from (2.11), the unknown F can be determined as

$$F(u,v) = -\frac{1}{2}u^2 v + \frac{1}{2}v^2. \tag{2.12}$$

Finally we obtain the following needed variational formulation:

$$J(u,v,\Psi) = \iint \left\{ v\Psi_t + (-uv - \alpha v_x)\Psi_x - \frac{1}{2}u^2 v + \frac{1}{2}v^2 \right\}. \tag{2.13}$$

Proof. Making the above functional equation (2.13) stationary with respect to Ψ, u, and v, we obtain the following Euler-Lagrange equations:

$$-v_t - (-uv - \alpha v_x)_x = 0, \tag{2.14}$$

$$-v\Psi_x - uv = 0, \tag{2.15}$$

$$\Psi_t - u\Psi_x + \alpha\Psi_{xx} - \frac{1}{2}u^2 + v = 0. \tag{2.16}$$

Equation (2.14) is equivalent to (1.2), and (2.15) is equivalent to (2.4); in view of (2.4), (2.16) becomes (2.3). □

Similary we can also begin with the following trial Lagrangian:

$$L_1(u,v,\Phi) = u\Phi_t + \left(-\frac{1}{2}u^2 - v + \alpha u_x\right)\Phi_x + G(u,v). \tag{2.17}$$

It is obvious that the stationary condition with respect to Φ is equivalent to (2.1) or (1.1). Now the Euler-Lagrange equations with respect to u and v are

$$\begin{aligned} \Phi_t - u\Phi_x - \alpha\Phi_{xx} + \frac{\delta G}{\delta u} &= 0, \\ -\Phi_x + \frac{\delta G}{\delta v} &= 0. \end{aligned} \tag{2.18}$$

In view of (2.5), we have

$$\begin{aligned} \frac{\delta G}{\delta u} &= -\Phi_t + u\Phi_x + \alpha\Phi_{xx} = 0, \\ \frac{\delta G}{\delta v} &= \Phi_x = -v. \end{aligned} \tag{2.19}$$

From (2.19), the unknown function $G(u, v)$ can be determined as

$$G(u, v) = -\frac{1}{2}v^2. \tag{2.20}$$

Therefore, we obtain another needed variational formulation:

$$J_1(u, v, \Phi) = \iint \left\{ u\Phi_t + \left(-\frac{1}{2}u^2 - v + \alpha u_x \right) \Phi_x - \frac{1}{2}v^2 \right\} dx\, dt. \tag{2.21}$$

3. Conclusion

We establish a family of variational formulations for the long water-wave problem using He's semi-inverse method. It is shown that the method is a powerful tool to the search for variational principles for nonlinear physical problems directly from field equations without using Lagrange multiplier. The result obtained in this paper might find some potential applications in future.

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