A CONVOLUTION APPROACH ON PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract: In this paper, we determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)*\psi(z)}{f_n(z)*\psi(z)}\right\}$ and $\operatorname{Re}\left\{\frac{f_n(z)*\psi(z)}{f(z)*\psi(z)}\right\}$. We extend the results of ([1] – [5]) and correct the conditions for the results of Frasin [2, Theorem 2.7], [1, Theorem 2], Rosy et al. [4, Theorems 4.2 and 4.3], as well as Raina and Bansal [3, Theorem 6.2].

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. A function $f(z)$ in S is said to be starlike of order α $(0 \le \alpha < 1)$, denoted by $S^*(\alpha)$, if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U),$$

and is said to be convex of order α $(0 \le \alpha < 1)$, denoted by $K(\alpha)$, if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U).$$

Let $T^*(\alpha)$ and $C(\alpha)$ be subclasses of $S^*(\alpha)$ and $K(\alpha)$, respectively, whose functions are of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \ge 0. \tag{1.2}$$

A sufficient condition for a function of the form (1.1) to be in $S^*(\alpha)$ is that

$$\sum_{k=2}^{\infty} (k-\alpha)|a_k| \le 1-\alpha \tag{1.3}$$

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and to be in $K(\alpha)$ is that

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| \le 1-\alpha. \tag{1.4}$$

For functions of the form (1.2), Silverman [6] proved that the above sufficient conditions are also necessary.

Let $\phi(z) \in S$ be a fixed function of the form

$$\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k, \quad (c_k \ge c_2 > 0, k \ge 2). \tag{1.5}$$

Very recently, Frasin [2] defined the class $H_\phi(c_k, \delta)$ consisting of functions $f(z)$, of the form (1.1) which satisfy the inequality

$$\sum_{k=2}^{\infty} c_k |a_k| \le \delta, \tag{1.6}$$

where $\delta > 0$.

He shows that for suitable choices of c_k and δ, $H_\phi(c_k, \delta)$ reduces to various known subclasses of S studied by various authors (for a detailed study, see [2] and the references therein).

In the present paper, we determine sharp lower bounds for $\operatorname{Re}\left\{\frac{f(z)*\psi(z)}{f_n(z)*\psi(z)}\right\}$ and $\operatorname{Re}\left\{\frac{f_n(z)*\psi(z)}{f(z)*\psi(z)}\right\}$, where

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k$$

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is a sequence of partial sums of a function

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

belonging to the class $H_\phi(c_k, \delta)$ and

$$\psi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k, \quad (\lambda_k \geq 0)$$

is analytic in open unit disc U and the operator "*" stands for the Hadamard product or convolution of two power series, which is defined for two functions $f, g \in A$, where $f(z)$ and $g(z)$ are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

as

$$(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In this paper, we extend the results of Silverman [5], Frasin ([1], [2]) Rosy et al. [4] as well as Raina and Bansal [3] and we point out that some conditions on the results of Frasin ([2, Theorem 2.7], [1, Theorem 2]), Rosy et al. ([4, Theorem 4.2, 4.3]), Raina and Bansal ([3, Theorem 6.2]) are incorrect and we correct them. It is seen that this study not only gives a particular case of the results ([1] – [5]) but also gives rise to several new results.

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2. Main Results

Theorem 2.1. *If* $f \in H_\phi(c_k, \delta)$ *and* $\psi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k$, $\lambda_k \geq 0$, *then*

$$\operatorname{Re}\left\{\frac{f(z) * \psi(z)}{f_n(z) * \psi(z)}\right\} \geq \frac{c_{n+1} - \lambda_{n+1}\delta}{c_{n+1}} \qquad (z \in U) \tag{2.1}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z) * \psi(z)}{f(z) * \psi(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + \lambda_{n+1}\delta} \qquad (z \in U), \tag{2.2}$$

where

$$c_k \geq \begin{cases} \lambda_k \delta & \text{if} \quad k = 2, 3, \dots, n, \\ \frac{\lambda_k c_{n+1}}{\lambda_{n+1}} & \text{if} \quad k = n+1, n+2, \dots \end{cases}$$

The results (2.1) *and* (2.2) *are sharp with the function given by*

$$f(z) = z + \frac{\delta}{c_{n+1}} z^{n+1}, \tag{2.3}$$

where $0 < \delta \leq \frac{c_{n+1}}{\lambda_{n+1}}$.

Proof. Define the function $\omega(z)$ by

$$\begin{aligned} \frac{1+\omega(z)}{1-\omega(z)} &= \frac{c_{n+1}}{(\lambda_{n+1})\delta}\left[\frac{f(z) * \psi(z)}{f_n(z) * \psi(z)} - \left(\frac{c_{n+1} - \delta\lambda_{n+1}}{c_{n+1}}\right)\right] \\ &= \frac{1 + \sum_{k=2}^{n} \lambda_k a_k z^{k-1} + \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}{1 + \sum_{k=2}^{n} \lambda_k a_k z^{k-1}}. \end{aligned} \tag{2.4}$$

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It suffices to show that $|\omega(z)| \leq 1$. Now, from (2.4) we can write

$$\omega(z) = \frac{\frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} \lambda_k a_k z^{k-1} + \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}.$$

Hence we obtain

$$|\omega(z)| \leq \frac{\frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k|}{2 - 2\sum_{k=2}^{n} \lambda_k |a_k| - \frac{c_{n+1}}{(\lambda_{n+1})\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k|}.$$

Now $|\omega(z)| \leq 1$ if

$$2\frac{c_{n+1}}{(\lambda_{n+1})\,\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k| \leq 2 - 2\sum_{k=2}^{n} \lambda_k |a_k|$$

or, equivalently,

$$\sum_{k=2}^{n} \lambda_k |a_k| + \frac{c_{n+1}}{(\lambda_{n+1})\,\delta} \sum_{k=n+1}^{\infty} \lambda_k |a_k| \leq 1. \tag{2.5}$$

It suffices to show that the L.H.S. of (2.5) is bounded above by $\sum_{k=2}^{\infty} \frac{c_k}{\delta} |a_k|$, which is equivalent to

$$\sum_{k=2}^{n} \left(\frac{c_k - \delta\lambda_k}{\delta}\right) |a_k| + \sum_{k=n+1}^{\infty} \left(\frac{\lambda_{n+1}c_k - c_{n+1}\lambda_k}{\lambda_{n+1}\delta}\right) |a_k| \geq 0. \tag{2.6}$$

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To see that the function given by (2.3) gives a sharp result we observe that for $z = re^{i\pi/n}$

$$\frac{f(z) * \psi(z)}{f_n(z) * \psi(z)} = 1 + \frac{\delta}{c_{n+1}}\lambda_{n+1}z^n \to 1 - \frac{\delta}{c_{n+1}}\lambda_{n+1}$$
$$= \frac{c_{n+1} - \delta\lambda_{n+1}}{c_{n+1}}$$

when $r \to 1^-$.

To prove the second part of this theorem, we write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{n+1} + \lambda_{n+1}\delta}{\lambda_{n+1}\delta}\left[\frac{f_n(z) * \psi(z)}{f(z) * \psi(z)} - \left(\frac{c_{n+1}}{c_{n+1} + \lambda_{n+1}\delta}\right)\right]$$
$$= \frac{1 + \sum_{k=2}^{n} \lambda_k a_k z^{k-1} - \frac{c_{n+1}}{\lambda_{n+1}\delta}\sum_{k=n+1}^{\infty} \lambda_k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} \lambda_k a_k z^{k-1}},$$

where

$$|\omega(z)| \le \frac{\left(\frac{c_{n+1}+\lambda_{n+1}\delta}{\lambda_{n+1}\delta}\right)\sum_{k=n+1}^{\infty} \lambda_k |a_k|}{2 - 2\sum_{k=2}^{n} \lambda_k |a_k| - \frac{c_{n+1}-\lambda_{n+1}\delta}{\lambda_{n+1}\delta}\sum_{k=n+1}^{\infty} \lambda_k |a_k|} \le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{n} \lambda_k |a_k| + \frac{c_{n+1}}{(\lambda_{n+1})\,\delta}\sum_{k=n+1}^{\infty} \lambda_k |a_k| \le 1.$$

Making use of (1.6), we get (2.6). Finally, equality holds in (2.2) for the function $f(z)$ given by (2.3). $\square$

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Taking $\psi(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following result given by Frasin in [2].

Corollary 2.2. *If $f \in H_\phi(c_k, \delta)$, then*

(2.7) $$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1}-\delta}{c_{n+1}} \qquad (z \in U)$$

and

(2.8) $$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+\delta} \qquad (z \in U),$$

where

$$c_k \geq \begin{cases} \delta & \textit{if} \quad k = 2, 3, \dots, n, \\ c_{n+1} & \textit{if} \quad k = n+1, n+2, \dots \end{cases}$$

The results (2.7) and (2.8) are sharp with the function given by (2.3).

If we put $\psi(z) = \frac{z}{(1-z)^2}$ in Theorem 2.1, we obtain:

Corollary 2.3. *If $f \in H_\phi(c_k, \delta)$, then*

(2.9) $$\operatorname{Re}\frac{f'(z)}{f_n'(z)} \geq \frac{c_{n+1}-(n+1)\delta}{c_{n+1}} \qquad (z \in U)$$

and

(2.10) $$\operatorname{Re}\frac{f_n'(z)}{f'(z)} \geq \frac{c_{n+1}}{c_{n+1}+(n+1)\delta} \qquad (z \in U),$$

where

(2.11) $$c_k \geq \begin{cases} k\delta & \textit{if} \quad k = 2, 3, \dots, n, \\ \frac{kc_{n+1}}{n+1} & \textit{if} \quad k = n+1, n+2, \dots \end{cases}$$

The results (2.9) and (2.10) are sharp with the function given by (2.3).

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Remark 1. Frasin has shown in Theorem 2.7 of [2] that for $f \in H_\phi(c_k, \delta)$, inequalities (2.9) and (2.10) hold with the condition

$$c_k \geq \begin{cases} k\delta & \text{if} \quad k = 2, 3, \dots, n, \\ k\delta\left(1 + \frac{c_{n+1}}{n+1}\right) & \text{if} \quad k = n+1, n+2, \dots \end{cases} \tag{2.12}$$

However, it can be easily seen that the condition (2.12) for $k = n+1$ gives

$$c_{n+1} \geq (n+1)\,\delta\left(1 + \frac{c_{n+1}}{(n+1)\,\delta}\right)$$

or, equivalently $\delta \leq 0$, which contradicts the initial assumption $\delta > 0$. So Theorem 2.7 of [2] does not seem suitable with the condition (2.12), but our condition (2.11) remedies this problem.

Taking $\psi(z) = \frac{z}{1-z}$, $c_k = \frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k}$, where $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$ and $\delta = 1$ in Theorem 2.1, we obtain the following result given by Rosy et al. in [4].

Corollary 2.4. *If f is of the form (1.1) and satisfies the condition $\sum_{k=2}^{\infty} c_k |a_k| \leq 1$, where $c_k = \frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k}$, $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1}-1}{c_{n+1}} \qquad (z \in U) \tag{2.13}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+1} \qquad (z \in U). \tag{2.14}$$

The results (2.13) and (2.14) are sharp with the function given by

$$f(z) = z + \frac{1}{c_{n+1}} z^{n+1}. \tag{2.15}$$

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Taking

$$\psi(z)=\frac{z}{(1-z)^2},\qquad c_k=\frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k},$$

where $\lambda\geq 0$, $\beta\geq 0$, $-1\leq\alpha<1$ and $\delta=1$ in Theorem 2.1, we obtain

Corollary 2.5. *If f is of the form (1.1) and satisfies the condition*

$$\sum_{k=2}^{\infty}c_k|a_k|\leq 1,$$

where

$$c_k=\frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\binom{k+\lambda-1}{k},\qquad (\lambda\geq 0,\ \beta\geq 0,\ -1\leq\alpha<1),$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\}\geq\frac{c_{n+1}-(n+1)}{c_{n+1}}\qquad (z\in U) \tag{2.16}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\}\geq\frac{c_{n+1}}{c_{n+1}+(n+1)}\qquad (z\in U), \tag{2.17}$$

where

$$c_k\geq\begin{cases} k & \text{if } k=2,3,\dots,n,\\ \frac{kc_{n+1}}{n+1} & \text{if } k=n+1,n+2,\dots \end{cases} \tag{2.18}$$

The results (2.16) and (2.17) are sharp with the function given by (2.15).

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Remark 2. Rosy et al. has obtained inequalities (2.16) & (2.17) in Theorem 4.2 & 4.3 of [4] without any restriction on c_k. However, when we critically observe the proof of Theorem 4.2 we find that inequality (4.16) of [4, Theorem 4.2]

$$\sum_{k=2}^{n}(c_k - k)\,|a_k| + \sum_{k=n+1}^{\infty}\left(c_k - \frac{c_{n+1}k}{n+1}\right)|a_k| \geq 0$$

cannot hold if condition (2.18) does not occur. So Theorems 4.2 & 4.3 of [4] are not proper and proper results are mentioned in Corollary 2.5.

Taking $\psi(z) = \frac{z}{1-z}$, $c_k = \lambda_k - \alpha\mu_k$, $\delta = 1-\alpha$, where $0 \leq \alpha < 1$, $\lambda_k \geq 0$, $\mu_k \geq 0$, and $\lambda_k \geq \mu_k$ $(k \geq 2)$ in Theorem 2.1, we obtain the following result given by Frasin in [1].

Corollary 2.6. *If f is of the form* (1.1) *and satisfies the condition*

$$\sum_{k=2}^{\infty}(\lambda_k - \alpha\mu_k)\,|a_k| \leq 1-\alpha,$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \qquad (z \in U) \tag{2.19}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{\lambda_{n+1} - \alpha\mu_{n+1} + 1 - \alpha} \qquad (z \in U), \tag{2.20}$$

where

$$\lambda_k - \alpha\mu_k \geq \begin{cases} 1-\alpha & \textit{if} \quad k = 2,3,\ldots,n, \\ \lambda_{n+1} - \alpha\mu_{n+1} & \textit{if} \quad k = n+1, n+2, \ldots \end{cases}$$

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The results (2.19) and (2.20) are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} z^{n+1}. \tag{2.21}$$

Taking $\psi(z) = \frac{z}{(1-z)^2}$, $c_k = \lambda_k - \alpha\mu_k$, $\delta = 1 - \alpha$ where $0 \le \alpha < 1$, $\lambda_k \ge 0$, $\mu_k \ge 0$, and $\lambda_k \ge \mu_k$ $(k \ge 2)$ in Theorem 2.1, we obtain:

Corollary 2.7. *If f is of the form (1.1) and satisfies the condition*

$$\sum_{k=2}^{\infty} (\lambda_k - \alpha\mu_k) |a_k| \le 1 - \alpha,$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \ge \frac{\lambda_{n+1} - \alpha\mu_{n+1} - (n+1)(1-\alpha)}{\lambda_{n+1} - \alpha\mu_{n+1}} \qquad (z \in U) \tag{2.22}$$

and

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \ge \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{\lambda_{n+1} - \alpha\mu_{n+1} + (n+1)(1-\alpha)} \qquad (z \in U), \tag{2.23}$$

where

$$\lambda_k - \alpha\mu_k \ge \begin{cases} k(1-\alpha) & \text{if} \quad k = 2, 3, \dots, n, \\ \frac{k(\lambda_{n+1} - \alpha\mu_{n+1})}{n+1} & \text{if} \quad k = n+1, n+2, \dots \end{cases} \tag{2.24}$$

The results (2.22) and (2.23) are sharp with the function given by (2.21).

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Remark 3. Frasin has obtained inequalities (2.22) & (2.23) in Theorem 2 of [1] under the condition

$$\lambda_{k+1}-\alpha\mu_{k+1}\geq\begin{cases}k(1-\alpha) & \text{if} \quad k=2,3,\dots,n,\\ k(1-\alpha)+\frac{k(\lambda_{n+1}-\alpha\mu_{n+1})}{n+1} & \text{if} \quad k=n+1,n+2,\dots\end{cases} \tag{2.25}$$

However, when we critically observe the proof of Theorem 2 of [1], we find that the last inequality of this theorem

$$\begin{aligned}\sum_{k=2}^{n}\left(\frac{\lambda_k-\alpha\mu_k}{1-\alpha}-k\right)|a_k| \\ +\sum_{k=n+1}^{\infty}\left(\frac{\lambda_k-\alpha\mu_k}{1-\alpha}-\left(1+\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)k\right)|a_k|\geq 0\end{aligned} \tag{2.26}$$

cannot hold for the function given by (2.21) for supporting the sharpness of the results (2.22) & (2.23). So condition 2.25 of Theorem 2 in [1] is incorrect and the corrected results are mentioned in Corollary 2.7.

Taking

$$\psi(z)=\frac{z}{1-z}, \qquad c_k=\frac{\{(1+\beta)k-(\alpha+\beta)\}\mu_k}{1-\alpha}$$

and $\delta=1$, where $-1\leq\alpha<1$, $\beta\geq 0$, $\mu_k\geq 0$ $(\forall k\in N\backslash\{1\})$ in Theorem 2.1, we obtain the following result given by Raina and Bansal in [3].

Corollary 2.8. *If f is of the form (1.2) and satisfies the condition $\sum_{k=2}^{\infty}c_k|a_k|\leq 1$, where*

$$c_k=\frac{\{(1+\beta)k-(\alpha+\beta)\}\mu_k}{1-\alpha},$$

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and $\langle \mu_k \rangle_{k=2}^{\infty}$ *is a nondecreasing sequence such that*

$$\mu_2 \geq \frac{1-\alpha}{2+\beta-\alpha} \left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1, \quad -1 \leq \alpha < 1, \beta \geq 0 \right),$$

then

$$\operatorname{Re}\left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{c_{n+1}-1}{c_{n+1}} \qquad (z \in U) \tag{2.27}$$

and

$$\operatorname{Re}\left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{c_{n+1}}{c_{n+1}+1} \qquad (z \in U). \tag{2.28}$$

The results (2.27) *and* (2.28) *are sharp with the function given by*

$$f(z) = z - \frac{1}{c_{n+1}} z^{n+1}. \tag{2.29}$$

Taking $\psi(z) = \frac{z}{(1-z)^2}$, $c_k = \frac{\{(1+\beta)k-(\alpha+\beta)\}\mu_k}{1-\alpha}$ and $\delta = 1$, where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\mu_k \geq 0$ $(\forall k \in N \setminus \{1\})$ in Theorem 2.1, we obtain the following result given by Raina and Bansal in [3].

Corollary 2.9. *If* f *is of the form* (1.2) *and satisfies the condition*

$$\sum_{k=2}^{\infty} c_k |a_k| \leq 1,$$

where

$$c_k = \frac{\{(1+\beta)k-(\alpha+\beta)\}\mu_k}{1-\alpha},$$

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and $\langle \mu_k \rangle_{k=2}^{\infty}$ *is a nondecreasing sequence such that*

$$\mu_2 \geq \frac{2(1-\alpha)}{2+\beta-\alpha} \qquad \left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1, \quad -1 \leq \alpha < 1, \beta \geq 0\right).$$

Then

$$\text{(2.30)} \qquad \operatorname{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq \frac{c_{n+1}-(n+1)}{c_{n+1}} \qquad (z \in U)$$

and

$$\text{(2.31)} \qquad \operatorname{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+(n+1)} \qquad (z \in U),$$

where

$$\text{(2.32)} \qquad c_k \geq \begin{cases} k & \text{if} \quad k = 2, 3, \dots, n, \\ \frac{kc_{n+1}}{n+1} & \text{if} \quad k = n+1, n+2, \dots \end{cases}$$

The results (2.30) *and* (2.31) *are sharp with the function given by* (2.29).

Remark 4. Raina and Bansal [3] have obtained inequalities (2.30) & (2.31) in Theorem 6.2 of [3] without any restriction on c_k . However, we easily see that condition (2.32) is must.

Remark 5. Taking $\psi(z) = \frac{z}{1-z}$, $c_k = (k-\alpha)$, $c_k = k(k-\alpha)$, $\delta = 1-\alpha, 0 \leq \alpha < 1$ in Theorem 2.1, we obtain Theorems 1-3 given by Silverman in [5].

Remark 6. Taking $\psi(z) = \frac{z}{(1-z)^2}$, $c_k = (k-\alpha)$, $c_k = k(k-\alpha)$, $\delta = 1-\alpha$, $0 \leq \alpha < 1$ in Theorem 2.1, we obtain Theorems 4-5 given by Silverman in [5].

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