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Memory Effects and Γ-Convergence: A Time Dependent Case

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We study the asymptotic behavior of the energy functional associated to the parametric equation $\partial_t u_\varepsilon(t,x) + a_\varepsilon(t,x)\, g\,(u_\varepsilon(t,x)) = f(t,x)$, $u_\varepsilon(0,x) = u_0(x)$. Techniques of Young measures are improved in order to characterize the Γ-limit of the sequence of energy functionals in terms of the oscillations of a_ε. We assume some sort of independence between time and the oscillating character of a_ε. The example of a periodic mixture of two materials with coefficients analytic in time is presented.

1. Introduction

Consider the problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t,x) + a_\varepsilon(t,x)\, g\,(u_\varepsilon(t,x)) = f\,(t,x) \text{ in } (0,\,T)\times\Omega, \\ \qquad\qquad\qquad\qquad u_\varepsilon(0,x) = u_0(x) \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $a_\varepsilon(t,x)$ are highly oscillating coefficients and ε is a decreasing sequence of real numbers converging to 0. In the early eighties L. Tartar studied the above problem for a linear function g and time independent coefficients a_ε (see [18]). A memory effect appeared in the limit equation, by means of a kernel expressed in terms of Young measures associated to the sequence $(a_\varepsilon)_\varepsilon$. In the case when the coefficients are time dependent the limit of problem (1.1) was studied by L. Mascarenhas in [13] for linear g. L. Tartar studied the same problem in [19] and obtained a simpler form of the kernel expressed in terms of weak limits of some integral expressions of a_ε. He also considered the case of a quadratic function g (see [19]). Y. Amirat, K. Hamdache and A. Ziani studied this kind of problems in the context of transport equations (see [3]).

A different approach, suggested by De Giorgi in [10], is based on Γ-convergence of the associated energy functionals. The case with time independent coefficients, for linear and non-linear g, was treated by L. Mascarenhas in [14] and [15], and also by L. Ambrosio, P. D'Ancona and S. Mortola in [2].

The goal of the present paper is to characterize, in the time dependent case, the Γ-limit of the sequence of energy functionals in terms of the oscillations of the coefficients a_ε. In order to do this, we assume a sort of independence between the time variable t and the oscillations of the coefficients $a_\varepsilon(t,x)$, in the sense of (3.2) below.

As a tool we improve the techniques of Young measures already introduced by L. Mascarenhas in [14] and [15] for the time independent case. Those techniques consist in a

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generalization, to the non periodic case, of the two scale convergence method introduced by G. Nguetseng [16], W. E [12] and G. Allaire [1]. Recently M. Valadier revealed the link between two scale convergence and Young measures by introducing the two scale Young measures (see [21] and [22]). In order to identify the two scale limit of a bounded sequence one has to consider the barycenter of the associated two-scale Young measure. In the problem under consideration we deal with general non periodic time dependent coefficients a_ε and we use classical Young measures as presented in [20]. This allows us to describe the Γ-limit of the energy functionals corresponding to (1.1), as ε goes to zero, only in terms of the Young measures associated to some sequence characterizing the oscillations (the sequence $(\alpha_\varepsilon)_\varepsilon$ introduced in (3.2) below).

For the sake of clarity, we first present in detail the linear case and after that the non-linear case, just pointing out some steps. In Section 2 we give some results on measure theory, integration and classical Young measures. The setting of the problem is presented in Section 3. In Section 4, we give a formula for the Γ-limit of the energy functionals, in the linear case. We prove some auxiliary results which will be used also for treating the non-linear case. In Section 5, the formula of the Γ-limit of the energy functionals is generalized for a non-linear case, proving only those steps which are essentially different from the linear case. In Section 6 we consider the example when the coefficients a_ε are analytic in t. We particularize further by taking a periodic mixture of two materials with coefficients analytic in t, and obtain a less abstract formula for the Γ-limit.

2. Preliminaries

We present here some properties and results about Young measures contained in [20]. The notations are the ones employed in [20].

Let $\mathcal{O}$ be a domain in $\mathbb{R}^N$ and S be a Souslin space (that is the image of a Polish space by a continuous function). We shall assume that S is a σ-compact locally compact space since in the sequel we shall deal with S as a metrizable compact space or as $\mathbb{R}^d$. Let m_N be the Lebesgue measure on $\mathbb{R}^N$. Let $\mathcal{F}$ be the σ-field of Lebesgue measurable subsets of $\mathcal{O}$. Let $\mathcal{B}(S)$ be the Borel σ-field of S and let $\mathcal{F}\otimes\mathcal{B}(S)$ be the σ-field product of $\mathcal{F}$ and $\mathcal{B}(S)$.

We call *Young measure* any positive measure μ on $\mathcal{O}\times S$ whose projection on $\mathcal{O}$ is m_N. Let $\mathcal{Y}(\mathcal{O};\, S)$ be the set of all Young measures.

We will not distinguish μ from its disintegration $(\mu_x)_{x\in\mathcal{O}}$ which is a measurable family of probabilities on S such that for any $\psi : \mathcal{O}\times S \longrightarrow \mathbb{R}$, μ-integrable,

$$\int_{\mathcal{O}\times S}\psi\, d\mu = \int_{\mathcal{O}}\int_S \psi(x,\xi)\, d\mu_x(\xi) dm_N(x) \quad \text{(see [6])}.$$

A positive bounded measure λ on $\mathcal{B}(S)$ is a *Radon measure* if $\forall A\in\mathcal{B}(S)$, $\forall\varepsilon>0$, $\exists K$ a compact subset $K\subset A$ such that $\lambda(A\setminus K)\le\varepsilon$.

Any positive bounded measure on a Souslin space is Radon (see [11]) and, consequently, since $\mathcal{O}\times S$ is a Souslin space, any measure in $\mathcal{Y}(\mathcal{O};\, S)$ is Radon.

A sequence of Young measures $(\mu_n)_n$ is said to be *tight* if $\forall\varepsilon>0\,\exists K_\varepsilon$ a compact subset of S such that $\sup_n \mu_n(\mathcal{O}\times(S\setminus K_\varepsilon))\le\varepsilon$.

A *Carathéodory integrand* on $\mathcal{O} \times S$ is a real $\mathcal{F} \otimes \mathcal{B}(S)$ measurable function such that for all $x \in \mathcal{O}$, $\psi(x, \cdot)$ is continuous and bounded on S and $x \longrightarrow \|\psi(x, \cdot)\|$ is Lebesgue integrable (the above norm is the norm of uniform convergence).

The set $\mathcal{Y}(\mathcal{O}; S)$ is endowed with the weakest topology which makes the maps $\mu \longrightarrow \int_{\mathcal{O}\times S} \psi \, d\mu$ continuous, for all Carathéodory integrands ψ. This topology will be called the *narrow topology*.

Let $u : \mathcal{O} \longrightarrow S$ be a measurable function. The Young measure associated to u is the image μ of m_N by the map $x \longrightarrow (x, u(x))$.

A sequence $(u_n)_n$ of measurable functions $u_n : \mathcal{O} \longrightarrow S$ is *tight* if the sequence of their associated Young measures is tight.

Note that if $S = \mathbb{R}^d$ and $(u_n)_n$ is bounded in $L^1(\mathcal{O}; \mathbb{R}^d)$ then $(u_n)_n$ is tight. If S is compact any sequence $(u_n)_n$, $u_n : \mathcal{O} \longrightarrow S$ is tight.

Theorem 2.1 (Theorem of compactness). (See Theorem 11 from [20])
If $(\mu_n)_n$ is a tight sequence of Young measures then there exists a subsequence $(\mu_{n_k})_k$ which narrow converges in $\mathcal{Y}(\mathcal{O}; S)$.

A sequence $(u_n)_n$ in $L^1(\mathcal{O}, m_N, \mathbb{R}^d)$ is *uniformly integrable* if: (a) $(u_n)_n$ is bounded in $L^1(\mathcal{O}, m_N, \mathbb{R}^d)$, and (b) $A \in \mathcal{F}$ with $m_N(A) \longrightarrow 0$ implies that $\sup_n \int_A \|u_n(x)\| \, dm_N(x) \longrightarrow 0$.

Theorem 2.2 (Fundamental theorem for Young measures). (See Theorems 16 and 17 from [20])
Let $(u_n)_n$ be a sequence of measurable functions, $u_n : \mathcal{O} \longrightarrow S$, such that the sequence of their Young measures narrow converges to μ. Then:

(a) *If $\psi : \mathcal{O}\times S \longrightarrow \mathbb{R}$ is a $\mathcal{F}\otimes\mathcal{B}(S)$ measurable function such that for all $x \in \mathcal{O}$, $\psi(x, \cdot)$ is lower semicontinuous and such that the sequence $(\psi(x, u_n(x))^-)_n$ is uniformly integrable in $\mathcal{O}$, then*

$$\int_{\mathcal{O}\times S} \psi \, d\mu \leq \liminf \int_{\mathcal{O}} \psi(x, u_n(x)) \, dx.$$

(b) *If $\psi : \mathcal{O} \times S \longrightarrow \mathbb{R}$ is a $\mathcal{F} \otimes \mathcal{B}(S)$ measurable function such that for all $x \in \mathcal{O}$, $\psi(x, \cdot)$ is continuous and such that $(\psi(x, u_n(x))_n$ is uniformly integrable in $\mathcal{O}$, then*

$$\int_{\mathcal{O}\times S} \psi d\mu = \lim_n \int_{\mathcal{O}} \psi(x, u_n(x)) \, dx.$$

3. Setting of the problem

Let Ω be a open bounded subset in $\mathbb{R}^N$.

Consider the problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t,x) + a_\varepsilon(t,x)\, u_\varepsilon(t,x) = f(t,x) \text{ in } (0, T) \times \Omega, \\ u_\varepsilon(0,x) = 0 \text{ on } \Omega, \end{cases} \tag{3.1}$$

where $f \in L^\infty((0, T) \times \Omega)$; for all $\varepsilon > 0$, $a_\varepsilon \in L^\infty((0, T) \times \Omega)$ and

$$0 < \alpha \leq a_\varepsilon(t, x) \leq \beta, \quad \text{for some } \alpha,\ \beta > 0.$$

We assumed that $u_0 \equiv 0$. For a general $u_0 \in L^2(\Omega)$ the analysis of the problem is similar by taking an appropriate space of admissible functions.

Let Π be a metrizable, compact space. Consider $(\alpha_\varepsilon)_\varepsilon$ a sequence of measurable functions, $\alpha_\varepsilon : \Omega \longrightarrow \Pi$, which is tight. Then, by the Theorem of compactness, there exists a subsequence still denoted by $(\alpha_\varepsilon)_\varepsilon$ such that the sequence of their associated Young measures narrow converges. We denote by $\mu \in \mathcal{Y}(\Omega;\, \Pi)$ its limit.

Consider the coefficients a_ε of the form:

$$a_\varepsilon(t, x) := c\,(t, x, \alpha_\varepsilon(x)) \tag{3.2}$$

where $c \in L^\infty([0, T] \times \Omega;\, C(\Pi))$ and, for some constants α, $\beta > 0$,

$$0 < \alpha \leq c\,(t, x, \lambda) \leq \beta \quad \forall\,(t, x, \lambda) \in [0, T] \times \Omega \times \Pi. \tag{3.3}$$

Here we denote by $C(\Pi)$ the space of real continuous functions on Π.

Remark 3.1. The coefficients a_ε above defined belong to $L^\infty([0, T] \times \Omega)$. Attending to (3.3), it suffices to prove that a_ε are measurable. As $L^\infty([0, T] \times \Omega;\, C(\Pi)) \subset L^1([0, T] \times \Omega;\, C(\Pi))$, by Lemma A3 in [20], page 178, c is actually a Carathéodory integrand, hence it is $\mathcal{F} \otimes \mathcal{B}\,(\Pi)$-measurable. Here we denote by $\mathcal{F}$ the set of Lebesgue measurable sets in $[0, T] \times \Omega$ and by $\mathcal{B}\,(\Pi)$ the set of Borel sets in Π. Since the map $(t, x) \mapsto (t, x, \alpha_\varepsilon(x))$ is $(\mathcal{F},\, \mathcal{F} \otimes \mathcal{B}\,(\Pi))$-measurable, the composition $(t, x) \mapsto c\,(t, x, \alpha_\varepsilon(x))$ is $\mathcal{F}$-measurable.

Let

$$H = \{v \in H^1(0, T;\, L^2(\Omega)),\ v\,(0, x) = 0 \quad m_N \text{ a.e. in } \Omega\}, \tag{3.4}$$

where m_N is the Lebesgue measure on $\mathbb{R}^N$. For each $\varepsilon > 0$, let J_ε be the energy functional

$$J_\varepsilon(v) := \frac{1}{2} \int_0^T \int_\Omega |v' + a_\varepsilon v|^2 dx\, dt - \int_0^T \int_\Omega (v' + a_\varepsilon v)\, f\, dx\, dt.$$

The energy functionals J_ε are strictly convex, lower semicontinuous and coercive; thus the problem

$$\min_{v \in H} J_\varepsilon(v) \tag{3.5}$$

has a unique solution. The following proposition proved in [15] still holds in the time dependent case under consideration. Its proof is based on a result by H. Brezis and I. Ekeland (see [4] and [5]).

Proposition 3.2. *For each ε, u_ε is solution of the problem* (3.1) *if and only if u_ε is solution of the minimization problem* (3.5).

In the following we introduce some notions about Γ-convergence which shall be used in the sequel. For details and for the missing proofs we refer G. Dal Maso [8].

Let X be a reflexive, separable Banach space. Let $(F_\varepsilon)_\varepsilon$ be a sequence of functions $F_\varepsilon : X \longrightarrow \mathbb{R}$. A sequence of functions $F_\varepsilon : X \longrightarrow \mathbb{R}$ is *equi-coercive* if there exists a coercive function $\Psi : X \longrightarrow \mathbb{R}$, in the sense that $\Psi(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$, such that $F_\varepsilon \geq \Psi$ for every ε.

If the sequence $(F_\varepsilon)_\varepsilon$ is equi-coercive then the sequential Γ-limits defined below coincide with the Γ-limits defined in Chapter 4 of [8] (see Proposition 8.16 in [8]).

Definition 3.3. We say that the function $F : X \longrightarrow \mathbb{R}$ is the *sequential lower* Γ*-limit* of the sequence $(F_\varepsilon)_\varepsilon$ with respect to the weak topology of X, if it satisfies conditions (a) and (b) below.

We say that the function $F : X \longrightarrow \mathbb{R}$ is the *sequential* Γ*-limit* of the sequence $(F_\varepsilon)_\varepsilon$ with respect to the weak topology of X, if it satisfies conditions (a) and (c) below.

(a) for all $v \in X$ and for each sequence $(v_\varepsilon)_\varepsilon$ converging weakly to v, one has $\liminf_{\varepsilon \longrightarrow 0} F_\varepsilon(v_\varepsilon) \geq F(v)$;

(b) for all $v \in X$ there exists a sequence $(v_\varepsilon)_\varepsilon$ converging weakly to v such that $\liminf_{\varepsilon \longrightarrow 0} F_\varepsilon(v_\varepsilon) = F(v)$;

(c) for all $v \in X$ there exists a sequence $(v_\varepsilon)_\varepsilon$ converging weakly to v such that $\lim_{\varepsilon \longrightarrow 0} F_\varepsilon(v_\varepsilon) = F(v)$.

The sequence $(J_\varepsilon)_\varepsilon$ is equi-coercive. Within this context the lower Γ-limit of the energy functionals J_ε may be calculated with the formula below

$$J(v) = \inf\big\{\liminf_{\varepsilon \longrightarrow 0} J_\varepsilon(v_\varepsilon) \ \mid\ v_\varepsilon \rightharpoonup v \text{ in } H\big\}. \tag{3.6}$$

We shall make use of the space:

$$\mathcal{H} = \{w \in H^1(0, T; L^2_\mu(\Omega \times \Pi)) \mid w(0, x, \lambda) = 0 \quad \mu \text{ a. e. in } \Omega \times \Pi\}.$$

4. Main result in the linear case

As we shall investigate the asymptotic behaviour of problem (3.1) via Γ-convergence of the energy functionals, we assume for the sake of simplicity that $f \equiv 0$. The energy functional becomes:

$$J_\varepsilon(v) = \frac{1}{2}\int_0^T\!\!\int_\Omega |v' + a_\varepsilon v|^2 dx\, dt.$$

Theorem 4.1. *The sequence of functionals* $(J_\varepsilon)_\varepsilon$ Γ*-converges; its* Γ*-limit is given by:*

$$J(v) = \frac{1}{2}\inf\Big\{\int_0^T\!\!\int_\Omega\int_\Pi \big(w' + c(t, x, \lambda)\, w\big)^2 d\mu_x(\lambda) dx\, dt : w \in \mathcal{H}, \int_\Pi w d\mu_x = v\Big\}. \tag{4.1}$$

Here, $(\mu_x)_{x\in\Omega}$ is the disintegration of μ with respect to the Lebesgue measure m_N. Recall that the sequence of Young measures associated to α_ε is assumed to narrow converge.

The above result generalizes the results in [14].

In order to perform the proof, we need the following auxiliary results. Their proofs will be given after the proof of Theorem 4.1.

Lemma 4.2. *For each bounded sequence $(v_\varepsilon)_\varepsilon$ in H there exists $w \in \mathcal{H}$ and a subsequence $\bar\varepsilon$ of ε such that, for all $\Phi \in L^2((0,T)\times\Omega;\, C_b(\Pi))$,*

$$\int_0^T\!\!\int_\Omega v_{\bar\varepsilon}(t,x)\Phi(t,x,\alpha_{\bar\varepsilon}(x))dx\,dt \longrightarrow \int_0^T\!\!\int_{\Omega\times\Pi} w(t,x,\lambda)\,\Phi\,(t,x,\lambda)\,d\mu(x,\lambda)\,dt, \tag{4.2}$$

and

$$\int_0^T\!\!\int_\Omega v'_{\bar\varepsilon}(t,x)\Phi(t,x,\alpha_{\bar\varepsilon}(x))dx\,dt \longrightarrow \int_0^T\!\!\int_{\Omega\times\Pi} w'(t,x,\lambda)\,\Phi\,(t,x,\lambda)\,d\mu(x,\lambda)\,dt. \tag{4.3}$$

In particular

$$v_{\bar\varepsilon} \rightharpoonup \int_\Pi w(t,x,\lambda)\,d\mu_x(\lambda) \;\; \textit{weakly in } H^1(0,T;\, L^2(\Omega)). \tag{4.4}$$

Proposition 4.3. *The space $C^1([0,T];\, C(\overline{\Omega}\times\Pi))$ is dense in $H^1((0,T);\, L^2_\mu(\Omega\times\Pi))$.*

Proof of Theorem 4.1. We shall prove that J is the lower Γ-limit of $(J_\varepsilon)_\varepsilon$. As the same proof holds for any subsequence of $(J_\varepsilon)_\varepsilon$, we obtain that J is the lower Γ-limit of any subsequence of $(J_\varepsilon)_\varepsilon$. This implies that J is the Γ-limit of $(J_\varepsilon)_\varepsilon$, as follows: Since $(J_\varepsilon)_\varepsilon$ is equi-coercive, so is every subsequence of it. Let $(J_{\varepsilon'})_{\varepsilon'}$ be a subsequence of $(J_\varepsilon)_\varepsilon$. Then there exists a further subsequence $(J_{\varepsilon''})_{\varepsilon''}$ of $(J_{\varepsilon'})_{\varepsilon'}$ which Γ-converges (see Proposition 8.12 in [8]). So the subsequence $(J_{\varepsilon''})_{\varepsilon''}$ Γ-converges to J. Then every subsequence of $(J_\varepsilon)_\varepsilon$ contains a further subsequence which Γ-converges to J therefore (see Proposition 8.17 in [8]) it turns out that $(J_\varepsilon)_\varepsilon$ Γ-converges to J.

We prove that J is the lower Γ-limit of the sequence $(J_\varepsilon)_\varepsilon$ by verifying properties (a) and (b) of Definition 3.3.

In order to prove property (a), we begin by showing that for every $v \in H$ and for every sequence $v_\varepsilon \rightharpoonup v$ in H there exists a subsequence $\bar\varepsilon$ of ε (the one obtained in Lemma 4.2) such that $\liminf_{\bar\varepsilon'\to 0} J_{\bar\varepsilon}(v_{\bar\varepsilon}) \geq J(v)$. Indeed, consider $v \in H$ and a sequence $(v_\varepsilon)_\varepsilon \subset H$ such that $v_\varepsilon \rightharpoonup v$ weakly in $H^1(0,T;\, L^2(\Omega))$. Let $(v_{\bar\varepsilon})_{\bar\varepsilon}$ be the subsequence of $(v_\varepsilon)_\varepsilon$ extracted in Lemma 4.2 and let $w \in \mathcal{H}$ associated to it.

Since $H^1(0,T;\, L^2(\Omega;\, C(\Pi)))$ is dense in $H^1(0,T;\, L^2_\mu(\Omega\times\Pi))$ (from Proposition 4.3) it turns out that there exists a sequence $v_n \in H^1(0,T;\, L^2(\Omega;\, C(\Pi)))$ such that

$$\lim_{n\to\infty} \|w - v_n\|_{H^1(0,T;\,L^2_\mu)} = 0. \tag{4.5}$$

Define $v^n_{\bar\varepsilon}(t,x) := v_n(t,x,\alpha_{\bar\varepsilon}(x))$. One may write $v_{\bar\varepsilon}(t,x) = v^n_{\bar\varepsilon}(t,x) + (v_{\bar\varepsilon}(t,x) - v^n_{\bar\varepsilon}(t,x))$ and then by using $\|(v'_{\bar\varepsilon} - (v^n_{\bar\varepsilon})') + a_{\bar\varepsilon}(v_{\bar\varepsilon} - v^n_{\bar\varepsilon})\|^2 \geq 0$, one gets the following inequality for all $n \in \mathbb{N}$:

$$\begin{aligned} J_{\bar\varepsilon}(v_{\bar\varepsilon}) \geq & \frac{1}{2}\int_0^T\!\!\int_\Omega \mid (v^n_{\bar\varepsilon})' + a_{\bar\varepsilon} v^n_{\bar\varepsilon} \mid^2 dx\,dt + \\ & + \int_0^T\!\!\int_\Omega \big((v^n_{\bar\varepsilon})' + a_{\bar\varepsilon} v^n_{\bar\varepsilon}\big)\big((v'_{\bar\varepsilon} - (v^n_{\bar\varepsilon})') + a_{\bar\varepsilon}(v_{\bar\varepsilon} - v^n_{\bar\varepsilon})\big)\,dx\,dt. \end{aligned} \tag{4.6}$$

For arbitrarily fixed $n \in \mathbb{N}$ the term $\frac{1}{2}\int_0^T\int_\Omega \mid (v_{\bar\varepsilon}^n)' + a_{\bar\varepsilon} v_{\bar\varepsilon}^n \mid dx\,dt$ passes to the limit as ε tends to 0: the function $\mid v_n'(t,x,\lambda) + c(t,x,\lambda) v_n(t,x,\lambda) \mid^2$ belongs to $L^1((0,T)\times\Omega;\, C(\Pi))$ and hence statement (b) of the fundamental Theorem for Young measures may be employed. The functions $\big(v_n' + c(t,x,\lambda)v_n\big)v_n'$ and $\big(v_n' + c(t,x,\lambda)v_n\big)c(t,x,\lambda)v_n$ belong to $L^1((0,T)\times\Omega;\, C(\Pi))$. Therefore one can apply the statement (b) of the fundamental Theorem for Young measures with these functions in order to pass to the limit the terms $\int_0^T\int_\Omega((v_{\bar\varepsilon}^n)'(t,x) + a_{\bar\varepsilon}(t,x)v_{\bar\varepsilon}^n(t,x))(v_{\bar\varepsilon}^n)'(t,x)dx\,dt$ and respectively $\int_0^T\int_\Omega((v_{\bar\varepsilon}^n)'(t,x) + a_{\bar\varepsilon}(t,x)v_{\bar\varepsilon}^n(t,x))a_{\bar\varepsilon}(t,x)v_{\bar\varepsilon}^n(t,x)dx\,dt$. Applying Lemma 4.2 with the sequence $(v_{\bar\varepsilon}')_{\bar\varepsilon}$ and the function $v_n' + c(t,x,\lambda)v_n \in L^2((0,T)\times\Omega;\, C(\Pi))$ and respectively with the sequence $(v_{\bar\varepsilon})_{\bar\varepsilon}$ and the function $(v_n' + c(t,x,\lambda)v_n)c(t,x,\lambda) \in L^2((0,T)\times\Omega;\, C(\Pi))$ one can pass to the limit the terms $\int_0^T\int_\Omega((v_{\bar\varepsilon}^n)' + a_{\bar\varepsilon}v_{\bar\varepsilon}^n)v_{\bar\varepsilon}'dx\,dt$ and respectively $\int_0^T\int_\Omega((v_{\bar\varepsilon}^n)' + a_{\bar\varepsilon}v_{\bar\varepsilon}^n)a_{\bar\varepsilon}v_{\bar\varepsilon}dx\,dt$. Hence, the second integral in the right hand side of (4.6) passes to the limit as follows:

$$\int_0^T\int_\Omega ((v_{\bar\varepsilon}^n)' + a_{\bar\varepsilon}v_{\bar\varepsilon}^n)[(v_{\bar\varepsilon}' - (v_{\bar\varepsilon}^n)') + a_{\bar\varepsilon}(v_{\bar\varepsilon} - v_{\bar\varepsilon}^n)]\,dx\,dt \longrightarrow$$
$$\int_0^T\int_\Omega\int_\Pi (v_n' + c(t,x,\lambda)v_n)[(w' - v_n') + c(t,x,\lambda)\,(w - v_n)]\,d\mu\,dt.$$

For fixed n, passing to the limit as $\bar\varepsilon \longrightarrow 0$ in inequality (4.6) we obtain:

$$\liminf_{\bar\varepsilon\longrightarrow 0} J_{\bar\varepsilon}(v_{\bar\varepsilon}) \geq \frac{1}{2}\int_0^T\int_\Omega\int_\Pi \mid v_n' + c(t,x,\lambda)v_n \mid^2 d\mu\,dt +$$
$$\int_0^T\int_\Omega\int_\Pi (v_n' + c(t,x,\lambda)v_n)[(w' - v_n') + c(t,x,\lambda)\,(w - v_n)]d\mu\,dt.$$

On the other hand, since v_n is close to w in the norm of $H^1(0,T;\, L_\mu(\Omega\times\Pi))$ (by (4.5)), one obtains:

$$\liminf_{\bar\varepsilon\longrightarrow 0} J_{\bar\varepsilon}(v_{\bar\varepsilon}) \geq \frac{1}{2}\int_0^T\int_\Omega\int_\Pi \mid w' + c(t,x,\lambda)w \mid^2 d\mu\,dt - O\left(\|w - v_n\|_{H^1(0,T;\,L^2_\mu)}\right).$$

When $n \longrightarrow \infty$ we obtain that

$$\liminf_{\bar\varepsilon\longrightarrow 0} J_{\bar\varepsilon}(v_{\bar\varepsilon}) \geq \frac{1}{2}\int_0^T\int_\Omega\int_\Pi \mid w' + c(t,x,\lambda)w \mid^2 d\mu\,dt \geq J(v),$$

where the last inequality holds since $J(v)$ is defined by (4.1) and w satisfies $\int_\Pi w(t,x,\lambda)\,d\mu(\lambda) = v(t,x)$ m_{N+1} a.e. in $(0,T)\times\Omega$. We just showed that $(v_\varepsilon)_\varepsilon$ has a subsequence $(v_{\bar\varepsilon})_{\bar\varepsilon}$ such that $\liminf_{\bar\varepsilon\longrightarrow 0} J_{\bar\varepsilon}(v_{\bar\varepsilon}) \geq J(v)$. The same arguments hold for any subsequence of $(v_\varepsilon)_\varepsilon$. Now, we prove that this implies $\liminf_{\varepsilon\longrightarrow 0} J_\varepsilon(v_\varepsilon) \geq J(v)$. Suppose $\liminf_{\varepsilon\longrightarrow 0} J_\varepsilon(v_\varepsilon) < J(v)$; this means that there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'}$ of $(v_\varepsilon)_\varepsilon$ such that $\lim_{\varepsilon'\longrightarrow 0} J_{\varepsilon'}(v_{\varepsilon'}) < J(v)$. On the other hand, there exists a further subsequence of it $(v_{\bar\varepsilon'})_{\bar\varepsilon'}$ such that $\liminf_{\bar\varepsilon'\longrightarrow 0} J_{\bar\varepsilon'}(v_{\bar\varepsilon'}) \geq J(v)$. Hence $J(v) \leq \lim_{\bar\varepsilon'\longrightarrow 0} J_{\bar\varepsilon'}(v_{\bar\varepsilon'}) = \liminf_{\bar\varepsilon'\longrightarrow 0} J_{\bar\varepsilon'}(v_{\bar\varepsilon'}) < J(v)$ which is contradictory. Consequently, $\liminf_{\varepsilon\longrightarrow 0} J_\varepsilon(v_\varepsilon) \geq J(v)$. As the sequence $(v_\varepsilon)_\varepsilon$ was arbitrary, property (a) is proved.

In order to complete the proof one has to show that property (b) in the Definition 3.3 of the lower Γ-limit holds. Consider $v \in H$. We shall prove that there exists a subsequence ε' of ε and a sequence $(v_{\varepsilon'})_{\varepsilon'}$ converging weakly to v in H such that $J(v) = \lim_{\varepsilon' \longrightarrow 0} J_{\varepsilon'}(v_{\varepsilon'})$. Then one can construct a sequence $(v_\varepsilon)_\varepsilon$ by: $v_\varepsilon := v_{\varepsilon'}$ if $\varepsilon = \varepsilon'$ and $v_\varepsilon := v$ if $\varepsilon \neq \varepsilon'$. By this construction, $v_\varepsilon \longrightarrow v$ weakly in H; employing property (a) proved above, we get $\liminf_{\varepsilon \longrightarrow 0} J_\varepsilon(v_\varepsilon) \geq J(v)$. Hence $J(v) = \lim_{\varepsilon' \longrightarrow 0} J_{\varepsilon'}(v_{\varepsilon'}) \geq \liminf_{\varepsilon \longrightarrow 0} J_\varepsilon(v_\varepsilon) \geq J(v)$, therefore $\liminf_{\varepsilon \longrightarrow 0} J_\varepsilon(v_\varepsilon) = J(v)$.

It remains now to construct a subsequence ε' of ε and a sequence $(v_{\varepsilon'})_{\varepsilon'}$ converging weakly to v such that $J(v) = \lim_{\varepsilon' \longrightarrow 0} J_{\varepsilon'}(v_{\varepsilon'})$. Having this end in view, let $(w_n)_n \subset \mathcal{H}$ be a minimizing sequence with respect to the infimum in the right hand side of (4.1). One has $\int_\Pi w_n(t, x, \lambda)\, d\mu_x(\lambda) = v(t, x)$ almost everywhere in $(0, T) \times \Omega$. Since $H^1((0, T); L^2(\Omega; C(\Pi)))$ is dense in $H^1((0, T); L^2_\mu(\Omega \times \Pi))$ (by Proposition 4.3), for each element w_n of the minimizing sequence, it implies that, there exists a sequence $(v_n^k)_k \subset H^1(0, T; L^2(\Omega; C(\Pi)))$ such that

$$\|w_n - v_n^k\|_{H^1(0, T; L^2_\mu)} < \frac{1}{k}. \tag{4.7}$$

Define $v_\varepsilon^{n,k}(t, x) := v_n^k(t, x, \alpha_\varepsilon(x))$. For $n \in \mathbb{N}$ fixed, applying statement (b) of the fundamental Theorem for Young measures and property (4.7), one can substract a subsequence $\varepsilon(n, k)$ of ε such that

$$\begin{aligned} &\lim_{k \longrightarrow \infty} \int_0^T \int_\Omega |{v_{\varepsilon(n,k)}^{n,k}}' + a_{\varepsilon(n,k)} v_{\varepsilon(n,k)}^{n,k}|^2 dx\, dt = \\ &\int_0^T \int_\Omega \int_\Pi (w_n' + c(t, x, \lambda) w_n)^2 d\mu\, dt \end{aligned} \tag{4.8}$$

and

$$\lim_{k \longrightarrow \infty} v_{\varepsilon(n,k)}^{n,k} = \lim_{k \longrightarrow \infty} v_n^k(t, x, \alpha_{\varepsilon(n,k)}(x)) = \int_\Pi w_n d\mu_x = v, \text{ weakly in } H.$$

For $k = k(n)$, employing (4.8), one gets:

$$\Big|\int_0^T \int_\Omega (v_{\varepsilon(n)}' + a_{\varepsilon(n)} v_{\varepsilon(n)})^2 dx\, dt - \int_0^T \int_\Omega \int_\Pi (w_n' + c(t, x, \lambda) w_n)^2 d\mu\, dt\Big| < \frac{1}{n},$$

where $\varepsilon(n) := \varepsilon(n, k(n))$. Since $(w_n)_n$ is a minimizing sequence, passing to the limit in the above inequality as $n \longrightarrow \infty$, one obtains:

$$J(v) = \lim_{n \longrightarrow \infty} \int_0^T \int_\Omega (v_{\varepsilon(n)}' + a_{\varepsilon(n)} v_{\varepsilon(n)})^2 dx\, dt,$$

and

$$v_{\varepsilon(n)} \longrightarrow v \text{ in the weak topology of } H,$$

that is, $\varepsilon(n)$ is a subsequence of ε for which $\lim_{n \longrightarrow \infty} J_{\varepsilon(n)}(v_{\varepsilon(n)}) = J(v)$. □

Proof of Lemma 4.2. Let σ_ε be the Young measure on $(0,\, T) \times \Omega \times \mathbb{R}^2$ associated to $(v_\varepsilon, v'_\varepsilon)$. The sequence $(v_\varepsilon, v'_\varepsilon)$ is bounded in $L^2((0,\, T) \times \Omega;\, \mathbb{R}^2)$, hence it is tight. Then by applying the Theorem of compactness with $\mathcal{O} = (0,\, T) \times \Omega$ and $S = \mathbb{R}^2$, there exists a subsequence (still denoted by σ_ε) which narrow converges to some $\sigma \in \mathcal{Y}((0,\, T) \times \Omega;\, \mathbb{R}^2)$.

The sequence $(\alpha_\varepsilon)_\varepsilon$ is tight and so is $(v_\varepsilon, v'_\varepsilon)_\varepsilon$. Since any pair of tight sequences is still tight, the sequence $(v_\varepsilon, v'_\varepsilon, \alpha_\varepsilon)_\varepsilon$ is tight. Let $(\theta_\varepsilon)_\varepsilon$ be the Young measure on $(0,\, T) \times \Omega \times \mathbb{R}^2 \times \Pi$ associated to $(v_\varepsilon, v'_\varepsilon, \alpha_\varepsilon)$. By applying the Theorem of compactness with $\mathcal{O} = (0,\, T) \times \Omega$ and $S = \mathbb{R}^2 \times \Pi$ there exists a subsequence still denoted by θ_ε which narrow converges to some $\theta \in \mathcal{Y}\left((0,\, T) \times \Omega;\, \mathbb{R}^2 \times \Pi\right)$. Note that the projection of θ on $(0,\, T) \times \Omega \times \Pi$ is $dt \otimes \mu$.

Applying the fundamental Theorem for Young measures, statement (b), with $\mathcal{O} = (0,\, T) \times \Omega$, $S = \mathbb{R}^2 \times \Pi$, $\psi\,(t, x, \xi, \zeta, \lambda) = \xi\, \Phi\,(t, x, \lambda)$, one obtains

$$\int_0^T\!\!\int_\Omega v_\varepsilon(t,x)\Phi\,(t, x, \alpha_\varepsilon(x))\, dx\, dt \longrightarrow \int_{(0,\,T)\times\Omega\times\mathbb{R}^2\times\Pi} \xi\Phi\,(t,x,\lambda)\, d\theta\,(t,x,\xi,\zeta,\lambda).$$

In the following we denote by $(\theta_{(t,x,\lambda)})_{(t,x,\lambda)\in(0,\,T)\times\Omega\times\Pi}$ the disintegration of θ with respect to $dt \otimes \mu$. Then:

$$\int_{(0,\,T)\times\Omega\times\mathbb{R}^2\times\Pi} \xi\Phi\,(t,x,\lambda)\, d\theta = \int_0^T\!\!\int_\Omega \int_\Pi \Phi\,(t,x,\lambda) \int_{\mathbb{R}^2} \xi d\theta_{(t,x,\lambda)}(\xi,\zeta)\, d\mu(x,\lambda)\, dt$$

and denoting $w\,(t,x,\lambda) := \int_{\mathbb{R}^2} \xi\, d\theta_{(t,x,\lambda)}(\xi,\zeta)$,

$$\int_0^T\!\!\int_\Omega v_\varepsilon(t,x)\Phi(t,x,\alpha_\varepsilon(x))\, dx\, dt \longrightarrow \int_0^T\!\!\int_{\Omega\times\Pi} w\,(t,x,\lambda)\, \Phi(t,x,\lambda)\, d\mu(x,\lambda) dt. \tag{4.9}$$

Applying Jensen's inequality to the probability measures $\theta_{(t,x,\lambda)}$,

$$\begin{aligned}\|w\|_{L^2(0,\,T;L^2_\mu(\Omega\times\Pi))} &= \int_0^T\!\!\int_\Omega \int_\Pi \Big|\int_{\mathbb{R}^2} \xi\, d\theta_{(t,x,\lambda)}(\xi,\zeta)\Big|^2 d\mu(x,\lambda)\, dt \leq \\ &\leq \int_0^T\!\!\int_\Omega \int_\Pi \int_{\mathbb{R}^2} \xi^2\, d\theta_{(t,x,\lambda)}(\xi,\zeta) d\mu(x,\lambda)\, dt\end{aligned}$$

and applying the fundamental Theorem for Young measures, statement (a), with $\mathcal{O} = (0,\, T) \times \Omega$, $S = \mathbb{R}^2 \times \Pi$, $\psi\,(t,x,\xi,\zeta,\lambda) = |\xi|^2$,

$$\int_{(0,\,T)\times\Omega\times\mathbb{R}^2\times\Pi} \xi^2 d\theta \leq \liminf_{\varepsilon\longrightarrow 0} \int_0^T\!\!\int_\Omega |v_\varepsilon(t,x)|^2\, dx\, dt < \infty,$$

wherefrom $w \in L^2(0,\, T;\, L^2_\mu(\Omega \times \Pi))$.

By using the same argument as above for the sequence of derivatives $(v'_\varepsilon)_\varepsilon$ we obtain

$$\int_0^T\!\!\int_\Omega v'_\varepsilon(t,x)\Phi(t,x,\alpha_\varepsilon(x)) dx\, dt \longrightarrow \int_0^T\!\!\int_{\Omega\times\Pi} \bar{w}\,(t,x,\lambda)\, \Phi(t,x,\lambda) d\mu(x,\lambda)\, dt. \tag{4.10}$$

where $\bar{w}\,(t,x,\lambda) := \int_{\mathbb{R}^2} \zeta \, d\theta_{(t,x,\lambda)}(\xi,\zeta)$ and $\bar{w} \in L^2(0,T;\, L^2_\mu(\Omega\times\Pi))$.

In order to complete the proof we verify that $\bar{w}$ is the time derivative of w and that $w \in \mathcal{H}$. Let $\Phi \in \mathcal{D}\,((0,\,T);\, C(\overline{\Omega}\times\Pi))$. Integrating by parts,

$$\int_0^T\!\!\int_\Omega v_\varepsilon'(t,x)\,\Phi\,(t,x,\alpha_\varepsilon(x))\,dx\,dt = -\int_0^T\!\!\int_\Omega v_\varepsilon(t,x)\,\Phi'(t,x,\alpha_\varepsilon(x))\,dx\,dt.$$

Passing to the limit in the above equality as $\varepsilon \longrightarrow 0$, like in (4.9) and (4.10), one obtains

$$\int_0^T\!\!\int_{\Omega\times\Pi} \bar{w}\,(t,x,\lambda)\,\Phi(t,x,\lambda)\,d\mu\,(x,\lambda)\,dt = -\int_0^T\!\!\int_{\Omega\times\Pi} w\,(t,x,\lambda)\,\Phi'(t,x,\lambda)\,d\mu\,(x,\lambda)\,dt.$$

Consider $\Phi(t,x,\lambda) = \varphi_1(t)\,\varphi_2(x,\lambda)$, with $\varphi_1 \in \mathcal{D}(0,\,T)$ and $\varphi_2 \in C(\overline{\Omega}\times\Pi)$. Then

$$\int_{\Omega\times\Pi}\Big[\int_0^T \big(\bar{w}\,(t,x,\lambda)\varphi_1(t) + w\,(t,x,\lambda)\varphi_1'(t)\big)dt\Big]\varphi_2(x,\lambda)\,d\mu\,(x,\lambda) = 0,$$

for every $\varphi_2 \in C(\overline{\Omega}\times\Pi)$, wherefrom

$$\int_0^T \big(\bar{w}\,(t,x,\lambda)\varphi_1(t) + w\,(t,x,\lambda)\varphi_1'(t)\big)dt = 0$$

μ almost everywhere in $\Omega\times\Pi$. Hence $\bar{w} = w'$ and then $w \in H^1(0,\,T;\, L^2_\mu(\Omega\times\Pi))$. Consider now $\Phi \in C^\infty([0,\,T[;\, C(\overline{\Omega}\times\Pi))$ such that $\Phi(t,x,\lambda) = \varphi_1(t)\,\varphi_2(x,\lambda)$, with $\varphi_1 \in C^\infty([0,\,T[)$, $\varphi_1(t) = 0$ in a neighborhood of T, and $\varphi_2 \in C(\overline{\Omega}\times\Pi)$. By arguments similar to the above ones, one proves that $w\,(0,x,\lambda) = 0$ μ almost everywhere in $\Omega\times\Pi$. □

Proof of Proposition 4.3. Consider $v \in H^1((0,\,T);\, L^2_\mu(\Omega\times\Pi))$ arbitrarily fixed. The space $C([0,\,T]\times\overline{\Omega}\times\Pi)$ is dense in $L^2((0,\,T);\, L^2_\mu(\Omega\times\Pi))$ (see Theorem 3.14 in [17]).Then, there exists a sequence $(\varphi_n)_n \subset C([0,\,T]\times\overline{\Omega}\times\Pi)$ such that

$$\varphi_n \longrightarrow v' \text{ in } L^2((0,\,T);\, L^2_\mu(\Omega\times\Pi)).$$

Since $H^1((0,\,T);\, L^2_\mu(\Omega\times\Pi))$ is imbedded in $C\,([0,\,T];\, L^2_\mu(\Omega\times\Pi))$ (see Théorème 1 pg. 570 in [9]), $v\,(0) := v\,(0,\cdot,\cdot) \in L^2_\mu(\Omega\times\Pi)$. The space $C(\overline{\Omega}\times\Pi)$ is dense in $L^2_\mu(\Omega\times\Pi)$, so there exists a sequence $(v_n)_n \in C(\overline{\Omega}\times\Pi)$ such that

$$v_n \longrightarrow v(0) \text{ in } L^2_\mu(\Omega\times\Pi).$$

Define $\psi_n \in C^1([0,\,T];\, C(\overline{\Omega}\times\Pi))$ by

$$\psi_n(t,x,\lambda) := \int_0^t \varphi_n(s,x,\lambda)\,ds + v_n(x,\lambda).$$

Writing

$$v\,(t,x,\lambda) = \int_0^t v'(s,x,\lambda)ds + v\,(0,x,\lambda),$$

one obtains

$$\|v-\psi_n\|^2_{L^2((0,T);\,L^2_\mu(\Omega\times\Pi))} =$$
$$= \int_0^T\!\!\int_\Omega\int_\Pi \Big[\int_0^t (v'(s,x,\lambda)-\varphi_n(s,x,\lambda))ds + v\,(0,x,\lambda) - v_n(x,\lambda)\Big]^2 d\mu\,(x,\lambda)dt \leq$$
$$2\int_0^T\!\!\int_\Omega\int_\Pi \Big\{\Big[\int_0^t (v'(s,x,\lambda)-\varphi_n(s,x,\lambda))ds\Big]^2 + [v\,(0,x,\lambda)-v_n(x,\lambda)]^2\Big\} d\mu\,(x,\lambda)dt \leq$$
$$\leq 2\left(T^2\|v'-\varphi_n\|^2_{L^2((0,T);\,L^2_\mu(\Omega\times\Pi))} + T\|v\,(0)-v_n\|^2_{L^2_\mu(\Omega\times\Pi)}\right).$$

Consequently $\psi_n \longrightarrow v$ strongly in $H^1((0,\,T);\;L^2_\mu(\Omega\times\Pi))$ and the proof is concluded. □

5. Main result in a non-linear case

Consider the non-linear problem (1.1) where, for all $\varepsilon > 0$, the coefficients a_ε belong to $L^\infty((0,\,T)\times\Omega)$, $0 < \alpha \leq a_\varepsilon(t,x) \leq \beta$ for some $\alpha,\,\beta > 0$, the function f belongs to $L^2((0,\,T)\times\Omega)$, u_0 belongs to $L^2(\Omega)$ and $g\,(y) = \dfrac{\mathrm{d}\psi}{\mathrm{d}y}$, with ψ such that:

$$\begin{aligned} &\psi\,:\,\mathbb{R}\longrightarrow\mathbb{R} \text{ is strictly convex,}\\ &\psi\in C^1(\mathbb{R}),\\ &\theta|y|^2-\gamma\leq\psi\,(y)\leq\rho|y|^2+\delta,\ \text{ for some } \theta,\ \rho>0,\ \gamma,\ \delta\geq 0.\end{aligned} \tag{5.1}$$

Denote by ψ^* the polar function associated to ψ.

We assume with no loss of generality that $u_0 = 0$. Adapting the results in [4] and [5], one proves that problem (1.1), which can be written as follows

$$\left(f-u'_\varepsilon\right)\in\partial G^t_\varepsilon(u_\varepsilon)\ \text{a. e. in } [0,\,T],\, u_\varepsilon(0)=u_0=0,$$

(here $G^t_\varepsilon(v) = \int_\Omega a_\varepsilon(t,x)\psi(v)dx$ and ∂G^t_ε is its subdifferential) is equivalent to the minimization problem $\min\limits_{v\in H} J_\varepsilon(v)$, where the energy functionals have the form below

$$J_\varepsilon(v) = \int_0^T\!\!\int_\Omega\left[a_\varepsilon\psi\,(v) + a_\varepsilon\psi^*\left(\frac{f-v'}{a_\varepsilon}\right) - f\,v\right]dxdt + \frac{1}{2}\int_\Omega|v\,(T)|^2dx.$$

The lower Γ-limit can be calculated with formula (3.6), for reasons similar to those in the linear case.

The following result characterizes the Γ-limit when the time dependent coefficients $a_\varepsilon(t,x)$ have the special form postulated in (3.2) and the function c satisfies (3.3). We shall use the same notations as in the previous section.

Theorem 5.1. *The sequence of functionals $(J_\varepsilon)_\varepsilon$ Γ-converges; its Γ-limit is given by:*

$$J\,(v) = \inf\left\{\int_0^T\!\!\int_\Omega\int_\Pi\left[c\,\psi\,(w) + c\,\psi^*\left(\frac{f-w'}{c}\right) - f\,w\right]d\mu_x dx\,dt + \frac{1}{2}\|w\,(T)\|^2_{L^2_\mu} :\right.$$
$$\left. w\in\mathcal{H},\ \int_\Pi w\,d\mu_x = v\right\} \tag{5.2}$$

Remark 5.2. The space $L^2_\mu(\Omega \times \Pi)$ is a separable Hilbert space (since μ is Radon and $\Omega \times \Pi$ is separable). Then a result of variational methods (see Théorème 1 pg. 570 in [9]) guarantees that the imbedding below holds

$$H^1(0,\,T;\,L^2_\mu(\Omega \times \Pi)) \hookrightarrow C([0,\,T];\,L^2_\mu(\Omega \times \Pi)).$$

Thus, the last term in (5.2) makes sense.

The proof is similar to the one in the linear case which was presented in the previous section. Therefore we present only some steps of the proof. Nevertheless, in the case under consideration, we need Lemma 5.3 (which is a consequence of Lemma 4.2) in order to be able to pass to the limit the terms generated by $\|w\,(T)\|^2_{L^2_\mu}$ in the estimates of the energy functionals. In the linear case this kind of term does not appear explicitly (it can be prevented from entering the computations).

Lemma 5.3. *In the context of* Lemma 4.2*, for the same subsequence $\bar{\varepsilon}$ of ε, for any function $\Phi \in L^2(\Omega;\, C(\Pi))$, the following convergence holds*

$$\int_\Omega v_{\bar{\varepsilon}}\,(T,x)\Phi(x,\alpha_{\bar{\varepsilon}}(x))\,dx \longrightarrow \int_\Omega \int_\Pi w(T,x,\lambda)\,\Phi\,(x,\lambda)\,d\mu_x(\lambda)\,dx.$$

Proof. Denote $\gamma_{\bar{\varepsilon}}(t) := \int_\Omega v_{\bar{\varepsilon}}(t,x)\Phi\,(x,\alpha_{\bar{\varepsilon}}(x))\,dx$ and $\gamma(t) := \int_\Omega \int_\Pi v\,(t,x,\lambda)\;\Phi\,(x,\lambda)\;d\mu_x(\lambda)\,dx$. From Lemma 4.2 one obtains $\gamma'_{\bar{\varepsilon}}(t) \rightharpoonup \gamma'(t)$ and $\gamma_{\bar{\varepsilon}}(t) \rightharpoonup \gamma(t)$ in the weak topology of $L^2(0,\,T)$. Passing to the limit as $\bar{\varepsilon} \longrightarrow 0$ in the right hand member of the equality $\gamma_{\bar{\varepsilon}}(t) - \gamma_{\bar{\varepsilon}}(0) = \int_0^t \gamma'_{\bar{\varepsilon}}(s)ds$, one obtains the pointwise convergence $\gamma_{\bar{\varepsilon}}(t) \longrightarrow \gamma(t)$, $\forall t \in [0,\,T]$. Taking in particular $t = T$ the conclusion is proved. □

Sketched proof of Theorem 5.1. Consider $v \in H$ and a sequence $(v_\varepsilon)_\varepsilon \subset H$, $v_\varepsilon \rightharpoonup v$ weakly in $H^1(0,\,T;\,L^2(\Omega))$. Let $(v_{\bar{\varepsilon}})_{\bar{\varepsilon}}$ be the subsequence of $(v_\varepsilon)_\varepsilon$ extracted in Lemma 4.2 and let $w \in \mathcal{H}$ associated to it. Let $(v_n)_n$, $v_n \in H^1(0,\,T;\,L^2(\Omega;\,C(\Pi)))$ be a sequence such that

$$\lim_{n\longrightarrow\infty} \|w - v_n\|_{H^1(0,T;\,L^2_\mu)} = 0.$$

Define $v^n_{\bar{\varepsilon}}(t,x) := v_n(t,x,\alpha_{\bar{\varepsilon}}(x))$ and write $v_{\bar{\varepsilon}}(t,x) = v^n_{\bar{\varepsilon}}(t,x) + (v_{\bar{\varepsilon}}(t,x) - v^n_{\bar{\varepsilon}}(t,x))$. Then $\forall n \in \mathbb{N}$ the following holds:

$$\begin{aligned} J_{\bar{\varepsilon}}(v_{\bar{\varepsilon}}) \geq & J_{\bar{\varepsilon}}(v^n_{\bar{\varepsilon}}) + \int_0^T\!\!\int_\Omega \big[a_{\bar{\varepsilon}}\psi'(v^n_{\bar{\varepsilon}})\,(v_{\bar{\varepsilon}} - v^n_{\bar{\varepsilon}}) + (\psi^*)'\big(\frac{f - (v^n_{\bar{\varepsilon}})'}{a_{\bar{\varepsilon}}}\big)\,(v_{\bar{\varepsilon}} - v^n_{\bar{\varepsilon}}) \\ & - f\,(v_{\bar{\varepsilon}} - v^n_{\bar{\varepsilon}})\big]\,dx\,dt + \int_\Omega v^n_{\bar{\varepsilon}}(T)\,(v_{\bar{\varepsilon}}(T) - v^n_{\bar{\varepsilon}}(T))\,dx. \end{aligned}$$

Using hypothesis (5.1) and passing to the limit as $\bar{\varepsilon} \longrightarrow 0$, we obtain:

$$\begin{aligned} & \liminf_{\bar{\varepsilon}\longrightarrow 0} J_{\bar{\varepsilon}}(v_{\bar{\varepsilon}}) \geq \lim_{\bar{\varepsilon}\longrightarrow 0} J_{\bar{\varepsilon}}(v^n_{\bar{\varepsilon}}) + \\ & + \int_0^T\!\!\int_\Omega \int_\Pi \big[c\psi'(v_n)\,(w - v_n) + (\psi^*)'\big(\frac{f - v_n'}{c}\big) + f\,(w - v_n)\big]\,d\mu_x dx\,dt + \\ & \qquad + \int_\Omega \int_\Pi v_n(T)\,(w(T) - v_n(T))\,d\mu_x\,dx\,dt. \end{aligned} \tag{5.3}$$

All terms in the above inequality appear from straightforward applications of the fundamental Theorem for Young measures and Lemma 4.2, except the last one, which results from the above Lemma 5.3. From the hypothesis (5.1) it turns out that the second and the third term in the righthand side of (5.3) are of the order of $\|w - v_n\|_{H^1(0,T;L^2_\mu)}$. So, as $n \longrightarrow \infty$ it implies that $\liminf_{\bar{\varepsilon} \longrightarrow 0} J_{\bar{\varepsilon}}(v_{\bar{\varepsilon}}) \geq J(v)$. By using Γ-convergence results (Propositions 8.12 and 8.14 from [8]) and reasoning as in the linear case, we obtain that J satisfies property (a) in Definition 3.3. Property (b) in Definition 3.3, as well as the conclusion that the lower Γ-limit is in fact Γ-limit, are proved analogously to the linear case. □

6. Example

We show how the case when the coefficients a_ε are analytic in t can be treated within the context of Theorem 4.1, respectively Theorem 5.1 (for the non-linear problem).

Let $\Omega \subset \mathbb{R}$ be an open interval. Consider $\Pi := [\alpha + \delta,\ \beta - \delta] \times \prod_{n=1}^{\infty}[-M_n,\ M_n]$ endowed with the product topology, where $0 < \delta < \frac{\beta-\alpha}{2}$ and $M_n > 0$ are constants such that $\limsup_{n\longrightarrow\infty} \sqrt[n]{M_n} < 1/T$ and $\sum_{n=1}^{\infty} M_n T^n < \delta$. Define $c : [0,\ T] \times \Omega \times \Pi \longrightarrow \mathbb{R}$ by

$$c\,(t, x, \lambda) = \hat{c}\,(t, \lambda) = \sum_{n=0}^{\infty} \lambda_n t^n; \tag{6.1}$$

the above conditions on δ and M_n insure the bounds of $\hat{c}$: $0 < \alpha \leq \hat{c}\,(t, \lambda) \leq \beta$ for all $(t, \lambda) \in [0,\ T] \times \Pi$.

Then, by (3.2) consider the coefficients $a_\varepsilon(t, x)$, analytic in t, in the following sense:

$$a_\varepsilon(t, x) = \sum_{n=0}^{\infty} \alpha_{n\varepsilon}(x)\, t^n, \tag{6.2}$$

where $(\alpha_\varepsilon)_\varepsilon$ is any sequence of m_N-measurable functions, $\alpha_\varepsilon : \Omega \longrightarrow \Pi$ (here α_ε means $(\alpha_{n\varepsilon})_{n\in\mathbb{N}}$). As Π is compact (by Tychonov's Theorem), any such sequence $(\alpha_\varepsilon)_\varepsilon$ is tight.

Therefore, Theorems 4.1 and 5.1 give the Γ-limit of the respective sequence of energy functionals $(J_\varepsilon)_\varepsilon$ for analytic coefficients a_ε in the sense of (6.2).

We shall compute a simpler form of the Γ-limit for a periodic (in space) mixture of two materials, the coefficient of each material being analytic in t. This is a generalization of Example 3.5 in [14]. Consider the linear problem (3.1) with analytic coefficients a_ε in the sense of (6.2), with $\alpha_{n\varepsilon}(x) = \rho_n \chi\big(\frac{x}{\varepsilon}\big) + \eta_n\big(1 - \chi\big(\frac{x}{\varepsilon}\big)\big)$, where χ is the characteristic function of the interval $[0,\ \gamma] \subset [0,\ 1]$, extended by periodicity to the whole $\mathbb{R}$. Here ρ_n and η_n belong to $[-M_n,\ M_n]$ (consequently, $\rho = (\rho_n)_{n\in\mathbb{N}}$ and $\eta = (\eta_n)_{n\in\mathbb{N}}$ are elements of Π) and γ is some real constant in $]0,\ 1[$. Then the Young measure μ associated to the sequence $(\alpha_\varepsilon)_\varepsilon$ will be $\mu = m_1 \otimes (\gamma\delta_\rho + (1-\gamma)\delta_\eta)$, where δ_ρ and δ_η are the Dirac measures concentrated in ρ and respectively in η. The space $L^2_\mu(\Omega \times \Pi)$ can be identified with $(L^2(\Omega))^2$, in the sense that there exists a bijection which associates, to each element $h \in L^2_\mu(\Omega \times \Pi)$ the pair $(h\,(\cdot, \rho),\ h\,(\cdot, \eta)) \in (L^2(\Omega))^2$. Consequently, the space $H^1(0,\ T;\ L^2_\mu(\Omega \times \Pi))$ is isomorphic to $(H^1(0,\ T;\ L^2(\Omega)))^2$. So, $\mathcal{H}$ can be identified to $\{(w_1,\ w_2) \in (H^1(0,\ T;\ L^2(\Omega)))^2 \mid w_1(0, x) = 0,\ w_2(0, x) = 0 \quad m_1\text{-a. e. } x \in \Omega\}$, through the bijection $w \mapsto (w_1, w_2)$, where

$w_1(t,x) := w\,(t,x,\rho)$ and $w_2(t,x) := w\,(t,x,\eta)$. Then, in formula (4.1) the condition $\int_\Pi w\,(t,x,\lambda)\,d\,\mu_x(\lambda) = v(t,x)$ means $\gamma\, w_1(t,x) + (1-\gamma)w_2(t,x) = v(t,x)$; therefore the infimum is to be taken over all $w_1 \in H^1(0,\,T;\,L^2(\Omega))$ such that $w_1(0,x) = 0$, m_1-a. e. $x \in \Omega$, obtaining:

$$J\,(v) = \inf_{\substack{w_1 \in H^1(0,\,T;\,L^2(\Omega))\\ w_1(0,\cdot)=0}} \int_0^T\!\!\int_\Omega \Big\{\gamma\big[w_1'(t,x) + \hat{c}\,(t,\rho)\,w_1(t,x)\big]^2 +$$

$$+\frac{1}{1-\gamma}\big[v'(t,x) + \hat{c}\,(t,\eta)\,v(t,x) - \gamma\,(w_1'(t,x) + \hat{c}\,(t,\eta)\,w_1(t,x))\big]^2\Big\}dx\,dt.$$

The above integral is convex in w_1, hence its minimum points, over the set $\{w_1 \in H^1(0,\,T;\,L^2(\Omega)) \mid w_1(0,x) = 0 \quad m_1\text{-a. e. } x \in \Omega\}$, are characterized by the Euler equations, which write:

$$\begin{cases} w_1'(0) - v'(0) = 0 \text{ in } \Omega, \\ w_1'(T) + A\,(T)\,w_1(T) - D\,(T) = 0 \text{ in } \Omega, \\ -w_1''(t,x) + (B(t,x) - A'(t,x))\,w_1(t,x) + E(t,x) = 0 \text{ in } (0,\,T)\times\Omega. \end{cases} \tag{6.3}$$

Here, $A(t,x) := (1-\gamma)\,\hat{c}\,(t,\rho) + \gamma\,\hat{c}\,(t,\eta)$, $B(t,x) := (1-\gamma)\,\hat{c}^2(t,\rho) + \gamma\,\hat{c}^2(t,\eta)$, $D(t,x) := v'(t,x) + \hat{c}\,(t,\eta)\,v(t,x)$ and $E(t,x) := v''(t,x) + [\hat{c}'(t,\eta) - \hat{c}^2(t,\eta)]\,v(t,x)$.

Let w_1^0 be solution of problem (6.3); then, the Γ-limit writes

$$J\,(v) = \int_0^T\!\!\int_\Omega \Big\{\gamma\big[w_1^{0\prime}(t,x) + \hat{c}\,(t,\rho)\,w_1^0(t,x)\big]^2 +$$

$$+\frac{1}{1-\gamma}\big[v'(t,x) + \hat{c}\,(t,\eta)\,v(t,x) - \gamma\,(w_1^{0\prime}(t,x) + \hat{c}\,(t,\eta)\,w_1^0(t,x))\big]^2\Big\}dx\,dt.$$

The above formula is quite simple; the space Π has been eliminated and the infimum has been replaced by the problem (6.3).

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References

[1] G. Allaire: Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (1992) 1482–1518.

[2] L. Ambrosio, P. D'Ancona, S. Mortola: Γ-convergence and the least squares method, Annali di Matematica Pura ed Applicata CLXVI(IV) (1994) 101–127.

[3] Y. Amirat, K. Hamdache, A. Ziani: On homogenization of ordinary differential equations and linear transport equations, Calculus of Variations, Homogenization and Continuum Mechanics, ed. by Bouchitté, G. Buttazzo, P. Suquet, Series on Advances in Mathematics for Applied Sciences 18 (1994) 29–50.

[4] H. Brezis, I. Ekeland: Un principe variationnel associé à certaines équations paraboliques, Le cas indépendant du temps, C. R. Acad. Sc. Paris 282(A) (1976) 971–974.

[5] H. Brezis, I. Ekeland: Un principe variationnel associé à certaines équations paraboliques, Le cas dépendant du temps, C. R. Acad. Sc. Paris 282(A) (1976) 1197–1198.

[6] N. Bourbaki: Intégration, Chapitre 9: Integration sur les espaces topologique séparés, Hermann, Paris, 1969.

[7] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Lecture Notes 580, Springer Verlag, Berlin, 1977.

[8] G. Dal Maso: An Introduction to Γ-Convergence, Birkhäuser, Boston, 1993.

[9] R. Dautray, J. L. Lions: Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques, Tome 3, Masson, Paris, 1985.

[10] E. De Giorgi: Some remarks on Γ-convergence and least square methods, Composite Media and Homogenization Theory, ed. by G. Dal Maso, G. F. Dell'Antonio, Birkhäuser (1991) 135–142.

[11] C. Dellacherie, P. A. Meyer: Probabilities and Potential, North-Holland, Amsterdam, 1978.

[12] W. E: Homogenization of linear and nonlinear transport equations, Comm. Pure Appl. Math. 45 (1992) 301–326.

[13] M. L. Mascarenhas: A linear homogenization problem with time dependent coefficient, Trans. Amer. Math. Soc. 281(1) (1984) 179–195.

[14] M. L. Mascarenhas: Memory effect phenomena and Γ-convergence, Proc. of the Royal Soc. of Edinburgh 123A (1993) 311–322.

[15] M. L. Mascarenhas: Γ-convergence and non local effect: a non linear case, Homogenization and Applications to Material Sciences, GAKUTO International Series, Mathematical Sciences and Applications 9 (1997) 279–290.

[16] G. Nguetseng: A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. 20 (1989) 608–623.

[17] W. Rudin: Real and Complex Analysis, McGraw-Hill Book Company, New York, 1966.

[18] L. Tartar: Nonlocal effects induced by homogenization, Partial Differential Equations and the Calculus of Variations, Essay in Honor of Ennio De Giorgi II, Birkhäuser, Boston (1989) 925–938.

[19] L. Tartar: Memory effects and homogenization, Arch. Rat. Mech. Anal. 111(2) (1990) 121–133.

[20] M. Valadier: Young measures, Methods of Nonconvex Analysis, ed. by A. Cellina, Lecture Notes in Math. 1446, Springer Verlag, p. 152.

[21] M. Valadier: Two-scale convergence and Young measures, Prépublication 02, Départ. de Math., Université de Montpellier (1995).

[22] M. Valadier: Admissible functions in two-scale convergence, Portugaliae Mathematica 54(2) (1997) 147–164.

[23] J. Warga: Optimal Control of Differential and Functional Equations, Academic Press, New York, 1972.