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Lipschitz Continuity of the State Function in a Shape Optimization Problem

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This paper deals with the existence and the regularity of state function u in an N-dimensional shape optimization problem. We use a variational approach to get the existence of a solution u of a variational problem. Then we prove a Lipschitz continuity result of u by a penalization argument.

1. Introduction

The aim of this paper is to present an existence and a Lipschitz continuity results for the following variational problem: Let D be an open subset of $\mathbb{R}^N$ and consider the functional J defined on the Sobolev space $H_0^1(D)$ as:

$$J(v) := \frac{1}{2}\int_D \left((A\nabla v, \nabla v) + a_0 v^2\right) dx \ - \langle f, v\rangle, \tag{1.1}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(D)$ and its dual space $H^{-1}(D)$, $f \in H^{-1}(D)$, $a_0 \in L^\infty(D)$ such that $a_0 \geq 0$ and A is a symmetric matrix of functions $a_{ij} \in L^\infty(D)$, $i,j = 1,\ldots,N$, satisfying for a suitable constant $\alpha > 0$ the usual ellipticity condition:

$$\forall x \in \overline{D},\ \forall \xi \in \mathbb{R}^N, \quad \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2. \tag{1.2}$$

For any $v \in H_0^1(D)$, set $\Omega_v := \{x \in D/\, v(x) \neq 0\}$. Consider the class of admissible functions $\mathbb{V} := \{v \in H_0^1(D)/\,|\Omega_v| = m\}$, where $|\cdot|$ denotes the Lebesgue measure and m is given constant in $]0, |D|[$ $(|D| \leq +\infty)$.

The considered problem is:

$$\begin{cases} \text{find } u \in \mathbb{V} \ \text{ such that:} \\ \forall v \in \mathbb{V}, \quad J(u) \leq J(v). \end{cases} \tag{$\mathcal{P}$}$$

An interest of the study of the continuity of u is that one can deduce an existence result for the following shape optimization problem:

$$\text{Min}\left\{E(\Omega)\ /\ \Omega \in \mathcal{O}_m\right\}, \tag{1.3}$$

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where $\mathcal{O}_m = \{\Omega \textit{ open subset of } D \,/\, \|\Omega| = m\}$ and $E(\Omega) = J(u_\Omega)$; the state function u_Ω being the unique solution of the Dirichlet problem:

$$\begin{cases} -\mathcal{A}u &= f \ \ in \ \Omega, \\ \quad u &\in H_0^1(\Omega), \end{cases} \tag{1.4}$$

where $\mathcal{A} : H_0^1(D) \longrightarrow H^{-1}(D)$ is the elliptic operator defined as $\mathcal{A}v = \mathrm{div}(A\nabla v) - a_0 v$.

In the case where $D = \mathbb{R}^2$ and $\mathcal{A}$ is the Laplace operator Δ, M. Crouzeix [3] proved the Lipschitz regularity of any solution of $(\mathcal{P})$ for $f \in L^\infty(\mathbb{R}^2)$ with compact support K satisfying $|K| < m$. He proved also that if the boundary $\partial\Omega_u$ of Ω_u is sufficiently smooth then (u, Ω_u) solves the free boundary problem: *find* $\Omega \in \mathcal{O}_m$ *and* $u \in H^1(\mathbb{R}^2)$ *satisfying*

$$\begin{cases} -\Delta u &= f \ \ in \ \Omega, \\ \quad u &= 0 \ \ on \ \partial\Omega, \\ \frac{1}{2}|\nabla u|^2 &= \Lambda \ \ on \ \partial\Omega, \end{cases} \tag{1.5}$$

where the constant Λ is an unknown of the problem.

For the same problem, it is proven in [9] that Ω_u is bounded, and $\partial\Omega_u$ is analytic when m is large enough and $\int f\,dx \neq 0$.

The problem (1.5) arises in in electromagnetic shaping of molten metals without surface tension; see [8] for another approach.

Since the technics used in [3] and [9] are specific to the Laplace operator in $\mathbb{R}^2$, we shall proceed differently. The structure of the paper is as follows:
In Section 2, we introduce more notations and we give the statement of the main result. In Section 3, we deal with the existence question; namely we prove a non existence result for (1.3) for non smooth data f. Section 4 is devoted to the study of an approximated variational problem $(\mathcal{P}_\varepsilon)$ where the the constraint $|\Omega| = m$ is regularized: we establish the existence of a minimizer u_ε; then we get the regularity of u_ε as a consequence of the necessary condition of optimality which is a semi-linear partial differential equation. In Section 5, we prove that (u_ε) converges strongly in H^1 to a solution u of the initial variational problem.

To get a Lipschitz regularity of this solution, a uniform Lipschitz estimate for u_ε is crucial. Nevertheless, by exploiting an idea of H. Berestycki, L. A. Caffarelli and L. Nirenberg [1], we establish, in Section 6, the desired estimate when u_ε does not change its sign.

2. Notations and the main result

In the rest of the paper, $D \subseteq \mathbb{R}^N$ is a given open set. The Lebesgue measure of a measurable subset E of $\mathbb{R}^N$, which we denote $|E|$ is given by $|E| = \displaystyle\int_D \chi_E(x)\,\mathrm{d}x$. χ_E being the characteristic function of E defined as: $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in E^{\mathrm{C}} := \mathbb{R}^N \setminus E$.

We shall use the notion of Sobolev capacity of a subset E of $\mathbb{R}^N$ defined as follows (see for instance [7]):

$$C_{1,2}(E) = \inf\left\{\|\varphi\|_{H^1(\mathbb{R}^N)} \,/\, \varphi \in \mathcal{U}_E\right\}$$

where $\mathcal{U}_E = \{\varphi \in H^1(\mathbb{R}^N)/\varphi \geq 1$ a.e. in a neighbourhood of $E\}$. We say that a property $P(x)$ holds *quasi everywhere* (shortly *q.e.*) on E if P holds for all $x \in E$ except for the elements of a set $G \subset E$ with $C_{1,2}(G) = 0$.

A subset Ω of $\mathbb{R}^N$ is said to be *quasi open* if for $\varepsilon > 0$, there exists a set G_ε such that $\Omega \cup G_\varepsilon$ is open and $C_{1,2}(G_\varepsilon) < \varepsilon$. A function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be *quasi continuous* if for $\varepsilon > 0$, there exists a continuous function $f_\varepsilon : \mathbb{R}^N \longrightarrow \mathbb{R}$ such that $C_{1,2}\left(\{x \in \mathbb{R}^N / f(x) \neq f_\varepsilon(x)\}\right) < \varepsilon$.

It is well known that every Sobolev function $u \in H^1(\mathbb{R}^N)$ has a *quasi continuous* representative which we still denote u. Therfore, level sets of Sobolev functions are quasi open sets; in particular $\Omega_v = \{x \in D/\, |v(x)| > 0\}$ is quasi open subset of D.

For an open subset Ω of $\mathbb{R}^N$, the usual definition of the Sobolev space $H_0^1(\Omega)$ is equivalent to the following (see for instance [6]):

$$H_0^1(\Omega) = \{v \in H^1(\mathbb{R}^N)\,/\, v = 0 \text{ } q.e. \text{ } on \text{ } \Omega^{\mathrm{C}}\}.$$

When Ω is only a *quasi open subset* of $\mathbb{R}^N$, we define the Sobolev space $H_0^1(\Omega)$ in the same way. If $|\Omega| < +\infty$ we will denote u_Ω the unique solution of the Dirichlet problem (1.4), which is to be understood in the following sense: *find* $u \in H_0^1(\Omega)$ *such that*

$$\forall \varphi \in H_0^1(\Omega), \quad \int_\Omega (A\nabla u, \nabla\varphi) + a_0 u\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x. \tag{2.1}$$

Recall that the existence and the uniqueness of u_Ω follow from Lax Milgramm Lemma thanks to (1.2) and the inequality:

$$\forall u \in H_0^1(\Omega), \quad \|\nabla u\|_{L^2(\Omega)} \leq C_0 |\Omega|^{\frac{1}{N}} \|u\|_{L^2(\Omega)}, \tag{2.2}$$

where $C_0 = C_0(N)$. Note that (2.2) is a consequence of Schwarz symmetrization principle [10]. Note also that u_Ω satisfies:

$$\forall v \in H_0^1(\Omega), \quad J(u_\Omega) \leq J(v). \tag{2.3}$$

In the above notation one can consider the following shape optimization problem which is a weak version of (1.3):

$$\mathrm{Min}\left\{E(\Omega)\,/\,\Omega \in \tilde{\mathcal{O}}_m\right\} \tag{2.4}$$

where $\tilde{\mathcal{O}}_m = \{\Omega$ quasi open subset of D $/|\Omega| = m\}$ and $E(\Omega) = J(u_\Omega)$.

G. Buttazzo and G. Dal Maso [2] proved an existence result for a class of shape optimization problems including (2.4). But, as we will see in the sequel (Theorem 3.11), a solution $\tilde{\Omega}$ of (2.4) is not always open set. So an interesting question is: under which conditions $\tilde{\Omega}$ is an open set and therefore a solution of (1.3)?

Our main result is:

Theorem 2.1. *Assume that the coefficients of* A *are in* $C^{0,1}(\overline{D})$, $f \in L^2(D) \cap L^q(D)$ *with* $q > N$ *and, if* D *is not bounded,*

$$\left\{\begin{array}{l} |f\,x| \in L^2(D), \ |a_0\,x| \in L^\infty(D) \text{ and} \\ \forall i,j = 1, \dots N, \ \nabla a_{ij}\cdot x \in L^\infty(D). \end{array}\right. \tag{2.5}$$

Assume also that D is of class $C^{1,1}$ and

$$\exists x_0 \in D \text{ such that}: \ |\partial^- D| = \int_{\partial^- D} \mathrm{d}\sigma < +\infty \tag{2.6}$$

where $\partial^- D := \{x \in \partial D \,/\, \nu(x)\cdot(x - x_0) < 0\}$, ν is the unit outward normal to ∂D and σ is $(N-1)$-dimensional area element of ∂D. Then, if f does not change its sign (i.e $f \geq 0$ or $f \leq 0$), the problem $(\mathcal{P})$ admit at least a solution $u \in C^{0,1}(\overline{D})$ satisfying:

$$\forall x \in \overline{D}, \quad |\nabla u(x)| \leq C\left(\|f\|^2_{L^2(D)} + \|f\|_{L^q(D)} + 1\right). \tag{2.7}$$

This theorem is a consequence of Corollary 6.4 and remark 6.5 in Section 6.

Remark 2.2. If D is not bounded, the assumption (2.5) is satisfied for example when f has a compact support and $\mathcal{A} = \Delta$ (i.e. $a_0 \equiv 0$ and $A = Id$).

The assumption (2.6) is satisfied for example if $|\partial D| < +\infty$ because $\partial^- D \subset \partial D$, in particular if D is bounded or if $D = K^{\mathrm{C}}$ for some compact set K with smooth boundary. Note also that (2.6) holds if D is star shaped with respect to some point $x_0 \in D$ since in this case $\partial^- D = \emptyset$.

Corollary 2.3. *Assume that the hypothesis of* Theorem 2.1 *hold. Let u be a solution of $(\mathcal{P})$ satisfying* (2.7)*. Then the open set Ω_u is a solution of the shape optimization problems* (2.4) *and* (1.3).

This corollary is a consequence of Theorem 2.1 and Corollary 3.8 in the next section.

3. Existence and non existence results

The main difficulty to prove an existence result for problem $(\mathcal{P})$ is that the class $\mathbb{V}$ is not closed for the weak topology of $H_0^1(D)$. To overcome this difficulty we introduce as in [3] the problem:

$$\begin{cases} \text{find } u \in \mathbb{V}_0 \ \text{ such that:} \\ \forall v \in \mathbb{V}_0, \quad J(u) \leq J(v), \end{cases} \tag{$\mathcal{P}_0$}$$

where $\mathbb{V}_0 = \{v \in H_0^1(D)/\, |\Omega_v| \leq m\}$.

Remark that $\mathbb{V} \subset \mathbb{V}_0$ and as it is shown in [3] or [9]:

Lemma 3.1. *The class $\mathbb{V}_0$ is weakly closed in $H_0^1(D)$.*

That is if a sequence of functions $v_n \in \mathbb{V}_0$ converges in weak topology of $H_0^1(D)$ to $v \in H_0^1(D)$, then $v \in \mathbb{V}_0$. Moreover we have:

Lemma 3.2. *The set $\mathbb{V}$ is dense in $\mathbb{V}_0$; that is:*

$$\forall v \in \mathbb{V}_0,\ \forall n \in \mathbb{N}^*,\ \exists\, v_n \in \mathbb{V} \text{ such that}: \ v_n \xrightarrow[n\to+\infty]{} v \text{ in } H_0^1(D).$$

Proceeding as in [9] where this lemma is proved for $D = \mathbb{R}^2$ one can prove it for an unbounded subset D of $\mathbb{R}^N$. Here is a proof in general case.

Proof. Let $v \in \mathbb{V}_0 \setminus \mathbb{V}$, i.e. $|\Omega_v| < m$. By Lebesgue measure theory, there exists an open set $\tilde{\omega}$ such that: $\Omega_v \subset \tilde{\omega} \subset D$ and $|\tilde{\omega}| \leq m < |D|$. Let $x_0 \in D$; since $r \longmapsto |B(x_0, r) \cup \tilde{\omega}|$ is an increasing continuous function from $[0, +\infty[$ to $[|\tilde{\omega}|, |D|]$, there exists some r_0 such that the open set $\omega := B(x_0, r_0) \cup \tilde{\omega}$ satisfies: $\Omega_v \subset \tilde{\omega} \subset D$ and $|\omega| = m$.

Now for $\varepsilon > 0$, consider the solution v_ε of the Dirichlet problem:

$$\begin{cases} v_\varepsilon - \varepsilon \Delta v_\varepsilon &= v + \varepsilon(1 - \chi_{\Omega_v}) \ in \ \omega, \\ v_\varepsilon &\in H_0^1(\omega). \end{cases}$$

Since $v + \varepsilon(1 - \chi_{\Omega_v}) \in L^2(\omega)$ we have $v_\varepsilon \in H^2_{loc}(\omega)$ and $\Delta v_\varepsilon \in L^2(\omega)$; moreover the equation is satisfied *a.e.* in ω. But $\Delta v_\varepsilon = 0$ *a.e.* on $\{x \in \omega / \, v_\varepsilon(x) = 0\}$ and $v + \varepsilon(1 - \chi_{\Omega_v}) \neq 0$ *a.e.* on ω. Thus $v_\varepsilon \neq 0$ *a.e.* on ω; extending v_ε by 0 outside ω we get $v_\varepsilon \in \mathbb{V}$.

Let Ψ be the solution of: $-\Delta \Psi = (1 - \chi_{\Omega_v})$ in ω and $\Psi \in H_0^1(\omega)$. The function $w_\varepsilon = (v_\varepsilon - v)$ satisfies:

$$\begin{cases} w_\varepsilon - \varepsilon \Delta w_\varepsilon &= \varepsilon \Delta(v + \Psi) \ in \ \omega, \\ w_\varepsilon &\in H_0^1(\omega). \end{cases}$$

Taking w_ε as test function, we get:

$$\int_\omega w_\varepsilon^2 \, \mathrm{d}x + \varepsilon \int_\omega |\nabla w_\varepsilon|^2 \, \mathrm{d}x = -\varepsilon \int_\omega \nabla w_\varepsilon \nabla (v + \Psi) \, \mathrm{d}x.$$

Since $|\omega| = m$ and thanks to (2.2), it follows that (w_ε) is bounded in $H_0^1(\omega)$ and therefore in $H_0^1(D)$. Thus, up to a subsequence, w_ε converges to 0 weakly in $H_0^1(D)$. Using the above equality once more, we get strong convergence in $H_0^1(D)$. □

Another interesting remark is given by the following lemma.

Lemma 3.3. *If u is a solution of $(\mathcal{P}_0)$ then u satisfies:*

$$\int_D (A \nabla u \nabla \varphi) + a_0 u \varphi \, \mathrm{d}x = \int_D f \varphi \, \mathrm{d}x,$$

for every function $\varphi \in H_0^1(D)$ such that $|\Omega_u \cup \Omega_\varphi| \leq m$.

Proof. Let u and φ as in Lemma 3.3. Then for every $t \in \mathbb{R}$, $u + t\varphi \in \mathbb{V}_0$ and therfore $J(u) \leq J(u + t\varphi)$. Thus the lemma follows from: $\left.\dfrac{\mathrm{d}J(u + t\varphi)}{\mathrm{d}t}\right|_{t=0} = 0.$ □

Remark 3.4. The condition $|\Omega_u \cup \Omega_v| \leq m$ is satisfied for all $\varphi \in H_0^1(\Omega_u)$. So if u is a solution of $(\mathcal{P}_0)$ then $u = u_{\Omega_u}$, i.e. u is the solution of the Dirichlet problem (2.1) with $\Omega = \Omega_u$.

From Lemma 3.3 one can prove that (see [5]):
$(f \not\equiv 0) \Longrightarrow (u \not\equiv 0)$ and $(f \geq 0 (\text{resp. } f \leq 0)) \Longrightarrow (u \geq 0 \, (\text{resp. } u \leq 0))$.

An immediate consequence of Lemmas 3.2 and 3.3 is the following lemma.

Lemma 3.5. *Assume that f satisfies the following condition:*

$$\text{there is no } u \in H_0^1(D) \text{ such that } \begin{cases} -\mathcal{A}u = f \ \text{in } D, \\ |\Omega_u| < m. \end{cases} \tag{3.1}$$

Then the problems $(\mathcal{P}_0)$ and $(\mathcal{P})$ are equivalent.

Proof. The proof is the same as for Lemma 1, in [3]. □

Remark 3.6. If (3.1) does not holds, i.e. $\exists\, u \in H_0^1(D)$ such that $-\mathcal{A}u = f$ *in* D and $|\Omega_u| < m$. Thus from (2.3) we have, $\forall v \in \mathbb{V}_0 \subset H_0^1(D)$, $J(u) \le J(v)$. In this case, $u = u_D$ is the unique solution of $(\mathcal{P}_0)$ (because of the uniqueness of the solution of Dirichlet problem).

Let us now give the existence result for problems $(\mathcal{P}_0)$ and $(\mathcal{P})$.

Theorem 3.7. *The problem $(\mathcal{P}_0)$ admit at least one solution. Moreover, if f satisfies* (3.1) *then any solution of $(\mathcal{P}_0)$ is also a solution of $(\mathcal{P})$.*

Proof. According to Lemma 3.5, we have only to prove the existence for $(\mathcal{P}_0)$. The proof is the same as for Theorem 1 in [3]. □

Corollary 3.8. *Let u be a solution of $(\mathcal{P}_0)$. Two cases could happen:*

(i) $u \in \mathbb{V}_0 \setminus \mathbb{V}$. *Then there exists at least an open set $\Omega^* \in \mathcal{O}_m$, satisfying $\Omega_u \subset \Omega^*$, which is a solution of shape optimization problems* (2.4) *and* (1.3).

(ii) $u \in \mathbb{V}$. *Then u solves $(\mathcal{P})$ and Ω_u is a solution of* (2.4). *If moreover u is continuous then Ω_u is a solution of* (1.3).

Remark 3.9. In the case (ii), a sufficient condition for u to be continuous is $f \in L^q(D)$ with $q > \frac{N}{2}$. Indeed, in this case u is Hölder continuous (see for instance [4]).

Proof. (i) If $u \in \mathbb{V}_0 \setminus \mathbb{V}$, i.e $|\Omega_u| < m$ then, as in the proof of Lemma 3.2, there exists $\Omega^* \in \mathcal{O}_m$ such that $\Omega_u \subset \Omega^*$ so that $u \in H_0^1(\Omega^*)$. Thus from (2.3) we have $E(\Omega^*) = J(u_{\Omega^*}) \le J(u)$ and, since $u_{\Omega^*} \in \mathbb{V}_0$, we have $J(u) = J(u_{\Omega^*})$. Now for all $\Omega \in \mathcal{O}_m$, $u_\Omega \in \mathbb{V}_0$; so $E(\Omega^*) = J(u) \le J(u_\Omega) = E(\Omega)$. That is the open set Ω^* is a solution of (1.3) and (2.4).

(ii) Proceeding as in (i) we get that the quasi open set Ω_u solves (2.4). It solves also (1.3) if Ω_u is an open set; this is in particular the case if u is continuous. □

Remark 3.10. A particular case of situation (i) is described in Remark 3.6. In the same way, the situation (ii) occur for example if f does not satisfy:

$$\text{there is no } u \in H_0^1(D) \text{ such that } \begin{cases} -\mathcal{A}u = f \text{ in } D, \\ |\Omega_u| = m. \end{cases} \tag{3.2}$$

By Corollary 3.8, the shape optimization problem (2.4) has always a solution. But in general this solution is not an open set. Indeed, when (3.2) does not hold we have the following non existence result for (1.3).

Theorem 3.11. *Let $D = B(0,1)$ be the unite ball of $\mathbb{R}^3$ and $\mathcal{A} = \Delta$. There exists m, with $0 < m < |D|$, and $f \in H^{-1}(D)$ such that the shape optimization problem* (1.3) *has no solution.*

To prove this theorem we need the next lemma.

Lemma 3.12. *Assume that Ω^* is a solution of the shape optimization problem* (1.3) *and that f does not satisfy* (3.2), *i.e. there exists $u \in H_0^1(D)$ such that $u = u_D$ and $|\Omega_u| = m$. Then*

$$u = u_{\Omega^*} \quad \textit{and} \quad \chi_{\Omega_u} = \chi_{\Omega^*} \quad a.e.$$

Proof. As in remark 3.6, $u = u_{\Omega_u}$ and $E(\Omega_u) = J(u) \le J(u_{\Omega^*}) = E(\Omega^*)$. On the other hand, Ω^* solves (1.3); so $E(\Omega^*) = J(u_{\Omega^*}) \le J(u_\Omega)$, $\forall \Omega \in \mathcal{O}_m$. For $\varepsilon > 0$, set $\Omega_\varepsilon := \{x \in D/\, |u(x)| > \varepsilon\}$. When $\varepsilon \searrow 0$, we have $|\Omega_\varepsilon| \nearrow m$. Moreover by the same argument as in the proof of Lemma 3.2, there exists an open set $\omega_\varepsilon \in \mathcal{O}_m$ with $\Omega_\varepsilon \subset \omega_\varepsilon \subset D$. So $E(\Omega^*) = J(u_{\Omega^*}) \le J(w_\varepsilon)$, where $w_\varepsilon = (u-\varepsilon)^+ - (u+\varepsilon)^- \in H_0^1(\omega_\varepsilon)$. Thus when $\varepsilon \searrow 0$, we get $J(u_{\Omega^*}) \le J(u)$ and therefore $J(u_{\Omega^*}) = J(u) \le J(v)$, $\forall v \in H_0^1(D)$. Then by strict convexity of J we obtain $u = u_{\Omega^*}$ and $\Omega_u \subset \Omega^*$. This finishes the proof because $|\Omega^*| = |\Omega_u| = m$. □

Proof of Theorem 3.11. According to Lemma 3.12 it is enough to find $u \in H_0^1(D)$ such that $0 < |\Omega_u| < |D|$ and that Ω_u (which is quasi open set) does not satisfy $\chi_{\Omega_u} = \chi_\Omega$ *a.e.* for any open set Ω with $|\Omega| = |\Omega_u|$. Indeed, Theorem 3.11 follows with $m = |\Omega_u|$ and $f = -\Delta u \in H^{-1}(D)$.

Let us first find the function u. Consider the function v defined on $\mathbb{R}^3$ as:

$$v(x) := \min(1, F(x)) \quad \text{with} \quad F(x) = \sum_{n \ge 0} \frac{\alpha_n}{|x - x_n|},$$

where (x_n) is a sequence of points of D which is dense on $\overline{D}$ and (α_n) is a sequence of sufficiently small non negative numbers such that:

$$\forall n \in \mathbb{N},\ B(x_n, \alpha_n) \subset D \quad \text{and} \quad \sum_{n \ge 0} \alpha_n < \frac{1}{16\pi}. \tag{3.3}$$

Observe that for all $n \in \mathbb{N}$, $v(x) = 1$ on $B(x_n, \alpha_n)$. From (3.3) it comes that $\|F\|_{L^1} < 1$ and therefore $|[v < 1]| \ne 0$ (here $[v < 1] = \{x \in D/\, v(x) < 1\}$). Moreover $v \in L^\infty(D)$ and it is superharmonic (as a minimum of two superharmonic functions); then $v \in H_0^1(D)$, see for example [7]. Note that v is l.s.c. but not continuous. Fix now a function $\Psi \in H_0^1(D)$ such that $\Psi > 0$ on D and $\Psi = 1$ on $B(0, \frac{1}{2})$. Set $u(x) := (1 - \eta - v(x))^+ \Psi$ where $\eta > 0$ is a fixed sufficiently small number so that $|[v < 1 - \eta]| \ne 0$ and therefore $|\Omega_u| \ne 0$. Moreover it is easily seen that $|\Omega_u| < |D|$ because $v \equiv 1$ on $\cup B(x_n, \alpha_n)$ and $|\cup B(x_n, \alpha_n)| \ne 0$. Note also that Ω_u is quasi open set.

It is remains to prove that there is no open set Ω such that $|\Omega| = |\Omega_u|$ and $\chi_{\Omega_u} = \chi_\Omega$ *a.e.* If this were the case, we will have $\Omega \ne \emptyset$ (because $|\Omega| \ne 0$). Then, by the density of (x_n), there exists $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $\varepsilon \le \alpha_{n_0}$ and $B(x_{n_0}, \varepsilon) \subset \Omega$. This implies that $v(x) < 1 - \eta$ in $B(x_{n_0}, \varepsilon)$ which is in contradiction with $v \equiv 1$ in $B(x_{n_0}, \alpha_{n_0})$. □

In conclusion, according to Theorem 3.7, Corollary 3.8 and remarks 3.6 and 3.10, it remains to study the case where f satisfies:

$$\text{there is no } u \in H_0^1(D) \text{ such that } \begin{cases} -\mathcal{A}u = f \text{ in } D, \\ |\Omega_u| \leq m. \end{cases} \tag{3.4}$$

In the rest of the paper we shall assume (3.4) as well as $f \in L^2(D)$ so that:

$$J(v) = J(v) := \frac{1}{2}\int_D \left((A\nabla v, \nabla v) + a_0 v^2\right) dx \; - \int_D f v \, \mathrm{d}x.$$

Remark 3.13. Thanks to maximum principle, (3.4) holds if $f \geq 0$ or $f \leq 0$.

4. An approximated variational problem

In this section we study a variational problem analog to $(\mathcal{P}_0)$ where the constraint $|\Omega_v| \leq m$ is regularized as follows:

Let $p : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a regular **even** function satisfying:

- $(p(r) = 1, \; \forall r \geq 1)$ and $(p'(r) \geq 0, \; \forall r \in \mathbb{R}^+)$,
- $p(0) = p'(0) = 0$ and $p''(0) > 0$.

Note that, for such a function, there exists a number $a > 0$, such that:

$$\forall r \in [-1, 1], \;\; p(r) \geq ar^2. \tag{4.1}$$

For $\varepsilon > 0$, let p_ε be the function defined on $\mathbb{R}$ as: $p_\varepsilon(r) := p(\frac{r}{\varepsilon})$ and consider the approximated variational problem:

$$\begin{cases} \text{find } u_\varepsilon \in \mathbb{V}_\varepsilon \text{ such that:} \\ J(u_\varepsilon) \leq J(v), \quad \forall v \in \mathbb{V}_\varepsilon, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $\mathbb{V}_\varepsilon := \{v \in H_0^1(D); \; \|p_\varepsilon(v)\|_{L^1(D)} \leq m\}$.

Remark 4.1. Note that, $\forall v \in H_0^1(D)$ with $\chi_{\Omega_v} \in L^1(D)$, $p_\varepsilon(v) \underset{\varepsilon \to 0}{\longrightarrow} \chi_{\Omega_v}$ in L^1-norm. Moreover, $\mathbb{V}_0 \subset \mathbb{V}_\varepsilon$; because $\forall v \in H_0^1(D)$, $p_\varepsilon(v) \leq \chi_{\Omega_v}$. Note also that by Fatou's Lemma, $\mathbb{V}_\varepsilon$ is weakly closed in $H_0^1(D)$.

Lemma 4.2. *Let $(v_\varepsilon)_\varepsilon$ be a sequence of functions such that: $\forall \varepsilon > 0$, $v_\varepsilon \in \mathbb{V}_\varepsilon$. If $(v_\varepsilon)_\varepsilon$ converges to v weakly in $H_0^1(D)$, then $v \in \mathbb{V}_0$.*

Proof. Let $(w_\varepsilon)_\varepsilon$ be the sequence of the functions $w_\varepsilon := (v_\varepsilon - \varepsilon)^+ - (v_\varepsilon + \varepsilon)^-$. It is obvious that $w_\varepsilon \in \mathbb{V}_0$. Since $v_\varepsilon \longrightarrow v$ weakly in $H_0^1(D)$, we get that $w_\varepsilon \longrightarrow v$ weakly in $H_0^1(D)$. Then Lemma 3.1 finishes the proof. □

Theorem 4.3. *There exists at least one solution u_ε of $(\mathcal{P}_\varepsilon)$.*

To prove this result, we need the following lemma:

Lemma 4.4. *For every $v \in H_0^1(D)$, we have:*

$$\|v\|_{L^2(D)} \leq C_0 \|p(v)\|_{L^1(D)}^{\frac{1}{N}} \|\nabla v\|_{L^2(D)} + 2 \left(\frac{\|p(v)\|_{L^1(D)}}{a} \right)^{\frac{1}{2}}, \tag{4.2}$$

where a is as in (4.1) *and C_0 is the constant in* (2.2).

Proof of Theorem 4.3. Remark first that from (1.2) and Hölder inequality, we have:

$$\forall v \in H_0^1(D), \quad J(v) \geq \frac{\alpha}{2} \|\nabla v\|_{L^2(D)}^2 - \|f\|_{L^2(D)} \|v\|_{L^2(D)}.$$

For every $v \in \mathbb{V}_\varepsilon$, the Lemma 4.4 applied to $\left(\frac{v}{\varepsilon}\right)$ gives:

$$\|v\|_{L^2(D)} \leq C_0 m^{\frac{1}{N}} \|\nabla v\|_{L^2(D)} + 2\varepsilon \left(\frac{m}{a}\right)^{\frac{1}{2}}. \tag{4.3}$$

Hence, for every $v \in \mathbb{V}_\varepsilon$,

$$J(v) \geq \frac{\alpha}{2} \left(\|\nabla v\|_{L^2(D)} - M_0 \right)^2 - \frac{\alpha}{2} M_0^2 - 2\varepsilon \left(\frac{m}{a}\right)^{\frac{1}{2}} \|f\|_{L^2(D)}, \tag{4.4}$$

where $M_0 = \frac{1}{\alpha} C_0 m^{\frac{1}{N}} \|f\|_{L^2(D)}$. Thus $J(v) > -\infty$, $\forall v \in \mathbb{V}_\varepsilon$.

Now, consider a minimizing sequence $(u_n)_n$. Since $0 \in \mathbb{V}_\varepsilon$ and $J(0) = 0$, we can assume that $J(u_n) \leq 0$, $\forall n \in \mathbb{N}$, so that (4.3) and (4.4) implies that:

$$\|u_n\|_{H_0^1(D)} \leq C_1 + c_\varepsilon,$$

where $C_1 = 2M_0(C_0 m^{\frac{1}{N}} + 1)$ and c_ε is a positive constant converging to 0 when $\varepsilon \to 0$. Then there exists $u_\varepsilon \in H_0^1(D)$ such that, using eventually a subsequence, we can assume that $(u_n)_n$ converges to u_ε weakly in $H_0^1(D)$. Moreover, u_ε satisfies:

$$\|u_\varepsilon\|_{H_0^1(D)} \leq C_1 + c_\varepsilon, \tag{4.5}$$

and, by the Remark 4.1, $u_\varepsilon \in \mathbb{V}_\varepsilon$. Hence the theorem follows from the lower semicontinuity of J. □

Now we have to prove the Lemma 4.4.

Proof of Lemma 4.4. Assume that $\|p(v)\|_{L^1(D)} < \infty$ (since (4.2) is obvious otherway). Set $\Omega := \{x \in D; |v(x)| > 1\}$ and $F := \{x \in D; |v(x)| \leq 1\}$. Thanks to (4.1), we have:

$$\|v\|_{L^2(F)}^2 \leq \frac{\|p(v)\|_{L^1(F)}^2}{a}. \tag{4.6}$$

On the other hand, inequality (2.2) applied to $w := (|v| - 1)^+$ gives:

$$\int_\Omega (v^2 - 2|v| + 1) dx \leq C_0^2 \|p(v)\|_{L^1(\Omega)}^{\frac{2}{N}} \|\nabla v\|_{L^2(\Omega)}^2.$$

Using Hölder inequality, we get:

$$\left(\|v\|_{L^2(\Omega)} - \|p(v)\|^{\frac{1}{2}}_{L^1(\Omega)}\right)^2 \le C_0^2 \|p(v)\|^{\frac{2}{N}}_{L^1(\Omega)} \|\nabla v\|^2_{L^2(\Omega)},$$

and therefore

$$\|v\|^2_{L^2(\Omega)} \le \left(C_0 \|p(v)\|^{\frac{1}{N}}_{L^1(\Omega)} \|\nabla v\|_{L^2(D)} + \|p(v)\|^{\frac{1}{2}}_{L^1(\Omega)}\right)^2. \tag{4.7}$$

Here we used $|\Omega| = \|p(v)\|_{L^1(\Omega)}$ and $\|\nabla v\|_{L^2(\Omega)} \le \|\nabla v\|_{L^2(D)}$.

Now to get (4.2) we put (4.6), (4.7) in

$$\|v\|^2_{L^2(D)} = \|v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(F)},$$

and we take into account that $\|p(v)\|_{L^1(D)} = \|p(v)\|_{L^1(\Omega)} + \|p(v)\|_{L^1(F)}$ and $0 < a \le 1$. □

Lemma 4.5. *Assume that (3.4) holds. Then there exists $\varepsilon_0 > 0$ such that*

$$\|p_\varepsilon(u_\varepsilon)\|_{L^1(D)} = m, \ \forall \varepsilon \in]0, \varepsilon_0]. \tag{4.8}$$

Proof. Assume that this is not the case, i.e. $\|p_{\varepsilon_i}(u_{\varepsilon_i})\|_{L^1(D)} < m$ for some subsequence $(\varepsilon_i)_i$. Then for ε_i fixed and for every $\varphi \in C_0^\infty(D)$, we have $u_{\varepsilon_i} + t\varphi \in \mathbb{V}_{\varepsilon_i}$, for $|t|$ small enough; so that $J(u_{\varepsilon_i}) \le J(u_{\varepsilon_i} + t\varphi)$ implies: $-\mathcal{A}u_{\varepsilon_i} = f$ in D. Thus, for all ε_i, $u_{\varepsilon_i} = u$ where u is the unique solution of the Dirichlet problem $-\mathcal{A}u = f$ in D, $u \in H_0^1(D)$. Hence

$$\|p_{\varepsilon_i}(u_{\varepsilon_i})\|_{L^1(D)} = \|p_{\varepsilon_i}(u)\|_{L^1(D)} \underset{\varepsilon_i \to 0}{\longrightarrow} \|\chi_u\|_{L^1(D)} \le m,$$

which is inconsistent with (3.4). □

Theorem 4.6. *Let u_ε be a solution of $(\mathcal{P}_\varepsilon)$. Then there exists a positive number λ_ε such that:*

$$-\mathcal{A}u_\varepsilon = f - \lambda_\varepsilon p'_\varepsilon(u_\varepsilon) \ \text{ in } D. \tag{4.9}$$

Moreover $\lambda_\varepsilon > 0$ whenever (3.4) holds.

Corollary 4.7. *Assume that A satisfies (1.2). Then, if $f \in L^q(D), q > \frac{N}{2}$, any solution u_ε of $(\mathcal{P}_\varepsilon)$ is locally Hölder continuous. Moreover, we have:*

$$\|u_\varepsilon\|_{L^\infty(D)} \le C\|f\|_{L^q(D)} + \varepsilon,$$

where C is a constant depending only on N, α, q, m and on the L^∞ norms of the coefficient of A and a_0.

Proof. To prove the estimate in Corollary 4.7, it is enough to see that from (4.9) we have $-\mathcal{A}u_\varepsilon = f$ in $\Omega_\varepsilon := \{x \in D;\ |u_\varepsilon(x)| > \varepsilon\}$, and use Theorem 8.16 in [4]. Note that $|\Omega_\varepsilon| \le m$. □

Corollary 4.8. *In addition to the hypothesis of* Corollary 4.7, *suppose that the coefficients of A are in $C^{0,1}(\overline{D})$. Then, if $f \in L^q(D), 1 < q < \infty$, any solution u_ε of $(\mathcal{P}_\varepsilon)$ is in $W^{2,q}_{loc}(D)$. In particular, if $q > N$ then $u_\varepsilon \in C^{1,\theta}_{loc}(D)$, for $0 < \theta < 1 - \frac{N}{q}$.*

Moreover, if D is of class $C^{1,1}$ then $u_\varepsilon \in C^{1,\theta}_{loc}(\overline{D})$.

Proof. This is a simple consequence of Theorems 9.11 and 9.13 in [4]. □

Proof of Theorem 4.6. Let $\varphi \in C_0^\infty(D)$, and consider the function $\Phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ defined as:

$$\Phi(t,\theta) := \int_D p_\varepsilon(\theta(u_\varepsilon + t\varphi))\,\mathrm{d}x - \int_D p_\varepsilon(u_\varepsilon)\,\mathrm{d}x.$$

It is easy to see that $\Phi \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\Phi(0,1) = 0$, moreover since u can not be identically 0 (because $f \not\equiv 0$) and p_ε is even, we have:

$$\partial_\theta \Phi(0,1) = \int_D p'_\varepsilon(u_\varepsilon) u_\varepsilon \,\mathrm{d}x > 0.$$

The implicit function theorem implies the existence of a positive number $\eta > 0$, and a function $\theta \in C^1(]-\eta,\eta[,\mathbb{R})$ such that: $\theta(0) = 1$, $\Phi(t,\theta(t)) = 0, \forall t \in]-\eta,\eta[$, and

$$\theta'(0) = -\frac{\partial_t \Phi(0,1)}{\partial_\theta \Phi(0,1)} = -\frac{\displaystyle\int_D p'_\varepsilon(u_\varepsilon)\varphi\,\mathrm{d}x}{\displaystyle\int_D p'_\varepsilon(u_\varepsilon)u_\varepsilon\,\mathrm{d}x}.$$

Thus, $\theta(t)(u_\varepsilon + t\varphi) \in \mathbb{V}_\varepsilon$ whenever $t \in]-\eta,\eta[$. Now writing that 0 is a local minimum of $t \mapsto J(\theta(t)(u_\varepsilon + t\varphi))$, we get:

$$\int_D (A\nabla u_\varepsilon, \nabla\varphi) + a_0 u_\varepsilon \varphi \,\mathrm{d}x = \int_D f\varphi\,\mathrm{d}x - \lambda_\varepsilon \int_D p'_\varepsilon(u_\varepsilon)\varphi\,\mathrm{d}x,$$

where

$$\lambda_\varepsilon = \frac{\displaystyle\int_D f u_\varepsilon\,\mathrm{d}x - \int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2\,\mathrm{d}x}{\displaystyle\int_D p'_\varepsilon(u_\varepsilon)u_\varepsilon\,\mathrm{d}x}.$$

This proves (4.9). Now, to prove that $\lambda_\varepsilon \geq 0$, we remark that for a sufficiently small number $t > 0$, we have $(1-t)u_\varepsilon \in \mathbb{V}_\varepsilon$. Then by making $t \to 0^+$ in $\frac{1}{t}\left(J((1-t)u_\varepsilon) - J(u_\varepsilon)\right) \geq 0$, we get:

$$\int_D f u_\varepsilon\,\mathrm{d}x - \int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2\,\mathrm{d}x \geq 0.$$

Proceeding as in Lemma 4.5, it comes that $\lambda_\varepsilon > 0$ whenever (3.4) holds. □

Another consequence of Theorem 4.6 is the following lemma.

Lemma 4.9. *In addition to the hypothesis of* Corollary 4.8*, suppose that D is of class $C^{1,1}$. Then*

$$\forall x \in \partial D, \quad |\nabla u_\varepsilon(x)| \leq C\left(\|f\|_{L^q(D)} + \varepsilon\right),$$

where the constant C does not depend on u_ε and ε.

Proof. In this proof, the value of C can change but does not depend on u_ε and ε. Remark first that by Kato's inequality it comes from (4.9) that:

$$\mathcal{A}|u_\varepsilon| \geq sign(u_\varepsilon)\mathcal{A}u_\varepsilon \geq -|f| \;\; in \; D.$$

Here we used that $\forall r \in \mathbb{R}$, $p'_\varepsilon(r)sign(r) \geq 0$ and $\lambda_\varepsilon \geq 0$.

Fix now $x_0 \in \partial D$. Without loss of generality we can assume $x_0 = 0$, the general case being recovered by the coordinate transformation $x \to x - x_0$. Let us denote B_1 the unite ball of $\mathbb{R}^N$ and ω_N its measure. By L^p-theory of P.D.E (see for instance chapter 8 in [4]), the Dirichlet problem

$$\begin{cases} -\mathcal{A}w = |f| \; in \; D \cap B_1, \\ \quad w = |u_\varepsilon| \; on \; \partial(D \cap B_1), \end{cases}$$

has a unique solution $w \in W^{2,q}_{loc}(D \cap B_1)$. Since $q > N > \frac{N}{2}$ and $D \cap B_1$ is bounded and according to Corollary 4.7, w satisfies

$$\|w\|_{L^\infty(D\cap B_1)} \leq C(\|f\|_{L^q(D)} + \|u_\varepsilon\|_{L^\infty(D)}) \leq C(\|f\|_{L^q(D)} + \varepsilon).$$

On the other hand by Lemma 9.16 in [4] and thanks to the smoothness of ∂D we have:

$$\|w\|_{W^{2,q}(D\cap B_1)} \leq C(\|w\|_{L^q(D\cap B_1)} + \|f\|_{L^q(D\cap B_1)}).$$

But for $q > N$, $\|w\|_{L^q(D\cap B_1)} \leq \omega_N \|w\|_{L^\infty(D\cap B_1)}$. Thus from Sobolev embedding theorem we get:

$$\|w\|_{C^1(\overline{D\cap B_{\frac{1}{2}}})} \leq C\left(\|f\|_{L^q(D\cap B_1)} + \varepsilon\right).$$

Now maximum principle implies that $w \leq |u_\varepsilon|$ in $D \cap B_1$. Taking into account that $u_\varepsilon(0) = w(0) = 0$ (since $0 \in \partial(D \cap B_1)$), we obtain that

$$|\frac{\partial u_\varepsilon(0)}{\partial \nu}| \leq |\frac{\partial w(0)}{\partial \nu}| \leq \|w\|_{C^1(\overline{D\cap B_1})}.$$

This finishes the proof since $u_\varepsilon = 0$ on ∂D and therefore the tangential derivative of u_ε vanishes on ∂D. □

5. Preliminary results when $\varepsilon \to 0$

In this section we shall derive some useful lemmas from the following result.

Theorem 5.1. *Let $(u_\varepsilon)_\varepsilon$ be a sequence of solutions of the corresponding problems $(\mathcal{P}_\varepsilon)$. Then, up to a subsequence, $(u_\varepsilon)_\varepsilon$ converges to a solution u of $(\mathcal{P}_0)$ in $H^1_0(D)$.*

Proof. Recall that from (4.5), $(u_\varepsilon)_\varepsilon$ is bounded in H^1-norm. Then by the Lemma 4.2, there exists $u \in \mathbb{V}_0$ and a subsequence, which we still denotes $(u_\varepsilon)_\varepsilon$, such that $(u_\varepsilon)_\varepsilon$ converges to u weakly in $H^1_0(D)$. Moreover, since $\mathbb{V}_0 \subset \mathbb{V}_\varepsilon$, u is a solution of $(\mathcal{P}_0)$. Now to get a strong convergence we have to prove that $u_\varepsilon \to u$ in $L^2(D)$ and $\nabla u_\varepsilon \to \nabla u$ in $L^2(D)$.

By (1.2) and using the fact that $a_0 \geq 0$ we get

$$\alpha\|\nabla u_\varepsilon - \nabla u\|_{L^2(D)} \leq \int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2 \,\mathrm{d}x + \int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2 \,\mathrm{d}x - 2\int_D (A\nabla u_\varepsilon, \nabla u) + a_0 u_\varepsilon u \,\mathrm{d}x,$$

Since $\lambda_\varepsilon > 0$ and $p'_\varepsilon(r)r \geq 0$, it comes from (4.9) that:

$$\int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2 \,\mathrm{d}x \leq \int_D f u_\varepsilon \,\mathrm{d}x.$$

Then the weak convergence implies that

$$\int_D (A\nabla u, \nabla u) + a_0 u^2 \,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2 \,\mathrm{d}x,$$

$$\limsup_{\varepsilon\to 0} \int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) + a_0 u_\varepsilon^2 \,\mathrm{d}x \leq \int_D f u dx = \int_D (A\nabla u, \nabla u) + a_0 u^2 \,\mathrm{d}x.$$

The last equality is the equality in Lemma 3.3 with $\varphi = u$. Thus $\nabla u_\varepsilon \to \nabla u$ in $L^2(D)$. Finally remark that, because of (4.1), $u_\varepsilon \to u$ in $L^2(D)$ is equivalent to $w_\varepsilon \to u$ in $L^2(D)$ where $w_\varepsilon = (u_\varepsilon - \varepsilon)^+ - (u_\varepsilon + \varepsilon)^-$. Note that we have also $\nabla w_\varepsilon \to \nabla u$ in $L^2(D)$. But $|\Omega_{(w_\varepsilon - u)}| \leq 2m$; therefore inequality (2.2) implies that

$$\|w_\varepsilon - u\|_{L^2(D)} \leq C_0 (2m)^{\frac{1}{N}} \|\nabla w_\varepsilon - \nabla u\|_{L^2(D)} \xrightarrow[\varepsilon\to 0]{} 0.$$

□

The next lemma gives another formulation of the necessary condition of optimality.

Lemma 5.2. *Let u_ε be a solution of $(\mathcal{P}_\varepsilon)$, and assume that D is of class $C^{1,1}$ and that the coefficients of A are in $C^{0,1}(\overline{D})$. Then, for every $\Phi \in \left(C_0^\infty(\mathbb{R}^N)\right)^N$, we have:*

$$\int_D ([D\Phi A]\nabla u_\varepsilon, \nabla u_\varepsilon) \,\mathrm{d}x - \frac{1}{2}\int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon) \operatorname{div}\Phi \,\mathrm{d}x - \frac{1}{2}\int_{\partial D} (A\nabla u_\varepsilon, \nabla u_\varepsilon)\, \Phi\cdot \nu \,\mathrm{d}\sigma - \frac{1}{2}\int_D ([A'\Phi]\nabla u_\varepsilon, \nabla u_\varepsilon) \,\mathrm{d}x - \int_{\partial D} a_0 u_\varepsilon \nabla u_\varepsilon \cdot \Phi \,\mathrm{d}x = \int_D f\nabla u_\varepsilon \Phi \,\mathrm{d}x + \lambda_\varepsilon \int_D p_\varepsilon(u_\varepsilon) \operatorname{div}\Phi \,\mathrm{d}x, \tag{5.1}$$

where $D\Phi$ is the Jacobian matrix of Φ and $[A'\Phi]$ is the matrix defined by $[A'\Phi] = (\nabla a_{ij}\Phi)_{1\leq i,j\leq N}$.

Proof. Let $\Phi = (\Phi_1, \ldots, \Phi_N) \in \left(C_0^\infty(\mathbb{R}^N)\right)^N$, and $\varphi_0 \in C^\infty([0,\infty[, \mathbb{R})$ such that:

$$0 \leq \varphi_0 \leq 1 \quad \text{and} \quad \varphi_0(t) = \begin{cases} 1 & \text{if } t \in [0,1], \\ 0 & \text{if } t \in [2,\infty[. \end{cases}$$

For $k \in \mathbb{N}^*$, consider the function φ_k defined on D by $\varphi_k(x) = \varphi_0(\frac{|x|}{k})$. Thanks to Corollary 4.8, $u_\varepsilon \in W^{2,2}_{loc}(D)$; thus we can choose $\Phi_l \varphi_k \partial_l u_\varepsilon$ as a test function in (4.9), with $l \in \{1, \ldots, N\}$. Integrating by part and taking the limit as $k \to \infty$, we get:

$$\frac{1}{2}\int_{\partial D} (A\nabla u_\varepsilon, \nabla u_\varepsilon)\, \Phi_l \nu_l \,\mathrm{d}\sigma - \int_{\partial D} (A\nabla u_\varepsilon, \nu)\, \Phi_l \partial_l u_\varepsilon \,\mathrm{d}\sigma + \int_D (A\nabla u_\varepsilon \nabla \Phi_l)\, \partial_l u_\varepsilon \,\mathrm{d}x - \frac{1}{2}\int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon)\, \partial_l \Phi_l \,\mathrm{d}x - \frac{1}{2}\int_D ([\partial_l A]\nabla u_\varepsilon, \nabla u_\varepsilon)\, \Phi_l \,\mathrm{d}x - \int_D a_0 u_\varepsilon \Phi_l \partial_l u_\varepsilon \,\mathrm{d}x = \int_D f \Phi_l \partial_l u_\varepsilon \,\mathrm{d}x + \lambda_\varepsilon \int_D p_\varepsilon(u_\varepsilon) \partial_l \Phi_l \,\mathrm{d}x,$$

where $[\partial_l A] := (\partial_l a_{i,j})_{1 \le i,j \le N}$. Hence, the lemma follows by taking the sum on l going from 1 to N. □

In the following lemma, we give another expression of λ_ε.

Lemma 5.3. *In addition to hypothesis of* Lemma 5.2*, suppose that* (2.5) *holds. Then*

$$\lambda_\varepsilon = \frac{1}{mN}\left(\frac{2-N}{2}\int_D (A\nabla u_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}x - \frac{1}{2}\int_{\partial D} (A\nabla u_\varepsilon, \nabla u_\varepsilon)\, \nu\cdot x \,\mathrm{d}\sigma - \frac{1}{2}\int_D ([A'x]\nabla u_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}x - \int_D a_0 u_\varepsilon \nabla u_\varepsilon \cdot x \,\mathrm{d}x - \int_D f \nabla u_\varepsilon \cdot x \,\mathrm{d}x\right). \tag{5.2}$$

Moreover, if $f \in L^q(D)$ with $q > N$ and if D satisfies (2.6) *then there exists a constant λ^* independent of $\varepsilon \le \varepsilon_0$ such that*

$$0 < \lambda_\varepsilon \le \lambda^*. \tag{5.3}$$

Proof. To proof (5.2), we put $\Phi = x\varphi_k$ in (5.1), where φ_k is as in the proof of the Lemma 4.3. Then, under the condition (2.5), we take the limit as $k \to \infty$. Now, to prove (5.3), remark first that from (5.2) we have:

$$0 < \lambda_\varepsilon \le \frac{1}{mN}\left(-\frac{1}{2}\int_D ([A'x]\nabla u_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}x - \int_D a_0 u_\varepsilon \nabla u_\varepsilon \cdot x \,\mathrm{d}x - \int_D f \nabla u_\varepsilon \cdot x \,\mathrm{d}x + \int_{\partial^- D} (A\nabla u_\varepsilon, \nabla u_\varepsilon) \nu\cdot x \,\mathrm{d}\sigma\right).$$

Hence (5.3) follows from (4.5) and the hypothesis (2.5) and (2.6). □

6. A uniform Lipschitz estimate

The aim of this section is to prove a uniform L^∞-gradient estimate for a solution u_ε of $(\mathcal{P}_\varepsilon)$. Following an idea from [1], and using standard elliptic estimates, we prove such estimate for any solution u_ε, of the equation (4.9), which does not change its sign. Namely the Harnack inequality compels us to assume the non negativity of u_ε. Note that if u_ε is a negative solution of $(\mathcal{P}_\varepsilon)$, then $-u_\varepsilon$ is a solution of the same problem with $-f$ instead f.

Since we are assuming that the coefficients of the matrix A are in $C^{0,1}(\overline{D})$, we shall consider a more general form of the equation (4.9):

$$Lv = f + g_\varepsilon(v) \ \text{ in } D, \tag{6.1}$$

where L is the elliptic operator:

$$Lv = \sum_{i,j=1}^{N} a_{ij}(x)\partial_{ij}v + \sum_{i=1}^{N} b_i(x)\partial_i v + c(x)v, \tag{6.2}$$

where the coefficients a_{ij}, b_i and c are functions defined on D. The functions a_{ij} satisfy $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, N$ and the ellipticity condition (1.2). The function f is in $L^q(D), q > N$, and the nonlinear term g_ε is a nonnegative L^∞ function satisfying:

$$g_\varepsilon \le \frac{B}{\varepsilon} \ \text{ and support of } g_\varepsilon \text{ is in } [0, \varepsilon], \tag{6.3}$$

where B is a constant independing on ε. We always assume that D is of class $C^{1,1}$ and that:

$$\begin{cases} a_{ij} \in C^0(\overline{D}), \ b_i \text{ and } c \text{ are measurable, and} \\ \|a_{ij}\|_{C^0(D)}, \|b_i\|_{L^\infty(D)}, \|c\|_{L^\infty(D)} \le K \ \ \forall i, j = 1, \ldots, N, \end{cases} \tag{6.4}$$

and

$$c \le 0. \tag{6.5}$$

Remark 6.1. Under the considerations in Lemma 5.3, the equation (4.9) comes from (6.1) by taking $g_\varepsilon = \lambda_\varepsilon p'_\varepsilon$ (so that $B = \lambda^* \|p'\|_{L^\infty(\mathbb{R})}$), $b_i = \sum_{j=1}^{N} \partial_j a_{ij}$ and changing f in $-f$.

The main result of this section is the following:

Theorem 6.2. *Let $v \in C^1(\overline{D})$ be a nonnegative solution of* (6.1), *and suppose that* (6.3)–(6.5) *are satisfied and that D is of class $C^{1,1}$. Then there exists a constant C, independing on ε, such that:*

$$\forall x_0 \in \overline{D}, \ \ |\nabla v(x_0)| \le C \left(\|v\|_{L^\infty(D)} + \|f\|_{L^q(D)} + B \right). \tag{6.6}$$

Remark 6.3. This result is analog to Theorem 3.1 in [1] where the studied problem is $Lv = g_\varepsilon(v)$ *in* D, $\mu \cdot \nabla v = 0$ *on* ∂D, with $\mu(x)$ is non tangent vector to ∂D.

An immediate consequence of this theorem is

Corollary 6.4. *Under the hypothesis of* Theorem 2.1*, any non negative (or equivalently non positive) solution* u_ε *of* $(\mathcal{P}_\varepsilon)$ *satisfies:*

$$\forall x \in \overline{D}, \quad |\nabla u_\varepsilon(x)| \leq C \left(\|f\|^2_{L^2(D)} + \|f\|_{L^q(D)} + 1 \right)$$

where C *is some constant idepending on* u_ε *and on* $\varepsilon \leq \varepsilon_0$.

Moreover if there exists a sequence (ε_n) *such that* $\varepsilon_n \to 0$ *and* $u_{\varepsilon_n} \geq 0$ *(or equivalently* $u_{\varepsilon_n} \leq 0$*), then the problem* $(\mathcal{P}_0)$ *admit at least one solution* $u \in C^{0,1}(\overline{D})$ *satisfying:*

$$\forall x \in \overline{D}, \quad |\nabla u(x)| \leq C \left(\|f\|^2_{L^2(D)} + \|f\|_{L^q(D)} + 1 \right).$$

Proof. The first estimate follows from Theorem 6.2 by the considerations in remark 6.1. Indeed, using (4.5) and (5.2) we see that $\lambda^* \leq C(\|f\|^2_{L^2(D)} + 1) + c_\varepsilon$. Recall also that from Corollary 4.7 it comes that $\|u_\varepsilon\|_{L^\infty} \leq C\|f\|_{L^q(D)} + \varepsilon$.

The second statement in Corollary 6.4 follows from the first one by Theorem 5.1 and Ascoli Theorem. □

Remark 6.5. Thanks to maximum principle, the condition of the second statement in Corollary 6.4 is in particular satisfied if $f \geq 0$ or $f \leq 0$.

The main tool of the proof of Theorem 6.2 is the interior gradient estimate for the solutions of the equation $Lu = f$ in $B_r := B(0, r)$:

$$|\nabla u(0)| \leq C \left(\frac{1}{r} \sup_{B_r} |u| + \|f\|_{L^q(B_r)} \right), \tag{6.7}$$

where C is a constant depending on N, α, K and on the moduli of continuity of the coefficients a_{ij}, i,j=1, ... , N, but does not depend on u and $r \leq 1$. This follows from the $W^{2,p}$-estimates, with $p > N$, and embedding theorem in [4]. An other important property of the nonnegative solution of the equation $Lu = f$ in B_1 is:

$$\sup_{B_{\frac{1}{4}}} u \leq C \left(u(0) + \|f\|_{L^N(B_1)} \right), \tag{6.8}$$

where C is a constant depending on N, α and K. This follows from Hölder and Harnack estimates, of Krylov and Safonov. See [4], chapter 9. We need also the following lemma which is a particular case of Theorem 2.2 in [1]:

Lemma 6.6. *Let* $u \in C^1(B_1) \cap C^0(\bar{B}_1)$ *be a nonnegative solution of* $Lu = 0$ *in the unite ball* B_1 *of* D*, and assume that*

$$u(\bar{x}) = 0 \qquad \textit{and} \qquad 0 \leq \lim_{t \to 0} \frac{u(t\bar{x})}{1 - t} \leq 1,$$

for some $\bar{x} \in \partial B_1$. *Then, there exists a constant* M *depending on* N *and the operator* L *such that:*

$$u(0) \leq M.$$

Remark 6.7. To prove Theorem 6.2, we have to show that $|\nabla v(x_0)|$ satisfies (6.6), for every $x_0 \in \overline{D}$. We shall distinguish three cases. In the first case, we consider $x_0 \in \Omega_\varepsilon := \{x \in D; v(x) > \varepsilon\}$, and we prove that the estimate of $|\nabla v(x_0)|$ follows from the second case where $x_0 \in \Omega_\varepsilon^C = \{x \in D; 0 \leq v \leq \varepsilon\}$. In this case we prove also that the estimate of $|\nabla v(x_0)|$ follows from the last case where we have to estimate $|\nabla v|$ on ∂D.

If L is the Laplace operator and f is sufficiently smooth, then the first step is a simple consequence of the following argument: Let E_N denotes the fundamental solution of the Laplace's equation, and consider $w = E_N * f$ its convolution with f. Obviously, we have $\Delta(v - w) = 0 \in \Omega_\varepsilon$, and therefore $|\nabla(v - w)|^2$ is a subharmonic function in Ω_ε, i.e. $\Delta(|\nabla(v - w)|^2) \geq 0$. Then,

$$\sup_{\Omega_\varepsilon} |\nabla(v - w)|^2 = \sup_{\partial\Omega_\varepsilon} |\nabla(v - w)|^2,$$

and consequently,

$$\sup_{\Omega_\varepsilon} |\nabla v|^2 \leq \sup_{\partial\Omega_\varepsilon} |\nabla v|^2 + 2\|\nabla w\|^2_{L^\infty(D)}.$$

Note that $\|\nabla w\|_{L^\infty(D)}$ is bounded by a constant depending on f and N, and that $\sup_{\partial\Omega_\varepsilon} |\nabla v|$ is bounded by $\sup_{\Omega_\varepsilon^C} |\nabla v|$.

Proof of Theorem 6.2. First case $x_0 \in \Omega_\varepsilon$:

As in the proof of Corollary 4.9, we can assume $x_0 = 0$. Let B_δ the ball centred at the origin with radius $\delta := d(0, \Omega_\varepsilon^C)$, the distance of the point 0 to Ω_ε^C, so that $B_\delta \subset \Omega \subset D$ and v satisfies: $v \geq \varepsilon$ on $\bar{B}_\delta$, and

$$Lv = f \text{ in } B_\delta.$$

Hence, if $\delta \geq 1$, then (6.6) follows from (6.7) with $r = 1$.

Now, for $\delta < 1$ we consider the scaled function: $w(x) := \dfrac{v(\delta x) - \varepsilon}{\delta}$, and see that w satisfies the scaled equation:

$$L_\delta w = f_\delta \text{ in } B_1 \quad \text{and} \quad w \geq 0 \text{ on } \bar{B}_1,$$

where f_δ is the function defined as $f_\delta(x) := \delta f(\delta x) - c(\delta x)\delta\varepsilon$, and L_δ is the operator defined as L with the coefficients: $a_{ij}^\delta(x) := a_{ij}(\delta x), b_i^\delta(x) = \delta b_i(\delta x)$ and $c^\delta(x) = \delta^2 c(\delta x)$, for $i, j = 1, \dots N$. Note that, since $\delta < 1$, L_δ satisfies the condition (1.2) and (6.4). Consider now the decomposition $w = w_1 + w_2$ on B_1, where w_1 and w_2 are given by:

$$\begin{cases} L_\delta w_1 = f_\delta \text{ in } B_1, \\ w_1 = 0 \text{ on } \partial B_1, \end{cases} \quad \text{and} \quad \begin{cases} L_\delta w_2 = 0 \text{ in } B_1, \\ w_2 = w \text{ on } \partial B_1. \end{cases}$$

From standard elliptic estimates, it comes that:

$$\|w_1\|_{C^1(\bar{B}_1)} \leq C(\|f\|_{L^q(B_1)} + \varepsilon), \tag{6.9}$$

where C is a constant independent of w_1 and δ (recall that $\delta < 1$). On the other hand, from the definition of δ, there exists $x^* \in \partial B_\delta \cap \Omega_\varepsilon^C$ such that $v(x^*) = \varepsilon$. So $w_2(\bar{x}) = 0$, with $\bar{x} = \delta x^* \in \partial B_1$. Moreover, by the maximum principle we have $w_2 \geq 0$ on $\bar{B}_1$; thus using (6.9) and writing $\nabla w_2 = \nabla w - \nabla w_1$, we get:

$$|\nabla w_2(\bar{x})| \leq C\|f\|_{L^q(B_1)} + |\nabla w(\bar{x})|.$$

Hence the Lemma 6.6, applied to w_2 normalized by the right hand side of the above inequality, gives:

$$0 \leq w_2(0) \leq M\left(C(\|f\|_{L^q(B_1)} + \varepsilon) + |\nabla w(\bar{x})|\right), \tag{6.10}$$

where M is the constant in the Lemma 5.5. Writing $w(0) = w_1(0) + w_2(0)$, we get:

$$0 \leq w(0) \leq C\left(\|f\|_{L^q(B_1)} + |\nabla w(\bar{x})| + \varepsilon\right),$$

where C is another constant as before. Now, writing (6.7) and (6.8) for w with $r = \frac{1}{4}$, we get:

$$|\nabla w(0)| \leq C\left(\sup_{B_{\frac{1}{4}}} w + \|f\|_{L^q(B_{\frac{1}{4}})} + \varepsilon\right),$$

$$\sup_{B_{\frac{1}{4}}} w \leq C\left(w(0) + \|f\|_{L^q(B_1)} + \varepsilon\right).$$

Here we use the fact that $\|f\|_{L^N(B_1)} \leq \omega_N^{\frac{q-N}{qN}}\|f_\delta\|_{L^q(B_1)}$ and $\|f_\delta\|_{L^q(B_1)} \leq \|f\|_{L^q(B_r)} + \omega_N^{\frac{1}{q}}\varepsilon$ (where $\omega_N = |B_1|$). Then, since $\nabla w(0) = \nabla v(0)$ and $\nabla w(\bar{x}) = \nabla v(x^*)$, we have:

$$|\nabla v(0)| \leq C\left(\|f\|_{L^q(D)} + |\nabla v(x^*)| + \varepsilon\right).$$

Thus, since $x^* \in \Omega_\varepsilon^C$, the gradient estimate in $x_0 \in \Omega$, with $d(x_0, \Omega_\varepsilon^C) < 1$, will follow from the gradient estimate on Ω_ε^C.

Second case $x_0 \in \Omega^C$:

As before, assume $x_0 = 0$, and consider the scaled function: $w(x) := \dfrac{v(\varepsilon x)}{\varepsilon}$, satisfying:

$$L_\varepsilon w = f_\varepsilon + \varepsilon g_\varepsilon(\varepsilon w) \quad \text{and} \quad w \geq 0 \text{ on } \bar{B}_1,$$

where L_ε is defined in the same way as for L_δ in the first case, and f_ε is given by $f_\varepsilon(x) = \varepsilon f(\varepsilon x)$. By (6.3), we have:

$$\|f_\varepsilon + \varepsilon g_\varepsilon(\varepsilon w)\|_{L^q(B_1)} \leq \varepsilon^{1-\frac{N}{q}}\|f\|_{L^q(B_\varepsilon)} + \omega_N^{\frac{1}{q}}B.$$

Then, proceeding as in the first case and taking into account that $w(0) \leq 1$, $\varepsilon \leq 1$ and $\nabla v(0) = \nabla w(0)$, we get:

$$|\nabla v(0)| \leq C\left(1 + \|f\|_{L^\infty(D)} + B\right).$$

This finishes the proof if $d(x_0, \partial D) \geq \varepsilon$.

Now if $x_0 \in \Omega_\varepsilon^C$ and $\delta = d(x_0, \partial D) < \varepsilon$, the function $w(x) = \dfrac{v(\delta x)}{\delta}$ satisfies

$$L_\delta w = f_\delta + \delta g_\varepsilon(\delta w) \text{ in } B_1 \quad \text{and} \quad w \geq 0 \text{ on } B_1.$$

Note that here $f_\delta(x) = \delta f(\delta x)$ so that

$$\|f_\delta + \delta g_\varepsilon(\delta w)\|_{L^q(B_1)} \leq \delta^{1-\frac{N}{q}} \|f\|_{L^q(B_1)} + \frac{\delta}{\varepsilon} \omega_N^{\frac{1}{q}} B \leq \|f\|_{L^q(B_1)} + \omega_N^{\frac{1}{q}} B,$$

since $\delta < \varepsilon < 1$. Using the same decomposition $w = w_1 + w_2$ as in the first case (with $f_\delta + \delta g_\varepsilon(\delta w)$ instate of f_δ) we get

$$|\nabla v(0)| \leq C \left(\|f\|_{L^q(B_\delta)} + \omega_N^{\frac{1}{q}} B + |\nabla v(x^*)| \right),$$

where x^* is some point of ∂D where $d(x_0, \partial D)$ is achieved.

Third case $x_0 \in \partial D$:

Here also assume that $x_0 = 0$ and consider the unite ball $B_1 = B(0,1)$ of center $0 \in \partial D$. Let w be the unique solution of

$$\begin{cases} Lw = -f^- & \text{in } B_1 \cap D, \\ w = v & \text{on } \partial(B_1 \cap D), \end{cases}$$

where f^- is given by $f = f^+ - f^-$ with $f^+, f^- \geq 0$. Note that, since $g_\varepsilon(v) \geq 0$, $Lw = -f \leq f + g_\varepsilon(v) = Lv$. Thus an argument similar to that in the proof of Corollary 4.9 gives the desired estimate:

$$|\nabla v(x_0)| \leq C(\|v\|_{L^\infty}(D) + \|f\|_{L^q(D)}).$$

□

Remark 6.8. In [5], it is shown that if u is a continuous solution of $(\mathcal{P})$ such that the open set Ω_u has a sufficiently smooth boundary $\partial\Omega_u$. Then (u, Ω_u) solves the following free boundary problem analog to (1.5):

$$\begin{cases} -\Delta u &= f \ \ in \ \ \Omega, \\ u &= 0 \ \ on \ \ \partial\Omega, \\ \frac{1}{2}|\nabla u|^2 &= \Lambda \ \ on \ \ \partial\Omega \cap D, \end{cases}$$

where the constant Λ is an unknown of the problem. Indeed Λ is the limit, up to a subsequence, of λ_ε.

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