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FIXED POINT AND APPROXIMATE FIXED POINT THEOREMS FOR NON–AFFINE MAPS

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Abstract. Let C be a non–empty subset of a linear topological space X, and T be a selfmap of C such that the range of $I-T$ is convex, where I denotes the identity map on X. We give conditions under which a map T has a fixed point or a V–fixed point (i.e. a point $x_0 \in C$ such that $Tx_0 \in x_0 + V$, where V is a neighborhood of the origin). Our theorems generalize the recent results of M. Edelstein and K.-K. Tan ([3], [4]). As an application we provide a simple proof of the Markov–Kakutani theorem. We also establish a common V–fixed point theorem for commuting affine maps (possibly discontinuous).

1. Introduction. In the paper [3] M. Edelstein and K.-K. Tan have obtained some fixed point theorems for affine maps on a normed space. In their recent work [4] the authors have extended these results by considering affine maps on linear topological spaces and linear spaces. Our purpose here is to show that similar results hold for a larger class of maps. These maps need to be neither affine nor defined on the whole space as was required in [3] and [4]. Instead, given a selfmap T on a subset C of a linear topological space X (not necessarily locally convex), we assume that the set $(I-T)(C)$

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is convex, where I denotes the identity map on X. Clearly, this assumption is automatically satisfied, if $C = X$ (more generally, if C is convex) and T is affine; this is exactly the case considered by Edelstein and Tan. Further, as an application of our main results we provide a very simple proof of the well–known Markov–Kakutani fixed point theorem. We emphasize that A. Markov's original proof uses the Schauder–Tychonoff principle (this proof can be found in [2], p. 75) and it is valid only for locally convex spaces. Though an easier proof inspired by F. Riesz's proof of the ergodic theorem was given by S. Kakutani (see, e.g., [1], p. 109), it seems that our proof is yet simpler. As another application we obtain results on approximate fixed points. Recall that a point a of a linear topological space X is said to be a *V–fixed point* of a map T if $Ta \in a + V$, where V is a given neighborhood of the origin (see [2], p. 104). If X is a normed space, then given $\varepsilon > 0$, a point a is said to be an *ε–fixed point* of T if $\|a - Ta\| \le \varepsilon$ (see [2], p. 56). Our motivation for investigating ε–fixed points derives from the fact that a continuous affine selfmap T on a closed bounded convex subset C of a Banach space X need not have fixed points unless X is reflexive (see Example 1). Nevertheless, our Theorem 4 shows that for any $\varepsilon > 0$, T has then an ε–fixed point even if T is discontinuous and the set C is not closed.

Finally, we prove a common *V–fixed* point theorem for affine maps (see Section 5). Throughout the paper any linear topological spaces are assumed to be Hausdorff.

2. Extensions of theorems by Edelstein and Tan. We begin with the following generalization of Theorem 1 [3].

Theorem 1. *Let C be a non–empty subset of a linear topological space X, and T be a selfmap on C such that the range of $I - T$ is convex. Then the following alternative holds: either*

(i) *for any neighborhood V of the origin, there exists a V–fixed point for T, or*

(ii) *for all $x \in C$,*

$$0 \notin \bigcap_{n\in\mathbb{N}} \operatorname{cl}\left\{\frac{T^n(x)}{n}, \frac{T^{n+1}(x)}{n+1}, \dots\right\}$$

(in particular, if X is a normed space, then $\liminf_{n\to\infty} \|T^n x\|/n > 0$ *so $\|T^n x\| \longrightarrow \infty$ for all $x \in C$).*

Proof. If condition (ii) holds, we are done. So assume that (ii) is not satisfied. Then there is an $x \in X$ such that $0 \in \operatorname{cl}\left\{\frac{T^n x}{n} : n \ge m\right\}$, for every $m \in \mathbb{N}$. For any neighborhood V of the origin there is a neighborhood U of the origin such that $U + U \subseteq V$. So we are able to find $k \in \mathbb{N}$

such that $k^{-1}x \in U$ and $T^kx/k \in -U$. Since $T(C) \subseteq C$, we get that $T^{n-1}x - T^nx \in (I-T)(C)$ for all $n \in \mathbb{N}$, where $T^0 = I$. Since $(I-T)(C)$ is convex and

$$\frac{1}{k}(x - T^kx) = \frac{1}{k}\sum_{i=1}^{k}(T^{i-1}x - T^ix),$$

we may conclude that $(x - T^kx)/k \in (I-T)(C)$. Hence, we infer that $V \cap (I-T)(C) \neq \emptyset$, which implies (i).

Finally, the first part of (ii) obviously implies the second. □

As an immediate consequence we obtain the following extension of Theorem 2 [3].

Corollary 1. *Under the assumptions of Theorem 1, if X is a normed space, the range of $I-T$ is closed and the map T is fixed–point free, then for all $x \in C$, $\|T^nx\| \longrightarrow \infty$ as $n \longrightarrow \infty$.*

Proof. Observe that condition (i) of Theorem 1 cannot hold; for otherwise, by hypothesis $0 \in (I-T)(C)$, which means that T has a fixed point, a contradiction. □

The next result generalizes Theorem 3 [3] and Theorem 3.3 [4] in that the assumptions on a map T are weakened and a new equivalent condition is added.

Theorem 2. *Let C be a non–empty subset of a linear topological space X, and T be a selfmap on C such that the range of $I-T$ is convex and closed. Then the following conditions are equivalent:*

- (i) *T has a fixed point;*
- (ii) *there is an $x \in C$ such that the sequence $\{T^nx\}_{n=1}^{\infty}$ has a convergent subsequence;*
- (iii) *there is an $x \in C$ such that the sequence $\{T^nx\}_{n=1}^{\infty}$ contains a bounded subsequence;*
- (iv) *there is an $x \in C$ such that the sequence $\left\{\frac{T^nx}{n}\right\}$ contains a subsequence convergent to 0;*
- (v) *there is an $x \in C$ such that*

$$0 \in \bigcap_{n\in\mathbb{N}} \operatorname{cl}\left\{\frac{T^n(x)}{n}, \frac{T^{n+1}(x)}{n+1}, \dots\right\}.$$

Proof. The implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(v) are obvious. That (v) implies (i), follows from Theorem 1 and the fact that $(I-T)(C)$ is closed. □

As an application of Theorem 2 we establish a fixed point theorem. We want to emphasize here that its assumptions deal with the sets only (the domain of T and the range of $I - T$) and not with the map itself.

Theorem 3. *Let C be a non–empty bounded subset of a linear topological space X, and T be a selfmap of C such that the range of $I - T$ is convex and closed. Then T has a fixed point.*

Proof. Clearly, condition (iii) of Theorem 2 is fulfilled so it suffices to apply Theorem 2. □

Theorem 3 and the Schauder–Tychonoff principle yield the following result.

Corollary 2. *Let C be a compact subset of a locally convex space X, and T be a continuous selfmap on C. If one of the sets C or $(I-T)(C)$ is convex, then T has a fixed point.*

We close this section with an approximate fixed point theorem, which immediately follows from Theorem 1 and its proof. The set of all V–fixed points of T is denoted by $\mathrm{Fix}_V\, T$.

Corollary 3. *Let C be a non–empty bounded subset of a linear topological space X, and T be a selfmap on C such that the range of $I - T$ is convex. Then, given a neighborhood V of the origin, there is an $n_V \in \mathbb{N}$ such that*

$$(I-T)^{-1}\left(\frac{1}{n}(I-T^n)(C)\right) \subseteq \mathrm{Fix}_V\, T, \quad \textit{for } n \geq n_V.$$

In particular, if X is a normed space, then for any $\varepsilon > 0$, T has an ε–fixed point; equivalently, $\inf_{x\in C} \|x - Tx\| = 0$.

Remark 1. Following the proof of Theorem 1, we obtain

$$\frac{1}{n}(I-T^n)(C) \subseteq (I-T)(C).$$

Therefore the set $(I-T)^{-1}\left(\frac{1}{n}(I-T^n)(C)\right)$ (and hence $\mathrm{Fix}_V\, T$) is non–empty.

3. A short proof of the Markov–Kakutani theorem.

Theorem (Markov–Kakutani). *Let C be a compact convex subset of a linear topological space X, and let $\mathcal{F}$ be a commuting family of continuous affine selfmaps on C. Then $\mathcal{F}$ has a common fixed point.*

Proof. We divide the proof into two parts.

1^0. We show that each $T \in \mathcal{F}$ has a fixed point. Since T is affine and C is convex, the set $(I-T)(C)$ is convex. Since T is continuous and C

is compact, the set $(I-T)(C)$ is compact, hence closed because X is a Hausdorff space. So it suffices to apply Theorem 3.

2^0. Now we take pattern by a part of the proof presented in [2], p. 75. By hypothesis, given $T \in \mathcal{F}$, Fix T, the set of all fixed points of T, is convex and compact. Moreover, given $S, T \in \mathcal{F}$, the set Fix T is S–invariant, i.e., $S(\text{Fix } T) \subset \text{Fix } T$ since S and T commute. Therefore, by part 1^0 with $C := \text{Fix } T$, the map $S|_{\text{Fix } T}$ has a fixed point, which means that Fix $S \cap$ Fix $T \neq \emptyset$. An easy induction shows that the family $\{\text{Fix } T : T \in \mathcal{F}\}$ has the finite intersection property. By compactness argument, we infer that $\bigcap_{T\in\mathcal{F}} \text{Fix } T \neq \emptyset$. □

Remark 2. A unified approach to several common fixed point theorems including the Markov–Kakutani result has been presented in [5].

4. Another applications to affine maps. Theorems 3 and Corollary 3 yield the following result for affine maps.

Theorem 4. *Let C be a non–empty bounded and convex (not necessarily closed) subset of a linear topological space X, and T be an affine (not necessarily continuous) selfmap on C. Then, given a neighborhood V of the origin, T has a V–fixed point. Moreover, if the range of $I-T$ is closed, then T has a fixed point.*

The following example (see [2], p. 35) shows that the assumption of Theorem 4 that $(I-T)(C)$ be closed cannot be omitted even if C is closed and T is continuous.

Example 1. Let $X := c_0$, the Banach space of all sequences convergent to 0, equipped with the sup–norm. Let C be the closed unit ball in X. For $x \in C$, $x = (x_1, x_2, \dots)$, define

$$Tx := (1, x_1, x_2, \dots).$$

Clearly, $T(C) \subset C$, T is affine and continuous, but T has no fixed point. Nevertheless, according to the first part of Theorem 4, T has an ε–fixed point x_ε for any $\varepsilon > 0$. We can ensure about it directly: given $\varepsilon \in (0,1)$, it suffices to put

$$x_\varepsilon := (1-\varepsilon, 1-2\varepsilon, \dots, 1-n_\varepsilon\varepsilon, 0, \dots),$$

where $n_\varepsilon := \max\{n \in \mathbb{N} : \ 1 - n\varepsilon > 0\}$.

5. A common V–fixed point theorem.

Theorem 5. *Let C be a non–empty bounded and convex subset of a linear topological space X, and $\mathcal{F}$ be a finite family of commuting affine selfmaps on C. Then, given a neighborhood V of the origin, $\mathcal{F}$ has a common V–fixed point.*

Proof. We employ the idea of Kakutani's proof (see [1], p. 106). Let $\mathcal{F} = \{T_k :\ 1 \le k \le p\}$. For $1 \le k \le p$, $n \in \mathbb{N}$ and $x \in C$, define

$$T_{kn}x := \frac{1}{n}\sum_{i=1}^{n} T_k^{i-1}x.$$

Then $T_{kn}(C) \subseteq C$ so if $y := T_{kn}x$, $x \in C$, then $y \in C$ and $y - T_k y = (x - T_k^n x)/n$ since T_k is affine. Therefore, we have the inclusion:

$$T_{kn}(C) \subseteq (I - T_k)^{-1}\left(\frac{1}{n}(I - T_k^n)(C)\right), \qquad \text{for } n \in \mathbb{N} \text{ and } 1 \le k \le p.$$

Hence and by Corollary 3, given a neighborhood V of the origin, there is an $n_V \in \mathbb{N}$ such that

$$T_{kn}(C) \subseteq \operatorname{Fix}_V T_k, \qquad \text{for } n \ge n_V \text{ and } 1 \le k \le p. \tag{1}$$

It is easy to verify that for each $n \in \mathbb{N}$ the family $\{T_{kn} :\ 1 \le k \le p\}$ is commuting. Hence, if

$$A_n := (T_{1n} \circ \ldots \circ T_{pn})(C),$$

then given k, $1 \le k \le p$, $A_n = (T_{kn} \circ S_{kn})(C)$, where S_{kn} denotes the superposition of all T_{in} with $i \ne k$. Since $S_{kn}(C) \subseteq C$, we get using (1) that

$$A_n \subseteq T_{kn}(C) \subseteq \operatorname{Fix}_V T_k \qquad \text{for } n \ge n_V.$$

Hence, we conclude that $A_n \subseteq \bigcap_{k=1}^{p} \operatorname{Fix}_V T_k$ for $n \ge n_V$. Since A_n is non–empty, so is the set $\bigcap_{k=1}^{p} \operatorname{Fix}_V T_k$. □

Remark 3. It can be easily shown that for any convex neighborhood V of the origin and affine map T, the set $\operatorname{Fix}_V T$ is convex. This fact could suggest to prove Theorem 5 by a slight modification of the proof presented in Section 3; that is, by considering the restriction $S|_{\operatorname{Fix}_V T}$, where S is another affine map, which commutes with T. However, while the set Fix T is S–invariant, that is not the case with $\operatorname{Fix}_V T$ unless S is nonexpansive. Moreover, the local convexity of X is necessary for the validity of such a modified proof. Consequently, we would obtain then a weaker result than Theorem 5.

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