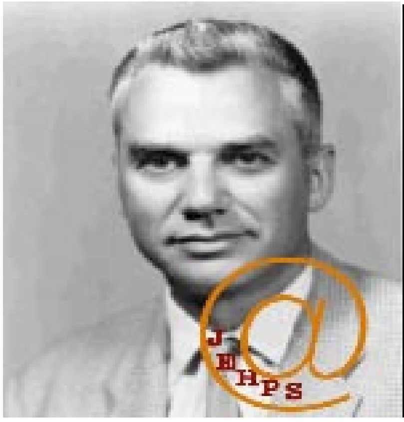

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Geometric properties of multivariate correlation in de Finetti's approach to insurance theory

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Resumé

Nous offrons des démonstrations et des commentaires sur certaines sections du papier de Bruno de Finetti sur les corrélations ("A proposito di correlazione"). En particulier nous nous concentrons sur son théorème sur les angles, qu'il a énoncé sans démonstration et qui est d'une simplicité et d'une puissance telle qu'elle méritrait d'être inséré dans les livres de texte de base sur la statistique multivariée. Nous établissons un lien entre le théorème sur les angles de de Finetti et la recherche contemporaine sur la frontière de l'ensemble convexe formée par les matrices de corrélation. Nous mettons en évidence, basé sur une section de l'article de de Finetti, qu'il a découvert les corrélations des rangs en même temps ou avant Kendall.

Abstract

We provide proofs and commentary on parts of Bruno De Finetti's paper "About Correlation". In particular we focus on De Finetti's angle theorem, which he stated without proof and whose striking simplicity and power would seem to make it a natural candidate for any multivariate statistics text. We establish a connection between de Finetti's angle theorem and ongoing research on the boundary of the convex set of correlation matrices. We also provide evidence from this article that de Finetti discovered rank correlation before or at the same time as Kendall.

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I. Introduction

I.1. Historical Context.

Due to the low regard with which Italian mathematicians held mathematics devoted to finance and insurance, the part of Bruno de Finetti's work devoted to this subject did not appear in his published collected works. This omission on the Italian side is reminiscent of that of Bachelier in France and for similar reasons. Recently, part of Bruno de Finetti's work in insurance has been rediscovered when it was mentioned by Claudio Albanese to Mark Rubinstein. Subsequently, one of us (L. B.), translated the paper by Bruno de Finetti and it was discovered that de Finetti had anticipated a good part of Markowitz's Nobel prize winning work on mean-variance portfolio selection by 12 years.

Bruno de Finetti (1906-1985) was an outstanding mathematician, statistician, philosopher and economist. Mark Rubinstein (2006) has recently written that "among de Finetti's papers is a treasure-trove of results in economics and finance written well before the work of the scholars that are traditionally credited with these ideas ... but perhaps most astounding is de Finetti's 1940 paper anticipating much of mean-variance portfolio theory later developed by Harry Markowitz in a series of three works (1952), (1956) and (1959) and A.D. Roy (1952). With the advantage of hindsight, we can see that Markowitz's work sparked the development of modern finance theory and practice. Yet, twelve years earlier, de Finetti had already quietly laid these foundations. de Finetti modelled portfolio variance as a sum of covariances, developed the concept of mean-variance efficiency, justified this criterion based on the normality of returns, considered the implications of fat tails, discussed bounds on negative correlation coefficients, and even worked out an early version of the critical line algorithm, the numerical method used to solve the portfolio selection problem." Bruno de Finetti's merits have been acknowledged by the Nobel prize laureate Harry Markowitz [13] in an article significantly titled "de Finetti Scoops Markowitz". In his 1940 article ("The Problem of Full-Risk Insurances"), Bruno de Finetti had claimed the originality of his ideas: "The extension, that seems so obvious to me, of the previous considerations on risk to the case of correlated events is almost never discussed in the research on this subject (to be more precise, I remember having seen it only in a paper by Dubois ... who however didn't face the problem of full-risk insurances)". Three years earlier, in 1937, de Finetti had published an article whose title is "About Correlations" and whose annotated translation, by two of us (L.B. and P.L.) is presented in this same volume. In "About Correlations", Bruno de Finetti set out a method to represent geometrically n correlated random variables. He stated that "it is always possible to represent n random variables by n vectors having a modulus equal to their standard deviations and whose pairwise correlations are given by the cosines of the angles between them". In particular, he pointed out that "n vectors can surely be orthogonal among each other (no-correlation case), or can form acute angles (positive correlation), but it is impossible that they form angles which are all obtuse (over a certain angle limit). For example, given three arbitrary vectors, the angle between any two of them cannot exceed 120 degrees. Then he claimed that the maximum negative correlation coefficient that we can simultaneously observe among n random variables is $\frac{-1}{n-1}$, otherwise the standard deviation of the sum would be imaginary. In this paper, we prove de Finetti's " angle theorem " and relate it to ongoing research on extreme correlation matrices. De Finetti was not the first to take a geometric view of correlation. This can be traced back as far as the pioneering paper by Fisher [9]. However we have no found no other prior source than de Finetti for the specific uses he makes of this tool, the angle theorem and restrictions on extreme correlation matrices.

In the last section we show that de Finetti discovered a form of rank correlation, ahead of or certainly, at the same time as Kendall.

It should also be noted that Fréchet in his 1957 paper [10] opened a new debate about correlations and laid the foundation for the modern theory of copulas, which has become very important in many applications of statistics in the applied sciences and in mathematical finance.

I.2. Contents of this contribution.

A striking feature of de Finetti's paper on correlation is his angle theorem. Given two random variables, discrete or continuous, X and Y, define the angle between two random variables by

$$\alpha(X,Y) = \arccos(\rho(X,Y)),$$

where $\rho(X,Y)$ is the correlation coefficient between X and Y. Then de Finetti proves that this angle shares very much the same properties as those of an angle between vectors in the plane. These properties are expressed by Theorem 1. below. If we define the scalar product of two random vectors by $X \cdot Y = E[(X - E[X])(Y - E[Y]]$, then using the Cauchy-Schwarz inequality, it immediately follows that

$$X \cdot Y \leq \sqrt{X \cdot X}\sqrt{Y \cdot Y} \qquad (1)$$

so that

$$\frac{|X \cdot Y|}{\sqrt{X \cdot X}\sqrt{Y \cdot Y}} \leq |\rho(X,Y)| \leq 1,$$

therefore, it has been mentioned by many authors that $\rho(X,Y)$ is *analogous* to the cosine of the angle between two vectors in the plane. However, until discovering de Finetti's paper, we have not seen this analogy developed to its full potential. In addition, de Finetti does not provide a proof of his main theorem in the cited paper. Nor have we been able to locate the proof of this theorem elsewhere in the literature. We fill this gap in the present paper and we describe some interesting connections between de Finetti's theorem and research on extreme correlation matrices.

II. Results and Discussion

II.1. de Finetti's angle theorem.

Definition 1. *Let X and Y be random variables and $\rho(X,Y)$ be their correlation . Define an angle $\alpha(X,Y)$ by*

$$\alpha(X,Y) = \arccos(\rho(X,Y)),$$

where $\arccos : [-1,1] \mapsto [0,\pi]$ *is the inverse of the cosine function.*

Then we have

Theorem 1. (de Finetti's angle theorem) *Let X, Y, Z be three random variables. Then we have*

$$\alpha(X,Y) \leq \alpha(X,Z) + \alpha(Y,Z) \leq 2\pi - \alpha(X,Y). \tag{2}$$

Moreover,

- *Equality holds in the lower inequality if and only if $Z = aX + bY$ where $a, b \geq 0$.*
- *Equality holds in the upper inequality if and only if $Z = -(aX + bY)$ where $a, b \geq 0$.*

Proof. Since subtracting a constant from the three random variables X, Y, Z does not alter their correlation, we may assume that the three random variables have mean zero.

Note that in the probability space $\Omega, \mathcal{F}, P$ the three random variables, with zero mean, generate a three dimensional Hilbert space for the inner product $< X, Y >= E_P[XY]$. Since any finite dimensional Hilbert space is isometrically isomorphic to $\mathbb{R}^3$ equipped with the standard inner product, results established for vectors in three dimensional space with the standard inner product carry over mutando-mutandis to this three dimensional Hilbert space.

Next observe that normalizing the three random vectors to have variance one, also does not change their correlation. Thus, to establish the lower bound, using the isometric isomorphism, it suffices to establish the result for three *unit* vectors in $\mathbb{R}^3$. Let x, y, z denote the points on the surface unit sphere in $\mathbb{R}^3$, that correspond to the tips of the three vectors X, Y, Z.

Great arcs are geodesics on the surface of the unit sphere. Given two points on the unit sphere we can always join them with the shortest such geodesic (a fortiori the shortest such curve), whose length is less than or equal to π and equals π if and only if the two points are antipodal. The surface of the unit sphere equipped with this notion of distance between points becomes a metric space. Therefore the triangle inequality holds and says

$$d(x,z) \leq d(x,y) + d(y,z)$$

But, as we have seen $d(x,y) = \alpha_{x,y}$ the angle between the vectors X and Y measured in radians. This completes the proof.

de Finetti's upper inequality

In order to establish the upper inequality for three random variables X, Y, Z, we note that, applying the lower inequality to the triplet $(X, Y, -Z)$, we have

$$\alpha(X,Y) \leq \alpha(X,-Z) + \alpha(Y,-Z). \tag{3}$$

So

$$\alpha(X,Y) \le \pi - \alpha(X,Z) + \pi - \alpha(Y,Z)$$
$$\alpha(X,Y) + \alpha(X,Z) + \alpha(Z,Y) \le 2\pi$$

Equality is achieved when $Z = -(aX + bY), a, b > 0$, because, in this case, as we saw in the proof of the lower inequality, equality is achieved in the inequality (3) above. □

II.2. Higher dimensional version of de Finetti's inequality.

On page 9 of [6] de Finetti, referring to his inequalities, says "analogous constraints subsist for four or more random variables," and adds "... and it can be interesting to extend the previous research to the case of several random variables with equal pairwise correlations". Concerning the first part of his sentence, a straightforward extension of the lower bound is the following:

Theorem 2. *Let $X_1, \cdots, X_n$ be random variables. Then, for any $3 \le k \le n$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$, we have*

$$\alpha(X_{i_1}, X_{i_k}) \le \sum_{p=1}^{k-1} \alpha(X_{i_p}, X_{i_{p+1}}). \tag{4}$$

In particular, when $k = n$ the inequality reads

$$\alpha(X_1, X_n) \le \alpha(X_1, X_2) + \alpha(X_1, X_3) + \cdots + \alpha(X_{n-1}, X_n).$$

Proof. The result is true for $k = 3$, so by induction, assume it is true for $k - 1$. Then we have

$$\alpha(X_{i_1}, X_{i_k}) \le \alpha(X_{i_1}, X_{i_{k-1}}) + \alpha(X_{i_{k-1}}, X_{i_k}) \le \sum_{p=1}^{k-2} \alpha(X_{i_p}, X_{i_{p+1}}) + \alpha(X_{i_{k-1}}, X_{i_k})$$

□

Question : What is the appropriate generalization of the *upper* inequality to n random variables, where $n > 3$?

Did de Finetti have something specific in mind when he mentions "analogous constraints"? Interestingly, on the bottom of page 9, as well as on page 10 he then specializes the discussion to the case where all correlations are *equal*. On the other hand the generalization we established above of the lower inequality, contains no such restriction. So, in our opinion, the following problem is interesting and to our knowledge open.

Problem: *What is the optimal inequality or family of inequalities, that takes the place of the upper inequality when $n > 3$?*

In the special case where all angles are equal, de Finetti establishes the following theorem:

Theorem 3. *Given n random variables, with pairwise equal correlations ρ, the maximum angle the n variables can make with one another is*

$$\arccos\left(\frac{1}{1-n}\right).$$

He gives two arguments to support his claim. The first, beginning at the bottom of page 9, is an argument that identifies the minimum value of the correlation. Although this result is known to some experts (see for instance the work of Laurent and Gregory [3]) it does not appear to be as well known as it deserves. The first proof we are alluding to, along the way, establishes the equally interesting result, which is the natural analogue of the fact that two random variables with correlation -1 and the same means and variance, must be opposites of each other

$$X_1 = -X_2$$

or equivalently $X_1 + X_2 = 0$. In the case of n random variables, de Finetti proves that :

Theorem 4. *If $X_1, \cdots, X_n$ are n random variables that have pairwise equal correlation ρ, then the minimum value of this correlation is the value given by $\rho = \frac{1}{1-n}$, for which the following identity holds*

$$X_1 + X_2 + \cdots + X_n = 0 \ .$$

This theorem is established using an argument by contradiction, that however seems to heavily rely on the correlations all being equal. But, de Finetti offers a second argument, at the bottom of page 10, without proof: "n vectors of S_n, in order to form two by two the same angle (as great as possible), must have the same direction of the rays as an equilateral simplex that connects the center to its vertices. The angle α of two of these vectors is given by $\alpha = \arccos\left(\frac{1}{1-n}\right)$." This second proof will be discussed below, see Theorem 5 below, which is in some sense a generalization of de Finetti's remark, but may not be the only one.

Remark 1. *Perhaps the appropriate generation of de Finetti's angle theorem to higher dimensional case is through the cosines of the angles, that is, going back to the correlation coefficients themselves. Motivated by the desire to generalize the famous Fermat-Torricelli theorem for triangles to n simplices Abu-Saymeh and Hajja establish several interesting inequalities in* [1]. *The following is a direct consequence of Abu-Saymeh and Hajja* [1] *(Corollary 2, page 374) and partially explains why the cosines of the angles might correspond to the sought for generalization:*

Theorem 5. *Let $X_1, X_2, \cdots, X_n$ be n random variables with mean zero and variance one. Let ρ_{ij} be the correlation coefficient between X_i and X_j, i.e., $\rho_{ij} = \mathbb{E}[X_i X_j] = \cos(\alpha_{ij})$ for $1 \leq i, j \leq n$. Then the ρ_{ij}'s satisfy, for any $2 \leq k \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$,*

$$\sum_{1 \leq a < b \leq k} \rho_{i_a i_b} \geq -\frac{k}{2}.$$

In particular, when $k = n$, we have $\rho_{12} + \rho_{13} + \cdots + \rho_{(n-1)n} \geq \frac{-n}{2}$.

Note that the bound is attained by taking $\rho_{ij} = \frac{1}{1-n}$ for all $i \neq j$. In this case, the angles α_{ij} are all the same and have the value $\arccos\left(\frac{1}{1-n}\right)$ as in de Finetti's example and their sums satisfy, for any $2 \leq k \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $\sum_{1 \leq a < b \leq k} \alpha_{i_a i_b} = \frac{k(k-1)}{2} \arccos\left(\frac{1}{1-k}\right)$. Note that when $n = 3$, $\frac{n(n-1)}{2} \arccos\left(\frac{1}{1-n}\right) = 3 \cdot \arccos\left(-\frac{1}{2}\right) = 2\pi$ which recovers the upper bound in de Finetti's angle theorem. For the case $n = 4$, the angles have the common value $\arccos\left(-\frac{1}{3}\right)$ which is exactly the value of the angle between *any two of the radial lines of a standard tetrahedron in* $\mathbb{R}^3$*. Therefore the vertices of the vector representations of the random variables coincide with the vertices of a tetrahedron as shown in Figure 1.*

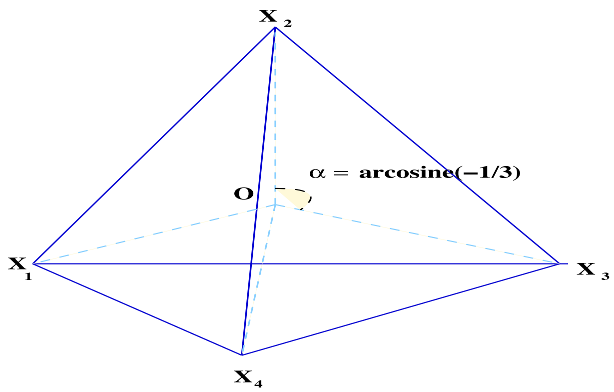

FIGURE 1. Tetrahedron and angle of radial lines in tetrahedron when $n = 4$.

III. EXTREME CORRELATION MATRICES

III.1. Spotting matrices that are not correlation matrices.

Suppose we are given an $n \times n$ matrix A. We wish to check that this matrix is a correlation matrix. In order for it to be a correlation matrix we need

- A is symmetric
- A has 1's along the diagonal
- A is positive semidefinite.

All of the above are trivial to check *except* for the last one. As an illustration, let us consider the following numerical 3×3 example.

$$A = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 1 & \rho_{2,3} \\ \rho_{1,3} & \rho_{2,3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & .866 & .61 \\ .866 & 1 & .9397 \\ .61 & .9397 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} &\alpha(X_1, X_3) = \arccos(.61) = 52.4^\circ \\ &> \arccos(X_1, X_2) + \arccos(X_2, X_3) \\ &= \arccos(.866) + \arccos(.9397) = 30^\circ + 20^\circ = 50^\circ, \end{aligned}$$

contradicting de Finetti's lower bound $\alpha(X_1, X_3) \leq \alpha(X_1, X_2) + \alpha(X_2, X_3)$. In fact we find that

$$\text{Eigenvalues} \begin{pmatrix} 1 & .866 & .61 \\ .866 & 1 & .9397 \\ .61 & .9397 & 1 \end{pmatrix} = \begin{pmatrix} 2.6335 \\ 0.3687 \\ -0.0022 \end{pmatrix}$$

so the matrix is not positive definite. But recall that $\arccos(\rho_{1,2}) = 30°$, $\arccos(\rho_{2,3}) = 20°$. Therefore the threshhold for the value of $\rho_{1,3}$ is $\cos(50°) = .6428$. So let's replace $\rho_{1,3}$ by $.6428$ and see what happens?

$$\rho_{1,3} \to .6428 \qquad (ie.\text{ replace } \rho_{1,3})$$
$$\text{Eigenvalues} \begin{pmatrix} 1 & .866 & .6428 \\ .866 & 1 & .9397 \\ .6428 & .9397 & 1 \end{pmatrix} = \begin{pmatrix} 0.0000 \\ 0.3615 \\ 2.6385 \end{pmatrix}$$

so the boundary of positive definiteness is reached *exactly* when

$$\arccos(\rho_{1,3}) = \arccos(\rho_{1,2}) + \arccos(\rho_{2,3}),$$

i.e., exactly when there is equality in de Finetti's lower bound.

To our knowledge, these kind of inequalities, (and the n entry generalization (4)) which put restrictions on the relative sizes of entries in a correlation matrix, are new in that setting since they do not (necessarily) involve directly the *principle minors*, but can be applied to arbitrary sub-blocks. For instance, consider a 4×4 matrix given by

$$\begin{pmatrix} a_{11} & a_{12} & \boxed{a_{13}} & \boxed{a_{14}} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & \boxed{a_{34}} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

One of the de Finetti's inequalities in (4) reads

$$\arccos(a_{14}) \leq \arccos(a_{13}) + \arccos(a_{34})$$

which does not involve directly any principle minors.

Consider next, de Finetti's choice of an $n \times n$ correlation matrix, the matrix

$$A^\rho = \begin{pmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \dots & 1 \end{pmatrix} \tag{5}$$

with $\rho = \frac{1}{1-n}$. We would like to provide a direct proof that this matrix is on the boundary of the set of correlation matrices by deriving a formula for the eigenvalues and eigenvectors of such a matrix [4]. Note that the complete set of eigenvalues of A^ρ is

$$\{1-\rho, 1-\rho, \dots, \dots, 1-\rho, 1+(n-1)\rho\}.$$

From this we see directly that the matrix A^ρ ceases to be positive definite *exactly when* $\rho = -\frac{1}{n-1}$. The eigenvectors associated to the above matrix do not depend on ρ. They are given as follows

[4] These formulas may well be in the literature, but we did not succeed in finding them.

- The eigenvector associated to the eigenvalue $1-\rho$ are of the form

$$[-1,0,\cdots,0,\cdots,+1,\cdots,0]^T, \qquad n-1 \text{ ways to place a single "+ 1".}$$

- The eigenvector corresponding to the eigenvalue $(1+(n-1)\rho)$ is of the form

$$[1,1,\cdots,1]^T.$$

Above, we have shown that the $n \times n$ correlation matrix with all off-diagonal values equal to $\frac{1}{1-n}$ is on the boundary of the convex set of correlation matrices. The set of all correlation matrices is a convex set. This can be seen as follows: If C_1 and C_2 are positive definite matrices, so are the convex combinations $(1-\lambda)C_1 + \lambda C_2$ of C_1 and C_2 for all $\lambda \in [0,1]$ since

$$\langle x, (1-\lambda)C_1 + \lambda C_2)x\rangle = (1-\lambda)\langle x, C_1 x\rangle + \lambda\langle x, C_2 x\rangle \geq 0 \qquad \text{for all } x \in \mathbb{R}^n,$$

where $\langle x, y\rangle$ is the dot product for $\mathbb{R}^n$. Second, if the diagonal entries of C_1 and C_2 are all $+1$'s then this is also the case for $(1-\lambda)C_1 + \lambda C_2$. A natural question to ask concerns the structure of the boundary of the convex set of correlation matrices. What does this boundary look like? This is still a topic of ongoing research. de Finetti's example $A^{\frac{1}{1-n}}$ and A^1 are examples of matrices which are *boundary points* of the set of correlation matrices, with rank $n-1$ and 1 respectively. Concerning the general structure of the latter here is an example of what is known. We begin with a definition:

Definition 2. *An element x in a convex set S is an extreme point if $x = ty + (1-t)z$ for $y, z \in S$ and $0 < t < 1$ implies y = z = x, that is, if x can be a convex combination of points of S in only trivial ways.*

Theorem 6. *There exist extreme points of rank k in the set of correlation matrices if and only if $k^2 + k \leq 2n$.*

This theorem is proved, for instance in Li and Tam [12]. Therefore de Finetti's choice of correlation matrix $A^{\frac{1}{1-n}}$ cannot be an extreme point if $n > 3$.

III.2. Faces and Vertices of set of correlation matrices.

- The convex set of the collection of all $n \times n$ correlation matrices is called the *elliptope*. It is non polyhedral and has both smooth and non smooth boundaries.
- Vertices of set of correlation matrices are the extreme points of this set possessing a full dimensional normal cone.
- Vertices have been fully characterized. They are the set of correlation matrices of rank one of the form, for any $0 \leq m \leq n$,

$$\begin{pmatrix} \mathbf{1}_{m\times m} & -\mathbf{1}_{m\times(n-m)} \\ -\mathbf{1}_{(n-m)\times m} & \mathbf{1}_{(n-m)\times(n-m)} \end{pmatrix},$$

where $\mathbf{1}_{k\times l}$ is the k by l matrix with all the entries equal to 1.

- **(Not a vertex)**
  In particular, de Finetti's

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

is not a vertex since in this case the vertices are

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Also note that in this example the de Finetti's matrix is not a convex combination of the above three vertices.

IV. DID DE FINETTI DISCOVER RANK CORRELATION?

At the end of his article, de Finetti says "One final remark is in order. Its purpose is to clarify that a concordance index can have a sign different from that of r, and to suggest what seems to me the simplest and most intrinsically meaningful index of concordance (and one that has not yet been considered, as far as I know)." The date of the published assertion "not yet considered, as far as I know ", is 1937. The date of Kendall's paper on rank correlation [11] is 1938. This would appear to give de Finetti precedence.

IV.1. De Finetti's definition.

De Finetti defines his notion of rank correlation as follows:

$$c = \int_D dF(x,y)dF(\xi,\eta)dxdyd\xi d\eta, \tag{6}$$

where the pairs (x, y) and (ξ, η) are independently chosen and have the same joint distributions $F(x, y)$ and where D is the subset of R^{2n} defined by $x > \xi, y > \eta$. De Finetti's (6) can clearly be written in the form

$$2\int_{\mathrm{R}^4} \mathbf{1}_{\xi<x}\mathbf{1}_{\eta<y} dF(x,y)dF(\xi,\eta)dxdyd\xi d\eta,$$

and, as is well known and easy to see, the latter can be written

$$RC^{DF} = 2\int_{\mathrm{R}^4} C(u,v)dC(u,v)\ ,$$

where $C(u, v)$ is the copula associated to the joint distribution. Compare this with the definition of rank correlation given by Kendall [11], see for instance [8]:

$$RC^{Ken} = 4\int_{\mathrm{R}^4} C(u,v)dC(u,v) - 1$$

so we have

$$RC^{DF} = \frac{RC^{Ken}+1}{2}\ .$$

Now the interesting thing is that de Finetti noticed that his definition of rank correlation has the value 1 for comonotonic distributions. This is *also the case* for the Kendall rank correlation. On the other hand, the Kendall rank correlation is equal to -1 for antimonotonic random variables and in this case, as de Finetti points out, his rank correlation equals zero.

IV.2. **An example of de Finetti.**

We also would like to supply a little detail, backing up the illustration given, also on page 19, by de Finetti, of a pair of random variables for which the correlation of (X, Y) can be of different sign than the correlation of $(g_a(X), Y)$ for an appropriately chosen, monotonically increasing function $f(X)$. The function de Finetti proposes is

$$g_a(X) = X + \frac{a-1}{2}X(X+1).$$

The distribution of the pair (X, Y) is as follows:

$$(X,Y) = \begin{cases} (-1,1) & \text{with probability } \frac{1}{4}, \\ (0,-1) & \text{with probability } \frac{1}{2}, \\ (a,1) & \text{with probability } \frac{1}{4}. \end{cases}$$

Note that

$$g_a(-1) = -1, \quad g_a(0) = 0 \quad \text{and} \quad g_a(a) = \frac{a}{2}(a^2+1).$$

For each of the three values $g_a(\cdot)$ either remains the same or increases when $a \in (0, +\infty)$.

$$E[g_a(X)Y] = -\frac{1}{4} + \frac{a^3+a}{8} = \rho(g_a(X), Y)\sigma_{g_a(X)}\sigma_Y$$

but the latter changes sign for $a = 1$. So the one parameter family of pairs $(g_a(X), X)$ constitutes an example of how the effect of applying a monotone function to one or more of the two random variables X and Y can change not only the size but also the *sign* of the correlation.

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Elie Cartan's proof of the angle theorem in 3 -space

In Cartan's treatment (page 30 [4]) the angle theorem emerges in the following context: What are the conditions on the angles betwee three vectors e_1, e_2, e_3 in order for them to constitute a valid oblique coordinate system in $\mathbb{R}^3$.

If we express the length of a vector V with respect to this coordinate system:

$$Xe_1 + Ye_2 + Ze_3, \tag{7}$$

and consider it's length, we get that

$$|V|^2 = X^2 + Y^2 + Z^2 + 2\cos\lambda YZ + 2\cos\mu(Z,X) + 2\cos\nu XY,$$

where $\cos\lambda = e_2 \cdot e_3, \cos\mu = e_1 \cdot e_3, \cos\nu = e_1 \cdot e_2$ are the cosines of the corresponding angles between these unit vectors.

Since, $V^2 \geq 0$, this leads to restrictions on the angles. Of course, from the point of view of the angle theorem, $\vec{X} = Xe_1, \vec{Y} = Ye_2, \vec{Z} = Ze_3$ are the vectors to which we apply the angle theorem and then (7) is the quadratic form associated to the 3×3 Grammian matrix, corresponding to the three vectors $\bar{X}, \bar{Y}$ and $\bar{Z}$.

Cartan's approach now consists in re-expressing (7) as a sum of squares as follows:

$$(X + Y\cos\nu + Z\cos\mu)^2 + (Y\sin\nu + Z\frac{\cos\lambda - \cos\mu\cos\nu}{\sin\nu})^2 + \frac{\sin^2\mu\sin^2\nu - (cos\lambda - \cos\mu\cos\nu)^2}{\sin^2\nu}Z^2 \tag{8}$$

Thus we seek conditions on the angles that ensure that

$$(cos\lambda - \cos\mu\cos\nu)^2 - \sin^2\mu\sin^2\nu < 0$$

This may be equivalently expressed as

$$[\cos(\mu+\lambda) - \cos\lambda]\,[\cos\lambda - \cos(\mu-\lambda)] > 0$$

Now, without loss of generality assume λ is the largest of the three angles (since otherwise we use an expression like (9) with the roles of the variables exchanged) and then the second factor of this inequality is itself negative and the condition we seek is

$$\cos\lambda > \cos(\mu+\nu)$$

or, equivalently

$$\lambda < \mu+\nu < 2\pi-\lambda$$

□