Quasi-boundary Value Method for Non-well Posed Problem for a Parabolic Equation with Integral Boundary Condition

M. DENCHE* and K. BESSILA

Departement de Mathematiques, Laboratoire: Equations Differentielles, Université Mentouri Constantine, 25000 Constantine, Algeria

In this paper we study the problem of control by the initial conditions of the heat equation with an integral boundary condition. This problem is ill-posed. Perturbing the final condition we obtain an approximate nonlocal problem depending on a small parameter. We show that the approximate problems are well posed. We also obtain estimates of the solutions of the approximate problems and a convergence result of these solutions. Finally, we give explicit convergence rates.

Keywords: Parabolic equation; Quasireversibility method; Quasi-boundary value problem; Integral boundary condition

AMS (MOS) Subject Classifications: 47A52, 65F22, 65J20, 65L08

*Corresponding author. e-mail: m_denche@hotmail.com

1. STATEMENT OF THE PROBLEM

Let $u(x, t; \xi)$ be a solution of the problem:

$$
\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2}; & x \in]0,1[, t > 0, \\[2mm]
u(0,t) = 0, & t \geq 0, \\[2mm]
\displaystyle\int_0^1 u(x,t)dx = 0, & t \geq 0, \\[2mm]
u(x,0) = \xi(x), & 0 \leq x \leq 1.
\end{cases}
\tag{1}
$$

We suppose that ξ verifies the compatibility conditions:

$$
\xi(0) = 0, \qquad \int_0^1 \xi(x)dx = 0.
\tag{2}
$$

In addition, we assume other hypotheses on ξ, which garantee the existence and uniqueness of the solution u of problem (1). A such problem was formulated by Samarskii [9] and solved for the first time by Ionkin [4]. A more general case was studied by Yurchuk [12].

Let us consider the following control problem: for $T > 0$ and $\varphi \in L^2(0,1)$ verifying the conditions (2), we try to minimize the functional:

$$
I(\xi) = \int_0^1 |u(x,T;\xi) - \varphi(x)|^2 dx.
\tag{3}
$$

This problem arises, for instance, from the control problem of heat propagation in a thin rod in which the law of variation of the total quantity of heat in the rod as well as its left end point are kept at zero temperature. In this case, the control is undertaken in terms of an unknown initial heat distribution $\xi(x)$.

An obvious solution to the problem (3) is to choose ξ such that $I(\xi) = 0$, i.e., $u(x, T; \xi) = \varphi(x)$.

Hence, in the rectangle $D_T = (0,1) \times (0,T)$, we consider u as a solution of the following final boundary value problem:

$$
\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2}; & \text{in } D_T \\[2mm]
u(0,t) = 0, & t \in [0,T] \\[2mm]
\displaystyle\int_0^1 u(x,t)dx = 0, & t \in [0,T] \\[2mm]
u(x,T) = \varphi(x), & x \in [0,1]
\end{cases}
\tag{4}
$$

and we take $\xi(x) = u(x, 0)$. Such problems are not well posed since even if a unique solution exists on $[0, T]$, it does not need to depend continuously on the final value φ.

One method for approaching such problems is quasireversibility, introduced by Lattes and Lions [6]. The idea consists in replacing the final boundary value problem with an approximate one which is well posed, then, the latter is used to construct approximate solutions of the final boundary value problem. In the original method of quasireversibility, Lattes and Lions [6] replaced the heat operator $(\partial/\partial t) - (\partial^2/\partial x^2)$ by a perturbed operator $P_\varepsilon = (\partial/\partial t) + A - \varepsilon AA^*$, well posed in the sense of a decreasing time, where A is an operator generated by the differential expression depending on x and the corresponding boundary conditions. Here, A^* denotes the adjoint operator. In our case, we cannot construct the operator A^* for the simple reason that $D(A)$ is not dense in $L_2(0, 1)$. A similar problem is considered in [2] where the authors give a certain modified quasi-reversibility method by taking the operator $P_\varepsilon = (\partial/\partial t) - (\partial^2/\partial x^2) - \varepsilon(\partial^3/\partial x^2 \partial t)$.

In this paper, we study the problem (1), where we perturb the final condition to form an approximate nonlocal problem depending on a small parameter, as follows:

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} \quad \text{in } D_T, \tag{5}$$

$$u(0, t) = 0, \quad t \in [0, T], \tag{6}$$

$$\int_0^1 u(x, t)dx = 0, \quad t \in [0, T], \tag{7}$$

$$\alpha u(x, 0) + (1 - \alpha)u(x, T) = \varphi(x), \quad x \in [0, 1], \tag{8}$$

where $\alpha \in {]}0, 1{[}$.

The same method is applied in [1] for the case where A is a self-adjoint operator. Following [3] and [11], this method is called quasi-boundary value method, and the related approximate problem is called quasi-boundary value problem (Q.B.V.P). A similar approach known as the method of auxilliary boundary conditions was given in [5], where the perturbed problem is defined in the interval $[0, T + \tau]$, with $\tau > 0$.

Now, we give a representation of the solution of the Q.B.V.P in the biorthogonal series with a parameter t of eigenfunctions and associated functions of the nonself-adjoint Sturm-Liouville operator corresponding to the problem (5)–(8). Then, we establish estimates of the solution of the Q.B.V.P. And so, we establish sufficient conditions of the existence of the solution of the Q.B.V.P. Moreover, we show the convergence of this method in the classical sense in the space $W_2^1(0, 1)$. With additional conditions on the function $\xi(x)$, we can prove stronger results concerning the convergence of the method. Furthermore, under some particular conditions on the function $\xi(x)$ we obtain the order of convergence of the method.

2. REPRESENTATION OF THE SOLUTION OF THE QUASI-BOUNDARY VALUE PROBLEM (5)–(8)

We start by giving the following definition

DEFINITION 1   We define a classical solution of (Q.B.V.P) to be a function u_α such that

(1) it is continuous in $\bar{D}_T$.
(2) in D_T it has a continuous first derivative with respect to t and a continuous second derivative with respect to x.
(3) it satisfies (5) with (6)–(8), in the usual classical sense.

Now, let us replace Q.B.V.P. (5)–(8) by the following equivalent Q.B.V.P.:

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} \quad \text{in } D_T, \tag{9}$$

$$u(0, t) = 0, \quad t \in [0, T], \tag{10}$$

$$u_x(0, t) = u_x(1, t), \quad t \in [0, T], \tag{11}$$

$$\alpha u(x, 0) + (1 - \alpha)u(x, T) = \varphi(x), \quad x \in [0, 1]. \tag{12}$$

We assume that the function $\varphi \in L^2(0, 1)$ satisfies the condition (2). Then, expanding this function in a biorthogonal series gives:

$$\varphi(x) = \varphi_0 X_0(x) + \sum_{k=1}^{+\infty}[\varphi_{2k-1}X_{2k-1}(x) + \varphi_{2k}X_{2k}(x)], \qquad (13)$$

with the respect to the formed basis of eigenfunctions and associated functions

$$X_0(x) = x; \ X_{2k-1}(x) = x\cos(2\pi kx), \ X_{2k}(x) = \sin(2\pi kx), \ k \geq 1, \tag{14}$$

of the non-self-adjoint Sturm-Liouville problem

$$\begin{cases} -\dfrac{d^2X(x)}{dx^2} = \lambda X(x), \quad 0 < x < 1, \\ X(0) = 0, \\ X'(0) = X'(1). \end{cases} \tag{15}$$

The latter corresponds to the problem (9)–(12). The coefficients φ_0, φ_{2k-1}, φ_{2k} are given by the formulae

$$\varphi_0 = \int_0^1 \varphi(x)Y_0(x)dx, \quad \varphi_{2k-1} = \int_0^1 \varphi(x)Y_{2k-1}(x)dx,$$

$$\varphi_{2k} = \int_0^1 \varphi(x)Y_{2k}(x)dx, \tag{16}$$

where the sequence $\{Y_k\}_0^\infty$ of eigenfunctions and associated functions for the adjoint problem of (15):

$$\begin{cases} -\dfrac{d^2Y(x)}{dx^2} = \lambda Y(x), \quad 0 < x < 1, \\ Y'(1) = 0, \\ Y(0) = Y(1). \end{cases} \tag{17}$$

is given by the formulae

$$Y_0(x) = 2; \quad Y_{2k-1}(x) = 4\cos(2\pi kx).$$
$$Y_{2k}(x) = 4(1-x)\sin(2\pi kx), \quad k \geq 1. \tag{18}$$

The eigenvalues of the problem (15) and those of the adjoint problem (17) are given by:

$$\lambda_k = (2\pi k)^2, \quad k \geq 0. \tag{19}$$

The bases (14) and (18) form a biorthogonal system.

Since $\int_0^1 \varphi(x)dx = 0$, we have $\varphi_0 = 0$ in (13).

By analogy with the Fourier method we seek the solution of the problem (9)−(12) in the form:

$$u_\alpha(x, t) = u_0(t)X_0(x) + \sum_{k=1}^{+\infty}[u_{2k-1}(t)X_{2k-1}(x) + u_{2k}(t)X_{2k}(x)]. \tag{20}$$

Now, if we can permute the sum $(\sum)$ with the first partial derivative $(\partial/\partial t)$ with respect to t, and similarly, with the second partial derivative $(\partial^2/\partial x^2)$ with respect to x in the formula (20), we can find

$$u_0(t) = 0,$$

$$u_{2k-1}(t) = \frac{\varphi_{2k-1}e^{-\lambda_k t}}{\alpha + (1-\alpha)e^{-\lambda_k T}},$$

$$u_{2k}(t) = \left[\frac{\varphi_{2k} + 2\sqrt{\lambda_k}\varphi_{2k-1}(T-t)}{\alpha + (1-\alpha)e^{-\lambda_k T}} - \frac{2\alpha\sqrt{\lambda_k}\varphi_{2k-1}T}{(\alpha + (1-\alpha)e^{-\lambda_k T})^2}\right]e^{-\lambda_k t}.$$

Therefore,

$$u_\alpha(x, t) = \sum_{k=1}^{+\infty}\frac{\varphi_{2k-1}e^{-\lambda_k t}}{\alpha + (1-\alpha)e^{-\lambda_k T}}X_{2k-1}(x) +$$

$$\left[\frac{\varphi_{2k} + 2\sqrt{\lambda_k}\varphi_{2k-1}(T-t)}{\alpha + (1-\alpha)e^{-\lambda_k T}}\right.$$

$$\left. - \frac{2\alpha\sqrt{\lambda_k}\varphi_{2k-1}T}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2}\right]e^{-\lambda_k t}X_{2k}(x). \tag{21}$$

This yields the following:

THEOREM 1   *If problem (5)–(8) admits a solution, then this can be represented in the form (21).*

3. EXISTENCE OF SOLUTION OF (Q.B.V.P) (5)–(8)

THEOREM 2   *If $\varphi \in W_2^2(0, 1)$ is such that:*

$$\varphi(0) = 0, \quad \int_0^1 \varphi(x)dx = 0, \quad \varphi'(0) = \varphi'(1) \tag{22}$$

then the function given by (21) is a classical solution of the (Q.B.V.P) (5)–(8), where the coefficients φ_{2k-1} and φ_{2k} are defined by (16).

Proof   The series (21) is the sum of the functions

$$
\begin{aligned}
u_{\alpha k}(x, t) = {} & \frac{\varphi_{2k-1}e^{-\lambda_k t}}{\alpha + (1 - \alpha)e^{-\lambda_k T}}X_{2k-1}(x) \\
& + \left[\frac{\varphi_{2k} + 2\sqrt{\lambda_k}\varphi_{2k-1}(T - t)}{\alpha + (1 - \alpha)e^{-\lambda_k T}} \right. \\
& \left. - \frac{2\alpha\sqrt{\lambda_k}\varphi_{2k-1}T}{[\alpha + (1 - \alpha)e^{-\lambda_k T}]^2} \right] e^{-\lambda_k t}X_{2k}(x), \tag{23}
\end{aligned}
$$

and it is easily verified that $u_{\alpha k}$ satisfies the Eq. (5) and the boundary conditions (7)–(8). We shall prove that, if $t \geq \varepsilon > 0$ (ε being an arbitrary positive number), the series $\sum_{k=1}^{+\infty}(\partial u_{\alpha k}/\partial t)$ and $\sum_{k=1}^{+\infty}(\partial^2 u_{\alpha k}/\partial x^2)$ (with $u_{\alpha k}$ as in (23)) are uniformly convergent. Setting $M = \int_0^1 |\varphi(x)|dx$ gives: $|\varphi_{2k-1}| \leq 4M$ and $|\varphi_{2k}| \leq 4M$. Thus, by virtue of (23) and the relations

$$
\begin{aligned}
\frac{\partial u_{\alpha k}(x,t)}{\partial t} = \frac{\partial^2 u_{\alpha k}(x,t)}{\partial x^2} = {} & \frac{-\lambda_k\varphi_{2k-1}}{[\alpha + (1-\alpha)e^{-\lambda_k T}]}e^{-\lambda_k t}X_{2k-1}(x) \\
& + \left[\frac{-\lambda_k\varphi_{2k} - 2\lambda_k^{(3/2)}\varphi_{2k-1}(T-t) - 2\sqrt{\lambda_k}\varphi_{2k-1}}{[\alpha + (1-\alpha)e^{-\lambda_k T}]} \right. \\
& \left. + \frac{2\alpha\lambda_k^{(3/2)}\varphi_{2k-1}T}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2} \right] e^{-\lambda_k t}X_{2k}(x).
\end{aligned}
$$

we conclude that

$$|u_{\alpha k}(x,t)| \leq \frac{1}{\alpha} \big[|\varphi_{2k-1}| + |\varphi_{2k}| + 4T\sqrt{\lambda_k}|\varphi_{2k-1}| \big] e^{-\lambda_k \varepsilon} \leq M_1 \lambda_k^{(1/2)} e^{-\lambda_k \varepsilon}$$

$$\left| \frac{\partial u_{\alpha k}(x,t)}{\partial t} \right| = \left| \frac{\partial^2 u_{\alpha k}(x,t)}{\partial x^2} \right| \leq \frac{1}{\alpha} \big[\lambda_k |\varphi_{2k-1}| + \lambda_k |\varphi_{2k}| + 4T\lambda_k^{(3/2)}|\varphi_{2k-1}|$$

$$+ 2\sqrt{\lambda_k}|\varphi_{2k-1}| \big] e^{-\lambda_k \varepsilon} \leq M_2 \lambda_k^{(3/2)} e^{-\lambda_k \varepsilon}.$$

Here, the constants M_1 and M_2 are positive and independent of k. Hence, the series $\sum_{k=1}^{+\infty}(\partial u_{\alpha k}/\partial t)$ and $\sum_{k=1}^{+\infty}(\partial^2 u_{\alpha k}/\partial x^2)$ are bounded by the absolutely convergent numerical series $\sum_{k=1}^{+\infty} M_3 k^3 e^{-4\pi^2 k^2 \varepsilon}$, where $M_3 > 0$ is independent of k. The Weierstrass criterion implies that the original series converge uniformly and determine continuous functions u_α, $(\partial u_\alpha/\partial t)$ and $(\partial^2 u_\alpha/\partial x^2)$ for $t \geq \varepsilon$.

It remains to prove that the series (21) converges uniformly in $\bar{D}_T$. The k-th term of this series is bounded by $|u_{\alpha k}(x,t)| \leq (4\max(1,T)/\alpha)$ $(|\varphi_{2k-1}| + |\varphi_{2k}| + \sqrt{\lambda_k}|\varphi_{2k-1}|)$, and integrations by parts show that

$$|\varphi_{2k-1}| = \frac{\sqrt{2}}{\pi k}|a_k|, \quad |\varphi_{2k}| \leq \frac{\sqrt{2}}{\pi k}(|c_k| + |d_k|) \quad \text{and}$$

$$\sqrt{\lambda_k}|\varphi_{2k-1}| = \frac{\sqrt{2}}{\pi k}|b_k|, \tag{24}$$

where $a_k = \int_0^1 \varphi'(x)\sqrt{2}\sin(2\pi kx)dx$, $c_k = \int_0^1 \varphi'(x)(1-x)\sqrt{2}\cos(2\pi kx)dx$, $d_k = \int_0^1 \varphi(x)\sqrt{2}\cos(2\pi kx)dx$ and $b_k = \int_0^1 \varphi''(x)\sqrt{2}\cos(2\pi kx)dx$, are the Fourier coefficients of $\varphi'(x)$, $\varphi'(x)(1-x)$, $\varphi(x)$ and $\varphi''(x)$ with respect to the orthonormal trigonometric system $\sqrt{2}\cos(2\pi kx)$, $\sqrt{2}\sin(2\pi kx)$, $k \geq 1$.

Using the elementary inequality, Bessel inequalities for the Fourier coefficients with respect to an orthonormal system and the relation $\sum_{k=1}^{+\infty}(1/k^2) = (\pi^2/6)$, we obtain

$$\sum_{k=1}^{+\infty} \frac{4\max(1,T)}{\alpha}(|\varphi_{2k-1}| + |\varphi_{2k}| + \sqrt{\lambda_k}|\varphi_{2k-1}|) \leq c. \tag{25}$$

Hence, the bounding series (25) is absolutely convergent; thus, the series (21) converges uniformly in $\bar{D}_T$ and its sum u_α is continuous in $\bar{D}_T$. ∎

4. SOLUTION ESTIMATES OF (Q.B.V.P)

THEOREM 3   *The solution of (Q.B.V.P) satisfies*

$$\|u_\alpha(x,t)\|_{L^2(0,1)} \leq \frac{1}{\alpha} C_1 \|\varphi\|_{W_2^1(0,1)},$$

with $C_1 = \sqrt{512 \max(1, T^2)}$.

Proof   From the Theorem 1 in [4], we know that for $\varphi \in L^2(0,1)$, we have:

$$r\|\varphi\|_{L^2(0,1)} \leq \sum_{k=0}^{+\infty} \varphi_k^2 \leq R\|\varphi\|_{L^2(0,1)}, \tag{26}$$

$$R^{-1}\|\varphi\|_{L^2(0,1)} \leq \sum_{k=0}^{+\infty} \bar{\varphi}_k^2 \leq r^{-1}\|\varphi\|_{L^2(0,1)}, \tag{27}$$

where $r = 3/4$, $R = 16$ and the coefficients φ_k and $\bar{\varphi}_k$ are given respectively by (16) and:

$$\bar{\varphi}_0 = \int_0^1 \varphi(x) X_0(x)\,dx, \quad \bar{\varphi}_{2k-1} = \int_0^1 \varphi(x) X_{2k-1}(x)\,dx,$$

$$\bar{\varphi}_{2k} = \int_0^1 \varphi(x) X_{2k}(x)\,dx. \tag{28}$$

Therefore,

$$\|u_\alpha(x,t)\|_{L^2(0,1)}^2 \leq \frac{32 \max(1,T^2)}{\alpha^2} \sum_{k=0}^{+\infty} [\varphi_{2k-1}^2 + \varphi_{2k}^2 + \lambda_k \varphi_{2k-1}^2]. \tag{29}$$

Using an integration by parts, we show that:

$$\lambda_k \varphi_{2k-1}^2 = 8 a_k^2, \tag{30}$$

where a_k are given in (24). By using the Bessel inequality, (26) and (30), we find:

$$\|u_\alpha(x,t)\|^2_{L^2(0,1)} \leq \frac{32\max(1,T^2)}{\alpha^2}\left[16\|\varphi\|^2_{L^2(0,1)} + 8\|\varphi'\|^2_{L^2(0,1)}\right]$$

Consequently,

$$\|u_\alpha(x,t)\|^2_{L^2(0,1)} \leq \frac{512\max(1,T^2)}{\alpha^2}\|\varphi\|^2_{W_2^1(0,1)}.$$

■

THEOREM 4   *If u_α is the solution of (Q.B.V.P), then*

$$\|u_\alpha(x,t)\|_{L^2(0,1)} \leq \alpha^{((t-T)/T)}C_2\|\varphi\|_{W_2^1(0,1)},$$

where $C_2 = \sqrt{512\max(1,T^2)}$.

Proof   This theorem is proved similarly to the preceding theorem by using the inequality: $[\alpha + (1-\alpha)e^{-\lambda_k T}]^{-2} \leq e^{2\lambda_k t}(\alpha^{((t-T)/T)})^2$, instead of the inequality $[\alpha + (1-\alpha)e^{-\lambda_k T}]^{-2} \leq \alpha^{-2}$. Hence, (29) is replaced by:

$$\|u_\alpha(x,t)\|^2_{L^2(0,1)} \leq 32\max(1,T^2)(\alpha^{((t-T)/T)})^2$$

$$\sum_{k=0}^{+\infty}[\varphi^2_{2k-1} + \varphi^2_{2k} + \lambda_k\varphi^2_{2k-1}].$$

■

THEOREM 5   *The solution of (Q.B.V.P) satisfies*

$$\|u_\alpha(x,t)\|_{C(\bar{D}_T)} \leq \frac{16\max(1,T)}{\sqrt{3}\alpha}\|\varphi\|_{W_2^2(0,1)}.$$

Proof   ¿From (23), it is easily shown that

$$\max_{\bar{D}_T}|u_{\alpha k}(x,t)| \leq \frac{4\max(1,T)}{\alpha}\left[|\varphi_{2k-1}| + |\varphi_{2k}| + \sqrt{\lambda_k}|\varphi_{2k-1}|\right]. \quad (31)$$

Using inequalities (24), Schwartz inequality and Bessel inequality yields the desired result.   ■

5. CONVERGENCE

THEOREM 6   *If u_α is the solution of (Q.B.V.P), then:*

(1) $\lim_{\alpha \to 1} \|u_\alpha(x,0) - \varphi(x)\|_{L^2(0,1)} = 0$.
(2) $\lim_{\alpha \to 0} \|u_\alpha(x,T) - \varphi(x)\|_{W_2^1(0,1)} = 0$.

Proof

(1) From (21), we have

$$
u_\alpha(x,0) = \sum_{k=1}^{+\infty} \frac{\varphi_{2k-1}}{\alpha + (1-\alpha)e^{-\lambda_k T}} X_{2k-1}(x)
$$
$$
+ \left[\frac{\varphi_{2k} + 2T\sqrt{\lambda_k}\varphi_{2k-1}}{\alpha + (1-\alpha)e^{-\lambda_k T}} - \frac{2\alpha T\sqrt{\lambda_k}\varphi_{2k-1}}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2} \right] X_{2k}(x).
$$

Then,

$$
u_\alpha(x,0) - \varphi(x) = \sum_{k=1}^{+\infty} \frac{(1-\alpha)\varphi_{2k-1}(1 - e^{-\lambda_k T})}{\alpha + (1-\alpha)e^{-\lambda_k T}} X_{2k-1}(x)
$$
$$
+ \left[\frac{(1-\alpha)\varphi_{2k}(1 - e^{-\lambda_k T})}{\alpha + (1-\alpha)e^{-\lambda_k T}} \right.
$$
$$
\left. + \frac{2(1-\alpha)T\sqrt{\lambda_k}\varphi_{2k-1}}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2} \right] X_{2k}(x)
$$

By using the inequality (26) and an elementary transformation, we obtain:

$$
\|u_\alpha(x,0) - \varphi(x)\|_{L^2(0,1)}^2 \leq \frac{32(1-\alpha)^2}{3\alpha^4} \sum_{k=1}^{+\infty} [\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2\lambda_k\varphi_{2k-1}^2].
$$

$$(32)$$

On the other hand, (30), (26) and Bessel inequality give:

$$
\sum_{k=1}^{+\infty} [\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2\lambda_k\varphi_{2k-1}^2] \leq 16\max(1, T^2)\|\varphi\|_{W_2^1(0,1)}^2. \qquad (33)
$$

Finally, (32) and (33) imply: $\lim_{\alpha \to 1} \|u_\alpha(x,0) - \varphi(x)\|_{L^2(0,1)} = 0$.

(2) It is easy to show that the series

$$u_\alpha(x, T) = \sum_{k=1}^{+\infty} \frac{\varphi_{2k-1} e^{-\lambda_k T}}{\alpha + (1-\alpha)e^{-\lambda_k T}} X_{2k-1}(x)$$

$$+ \left[\frac{\varphi_{2k}}{\alpha + (1-\alpha)e^{-\lambda_k T}} - \frac{2\alpha\sqrt{\lambda_k}\varphi_{2k-1}T}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2} \right] e^{-\lambda_k T} X_{2k}(x), \tag{34}$$

converges pointwise to φ as α tends to zero.

Let us prove that the convergence holds in $W_2^1(0,1)$ norm.
¿From (34) and (13), we have:

$$\varphi(x) - u_\alpha(x, T) = \sum_{k=1}^{+\infty} [S_{2k-1}X_{2k-1}(x) + S_{2k}X_{2k}(x)], \tag{35}$$

where

$$S_{2k-1} = \frac{\alpha\varphi_{2k-1}e^{-\lambda_k T}}{\alpha + (1-\alpha)e^{-\lambda_k T}}, \tag{36}$$

$$S_{2k} = \frac{\alpha\varphi_{2k}(1 - e^{-\lambda_k T})}{\alpha + (1-\alpha)e^{-\lambda_k T}} + \frac{2\alpha T\sqrt{\lambda_k}\varphi_{2k-1}e^{-\lambda_k T}}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2}. \tag{37}$$

Using inequality (26) and the inequalities:

$$S_{2k-1}^2 \le \frac{\alpha^2\varphi_{2k-1}^2}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2}, \tag{38}$$

$$S_{2k-1}^2 \le \frac{2\alpha^2\varphi_{2k}^2 + 8\alpha^2 T^2 \lambda_k^2 \varphi_{2k-1}^2}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2}, \tag{39}$$

we find that:

$$\|u_\alpha(x, T) - \varphi(x)\|_{L^2(0,1)}^2 \le \frac{32}{3} \sum_{k=1}^{+\infty} \frac{\alpha^2[\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2\lambda_k\varphi_{2k-1}^2]}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2}. \tag{40}$$

Hence, by virtue of (33), for fixed $\varepsilon > 0$ a positive integer N can be chosen such that

$$\sum_{k=N+1}^{+\infty} [\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2 \lambda_k \varphi_{2k-1}^2] \leq \frac{3\varepsilon}{64}. \tag{41}$$

Therefore, from (40) and (41) we have:

$$\|u_\alpha(x, T) - \varphi(x)\|_{L^2(0,1)}^2$$

$$\leq \frac{32\alpha^2}{3} \sum_{k=1}^{N} [\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2 \lambda_k \varphi_{2k-1}^2] e^{2\lambda_k T} + \frac{\varepsilon}{2}.$$

It remains to take α such that:

$$\alpha^2 < \frac{3\varepsilon}{64} \left(\sum_{k=1}^{N} [\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2 \lambda_k \varphi_{2k-1}^2] e^{2\lambda_k T} \right)^{-1},$$

to get $\lim_{\alpha \to 0} \|u_\alpha(x, T) - \varphi(x)\|_{L^2(0,1)} = 0$.

On the other hand, from (35) we have:

$$\frac{\partial [u_\alpha(x, T) - \varphi(x)]}{\partial x} = \Phi_1(x) + \Phi_2(x),$$

where

$$\Phi_1(x) = \frac{1}{4} \sum_{k=1}^{+\infty} [S_{2k-1} + \sqrt{\lambda_k} S_{2k}] Y_{2k-1}(x) + \sqrt{\lambda_k} S_{2k-1} Y_{2k}(x),$$

$$\Phi_2(x) = - \sum_{k=1}^{+\infty} \sqrt{\lambda_k} S_{2k-1} X_{2k}(x),$$

(27), (38) and (39) yield:

$$\|\Phi_1(x)\|_{L^2(0,1)}^2$$

$$\leq \sum_{k=1}^{+\infty} \alpha^2 \frac{[2\varphi_{2k-1}^2 + 4\lambda_k \varphi_{2k}^2 + 16T^2 \lambda_k^2 \varphi_{2k-1}^2 + \lambda_k \varphi_{2k-1}^2]}{[\alpha + (1 - \alpha)e^{-\lambda_k T}]^2},$$

whereas (26), (38) and (39) give:

$$\|\Phi_2(x)\|_{L^2(0,1)}^2 \leq \frac{4}{3} \sum_{k=1}^{+\infty} \alpha^2 \frac{\lambda_k \varphi_{2k-1}^2}{[\alpha + (1 - \alpha)e^{-\lambda_k T}]^2}.$$

Since

$$\left\| \frac{\partial [u_\alpha(x,T) - \varphi(x)]}{\partial x} - \varphi(x) \right\|_{L^2(0,1)}^2$$

$$\leq 2 \left[\|\Phi_1(x)\|_{L^2(0,1)}^2 + \|\Phi_2(x)\|_{L^2(0,1)}^2 \right],$$

we have:

$$\left\| \frac{\partial(u_\alpha(x,T) - \varphi(x))}{\partial x} \right\|_{L^2(0,1)}^2$$

$$\leq 32 \sum_{k=1}^{+\infty} \alpha^2 \frac{[\varphi_{2k-1}^2 + \lambda_k(\varphi_{2k-1}^2 + \varphi_{2k}^2) + T^2\lambda_k^2\varphi_{2k-1}^2]}{[\alpha + (1-\alpha)e^{-\lambda_k T}]^2}. \quad (42)$$

Furthermore, using integration by parts, conditions (22) and Bessel inequality, we show that:

$$\sum_{k=1}^{+\infty} [\varphi_{2k-1}^2 + \lambda_k(\varphi_{2k-1}^2 + \varphi_{2k}^2) + T^2\lambda_k^2\varphi_{2k-1}^2] \leq 24 \max(1,T^2)\|\varphi\|_{W_2^2(0,1)}^2.$$

Hence, for fixed $\varepsilon > 0$ a positive integer N can be chosen such that

$$\sum_{k=N+1}^{+\infty} [\varphi_{2k-1}^2 + \lambda_k(\varphi_{2k-1}^2 + \varphi_{2k}^2) + T^2\lambda_k^2\varphi_{2k-1}^2] \leq \frac{\varepsilon}{64}, \quad (43)$$

From (42) and (43), we find:

$$\left\| \frac{\partial(u_\alpha(x,T) - \varphi(x))}{\partial x} \right\|_{L^2(0,1)}^2 \leq 32\alpha^2 \sum_{k=1}^{N} [\varphi_{2k-1}^2 + \lambda_k(\varphi_{2k-1}^2 + \varphi_{2k}^2)$$

$$+ T^2\lambda_k^2\varphi_{2k-1}^2]e^{2\lambda_k T} + \frac{\varepsilon}{2}.$$

Now, if we take α such that:

$$\alpha^2 \leq \frac{\varepsilon}{64} \left(\sum_{k=1}^{N} [\varphi_{2k-1}^2 + \lambda_k(\varphi_{2k-1}^2 + \varphi_{2k}^2) + T^2\lambda_k^2\varphi_{2k-1}^2]e^{2\lambda_k T} \right)^{-1},$$

we get $\lim_{\alpha \to 0} \|(\partial(u_\alpha(x,T) - \varphi(x))/\partial x)\|_{L^2(0,1)} = 0$.
   This ends the proof of convergence in $W_2^1(0,1)$ norm. ∎

THEOREM 7   *If* $\varphi \in W_2^3(0,1)$ *verifies condition* (22) *and the condition*:

$$\varphi''(0) = 0, \tag{44}$$

then,

$$\lim_{\alpha \to 0} \left\| \frac{\partial^2 (u_\alpha(x,T) - \varphi(x))}{\partial x^2} \right\|_{L^2(0,1)}^2 = 0,$$

where u_α is the solution of Q.B.V.P (5)−(8).

Proof   Let $\varphi \in W_2^3(0,1)$, verifying conditions (22) and (44). From (35), we have:

$$\frac{\partial^2 (u_\alpha(x,T) - \varphi(x))}{\partial x^2}$$
$$= -\sum_{k=1}^{+\infty} [\lambda_k S_{2k-1} X_{2k-1}(x) + (\lambda_k S_{2k-1} + 2\sqrt{\lambda_k} S_{2k-1}) X_{2k}(x)],$$

where the coefficients S_{2k-1} and S_{2k} are given by (36) and (37) respectively. From (26), (38) and (39), we get:

$$\left\| \frac{\partial^2 (u_\alpha(x,T) - \varphi(x))}{\partial x^2} \right\|_{L^2(0,1)}^2 \leq \frac{64}{3} \sum_{k=1}^{+\infty} \alpha^2 [\alpha + (1-\alpha)e^{-\lambda_k T}]^{-2}$$
$$[\lambda_k \varphi_{2k-1}^2 + \lambda_k^2 (\varphi_{2k-1}^2 + \varphi_{2k}^2)$$
$$+ T^2 \lambda_k^3 \varphi_{2k-1}^2] \tag{45}$$

Now, using integrations by parts, condition (22), and Bessel inequality, we show that:

$$\sum_{k=1}^{+\infty} [\lambda_k \varphi_{2k-1}^2 + \lambda_k^2 (\varphi_{2k-1}^2 + \varphi_{2k}^2) + T^2 \lambda_k^3 \varphi_{2k-1}^2]$$
$$\leq 72 \max(1, T^2) \|\varphi\|_{W_2^3(0,1)}^2.$$

Hence, for fixed $\varepsilon > 0$ a positive integer N can be chosen such that:

$$\sum_{k=1}^{+\infty} [\lambda_k \varphi_{2k-1}^2 + \lambda_k^2 (\varphi_{2k-1}^2 + \varphi_{2k}^2) + T^2 \lambda_k^3 \varphi_{2k-1}^2] \leq \frac{3\varepsilon}{128}, \tag{46}$$

¿From (45) and (46), we find:

$$\left\| \frac{\partial^2 (u_\alpha(x,T) - \varphi(x))}{\partial x^2} \right\|^2_{L^2(0,1)}$$

$$\leq \frac{64}{3} \alpha^2 \sum_{k=1}^N [\lambda_k \varphi^2_{2k-1} + \lambda_k^2 (\varphi^2_{2k-1} + \varphi^2_{2k}) + T^2 \lambda_k^3 \varphi^2_{2k-1}] e^{2\lambda_k T} + \frac{\varepsilon}{2}$$

Now, if we take α such that:

$$\alpha^2 \leq \frac{3\varepsilon}{128} \left(\sum_{k=1}^N [\lambda_k \varphi^2_{2k-1} + \lambda_k^2 (\varphi^2_{2k-1} + \varphi^2_{2k}) + T^2 \lambda_k^3 \varphi^2_{2k-1}] e^{2\lambda_k T} \right)^{-1}.$$

we get $\lim_{\alpha \to 0} \|(\partial^2 (u_\alpha(x,T) - \varphi(x))/\partial x^2)\|_{L^2(0,1)} = 0$. ∎

THEOREM 8   If $\varphi \in W_2^2(0,1)$ satisfies (22) and there exists $\varepsilon \in (0,2)$ such that: $\sum_{k=1}^{+\infty} [\varphi^2_{2k-1} + \varphi^2_{2k} + T^2 \lambda_k^2 \varphi^2_{2k-1}] e^{\varepsilon \lambda_k T}$ converges, then $\int_0^1 |u_\alpha(x,T) - \varphi(x)|^2 dx$ converges to zero with order $\alpha^\varepsilon \varepsilon^{-2}$.

Proof   Let ε be in $(0,2)$ such that $\sum_{k=1}^{+\infty} [\varphi^2_{2k-1} + \varphi^2_{2k} + T^2 \lambda_k^2 \varphi^2_{2k-1}] e^{\varepsilon \lambda_k T}$ converges, and let β be in $(0,2)$. Fix a natural integer k and define $g_k(\alpha) = (\alpha^\beta / [\alpha + (1-\alpha) e^{-\lambda_k T}]^2)$. It can be shown that:

$$g_k(\alpha) \leq C_0 \left(\frac{\beta}{2-\beta} \right)^\beta e^{(2-\beta)\lambda_k T}, \tag{47}$$

where $C_0 = (1 - e^{-4\pi^2 T})^{-2}$. Furthermore from (40), we have

$$\|u_\alpha(x,T) - \varphi(x)\|^2_{L_2(0,1)} \leq \frac{32}{3} \sum_{k=1}^\infty \frac{\alpha^2 [\varphi^2_{2k-1} + \varphi^2_{2k} + T^2 \lambda_k \varphi^2_{2k-1}]}{[\alpha + (1-\alpha) e^{-\lambda_k T}]^2}.$$

Therefore,

$$\|u_\alpha(x,T) - \varphi(x)\|^2_{L_2(0,1)}$$

$$\leq \frac{32}{3} \alpha^{2-\beta} \sum_{k=1}^\infty (\varphi^2_{2k-1} + \varphi^2_{2k} + T^2 \lambda_k \varphi^2_{2k-1}) g_k(\alpha). \tag{48}$$

Hence, from (47) and (48) we get

$$\|u_\alpha(x,T) - \varphi(x)\|^2_{L_2(0,1)}$$

$$\leq \frac{32}{3} C_0 \left(\frac{\beta}{2-\beta} \right)^\beta \sum_{k=1}^\infty [\varphi^2_{2k-1} + \varphi^2_{2k} + T^2 \lambda_k \varphi^2_{2k-1}] e^{(2-\beta)\lambda_k T}$$

If we choose $\beta = 2 - \varepsilon$, we obtain

$$\|u_\alpha(x, T) - \varphi(x)\|_{L_2(0,1)}^2 \le C\alpha^\varepsilon \varepsilon^{-2}$$

where $C = (128(1 - e^{-4\pi^2 T})^{-2}/3) \sum_{k=1}^\infty [\varphi_{2k-1}^2 + \varphi_{2k}^2 + T^2 \lambda_k \varphi_{2k-1}^2] e^{\varepsilon \lambda_k T}$.

∎

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