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A GEOMETRIC INEQUALITY OF THE GENERALIZED ERDÖS-MORDELL TYPE

YU-DONG WU, CHUN-LEI YU, AND ZHI-HUA ZHANG

DEPARTMENT OF MATHEMATICS
ZHEJIANG XINCHANG HIGH SCHOOL
SHAOXING 312500, ZHEJIANG
PEOPLE'S REPUBLIC OF CHINA
yudong.wu@yahoo.com.cn

seetill@126.com

DEPARTMENT OF MATHEMATICS
SHILI SENIOR HIGH SCHOOL IN ZIXING
CHENZHOU 423400, HUNAN
PEOPLE'S REPUBLIC OF CHINA
zxzh1234@163.com

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Dedicated to Mr. Ting-Feng Dong on the occasion of his 55th birthday.

ABSTRACT. In this short note, we solve an interesting geometric inequality problem relating to two points in triangle posed by Liu [7], and also give two corollaries.

Key words and phrases: Geometric inequality, triangle, Erdös-Mordell inequality, Hayashi's inequality, Klamkin's inequality.

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1. INTRODUCTION AND MAIN RESULTS

Let P, Q be two arbitrary interior points in $\triangle ABC$, and let a, b, c be the lengths of its sides, S the area, R the circumradius and r the inradius, respectively. Denote by R_1, R_2, R_3 and r_1, r_2, r_3 the distances from P to the vertices A, B, C and the sides BC, CA, AB, respectively. For the interior point Q, define D_1, D_2, D_3 and d_1, d_2, d_3 similarly (see Figure 1.1).

The following well-known and elegant result (see [1, Theorem 12.13, pp.105])

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \tag{1.1}$$

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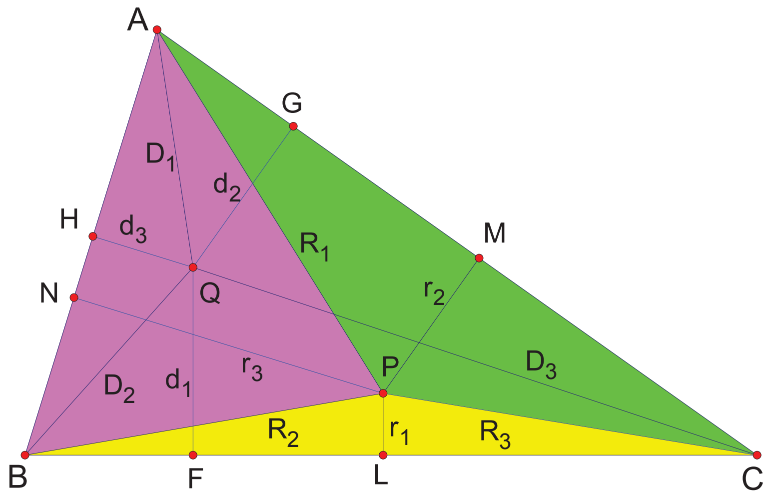

Figure 1.1:

concerning R_i and r_i $(i = 1, 2, 3)$ is called the **Erdös-Mordell inequality**. Inequality (1.1) was generalized as follows [9, Theorem 15, pp. 318]:

$$R_1x^2 + R_2y^2 + R_3z^2 \geq 2(r_1yz + r_2zx + r_3xy) \tag{1.2}$$

for all $x, y, z \geq 0$.

And the special case $n = 2$ of [9, Theorem 8, pp. 315-316] states that

$$\sqrt{R_1D_1} + \sqrt{R_2D_2} + \sqrt{R_3D_3} \geq 2\left(\sqrt{r_1d_1} + \sqrt{r_2d_2} + \sqrt{r_3d_3}\right), \tag{1.3}$$

which also extends (1.1).

Recently, for all $x, y, z \geq 0$, J. Liu [8, Proposition 2] obtained

$$\sqrt{R_1D_1}x^2 + \sqrt{R_2D_2}y^2 + \sqrt{R_3D_3}z^2 \geq 2\left(\sqrt{r_1d_1}yz + \sqrt{r_2d_2}zx + \sqrt{r_3d_3}xy\right) \tag{1.4}$$

which generalizes inequality (1.3).

In 2008, J. Liu [7] posed the following interesting geometric inequality problem.

Problem 1.1. For a triangle ABC and two arbitrary interior points P, Q, prove or disprove that

$$R_1D_1 + R_2D_2 + R_3D_3 \geq 4(r_2r_3 + r_3r_1 + r_1r_2). \tag{1.5}$$

We will solve Problem 1.1 in this paper.

From inequality (1.5), we get

$$R_1D_1 + R_2D_2 + R_3D_3 \geq 4(d_2d_3 + d_3d_1 + d_1d_2).$$

Hence, we obtain the following result.

Corollary 1.1. *For any $\triangle ABC$ and two interior points P, Q, we have*

$$R_1D_1 + R_2D_2 + R_3D_3 \geq 4\sqrt{(r_2r_3 + r_3r_1 + r_1r_2)(d_2d_3 + d_3d_1 + d_1d_2)}. \tag{1.6}$$

From inequality (1.5), and by making use of an inversion transformation [2, pp.48-49] (see also [3, pp.108-109]) in the triangle, we easily get the following result.

Corollary 1.2. *For any $\triangle ABC$ and two interior points P, Q, we have*

$$\frac{D_1}{R_1 r_1}+\frac{D_2}{R_2 r_2}+\frac{D_3}{R_3 r_3}\geq 4\cdot|PQ|\cdot\left(\frac{1}{R_1R_2}+\frac{1}{R_2R_3}+\frac{1}{R_3R_1}\right). \tag{1.7}$$

Remark 1. With one of Liu's theorems [8, Theorem 3], inequality (1.2) implies (1.4). However, we cannot determine whether inequalities (1.1) and (1.3) imply inequality (1.5) or inequality (1.6), or inequalities (1.5) and (1.3) imply inequality (1.1).

2. PRELIMINARY RESULTS

Lemma 2.1. *We have for any $\triangle ABC$ and an arbitrary interior point P that*

$$aR_1 \geq br_2 + cr_3, \tag{2.1}$$

$$bR_2 \geq cr_3 + ar_1, \tag{2.2}$$

$$cR_3 \geq ar_1 + br_2. \tag{2.3}$$

Proof. Inequalities (2.1) – (2.3) directly follow from the obvious fact

$$ar_1 + br_2 + cr_3 = 2S,$$

the formulas of the altitude

$$h_a = \frac{2S}{a}, \qquad h_b = \frac{2S}{b}, \qquad h_c = \frac{2S}{c},$$

and the evident inequalities [11]

$$R_1 + r_1 \geq h_a,$$
$$R_2 + r_2 \geq h_b,$$
$$R_3 + r_3 \geq h_c.$$

□

Lemma 2.2 ([4, 5])**.** *For real numbers $x_1, x_2, x_3, y_1, y_2, y_3$ such that*

$$x_1x_2 + x_2x_3 + x_3x_1 \geq 0$$

and

$$y_1y_2 + y_2y_3 + y_3y_1 \geq 0,$$

the inequality

$$(y_2+y_3)x_1 + (y_3+y_1)x_2 + (y_1+y_2)x_3 \geq 2\sqrt{(x_1x_2+x_2x_3+x_3x_1)(y_1y_2+y_2y_3+y_3y_1)} \tag{2.4}$$

holds, with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}$.

Lemma 2.3 (Hayashi's inequality, [9, pp.297, 311]). *For any* $\triangle ABC$ *and an arbitrary point* P*, we have*

$$\frac{R_1R_2}{ab} + \frac{R_2R_3}{bc} + \frac{R_3R_1}{ca} \geq 1. \tag{2.5}$$

Equality holds if and only if P *is the orthocenter of the acute triangle* ABC *or one of the vertexes of triangle* ABC.

Lemma 2.4 (Klamkin's inequality, [6, 10]). *Let* A, B, C *be the angles of* $\triangle ABC$. *For positive real numbers* u, v, w*, the inequality*

$$u \sin A + v \sin B + w \sin C \leq \frac{1}{2}(uv + vw + wu)\sqrt{\frac{u+v+w}{uvw}} \tag{2.6}$$

holds, with equality if and only if $u = v = w$ *and* $\triangle ABC$ *is equilateral.*

Lemma 2.5. *For any* $\triangle ABC$ *and an arbitrary interior point* P*, we have*

$$\sqrt{abr_1r_2 + bcr_2r_3 + car_3r_1} \geq 2(r_2r_3 + r_3r_1 + r_1r_2). \tag{2.7}$$

Proof. Suppose that the actual barycentric coordinates of P are (x, y, z), Then $x =$ area of $\triangle PBC$, and therefore

$$\frac{x}{x+y+z} = \frac{\text{area}(\triangle PBC)}{S} = \frac{r_1 a}{bc \sin A} = \frac{2r_1}{bc} \cdot \frac{a}{2\sin A} = \frac{2Rr_1}{bc}.$$

Therefore

$$r_1 = \frac{bc}{2R} \cdot \frac{x}{x+y+z},$$
$$r_2 = \frac{ca}{2R} \cdot \frac{y}{x+y+z},$$
$$r_3 = \frac{ab}{2R} \cdot \frac{z}{x+y+z}.$$

Thus, inequality (2.7) is equivalent to

$$\frac{abc}{2R(x+y+z)}\sqrt{xy+yz+zx} \geq \frac{abc}{R(x+y+z)^2}\left(\frac{a}{2R}yz + \frac{b}{2R}zx + \frac{c}{2R}xy\right) \tag{2.8}$$

or

$$\frac{1}{2}(x+y+z)\sqrt{xy+yz+zx} \geq yz\sin A + zx \sin B + xy \sin C. \tag{2.9}$$

Inequality (2.9) follows from Lemma 2.4 by taking

$$(u, v, w) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right).$$

This completes the proof of Lemma 2.5. ☐

3. SOLUTION OF PROBLEM 1.1

Proof. In view of Lemmas 2.1 – 2.3 and 2.5, we have that

$$\begin{aligned}
&R_1D_1 + R_2D_2 + R_3D_3 \\
&\quad = aR_1 \cdot \frac{D_1}{a} + bR_2 \cdot \frac{D_2}{b} + cR_3 \cdot \frac{D_3}{c} \\
&\quad \geq (br_2 + cr_3) \cdot \frac{D_1}{a} + (cr_3 + ar_1) \cdot \frac{D_2}{b} + (ar_1 + br_2) \cdot \frac{D_3}{c} \\
&\quad \geq 2\sqrt{(abr_1r_2 + bcr_2r_3 + car_3r_1)\left(\frac{D_1D_2}{ab} + \frac{D_2D_3}{bc} + \frac{D_3D_1}{ca}\right)} \\
&\quad \geq 2\sqrt{abr_1r_2 + bcr_2r_3 + car_3r_1} \\
&\quad \geq 4(r_2r_3 + r_3r_1 + r_1r_2).
\end{aligned}$$

The proof of inequality (1.5) is thus completed. □

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