# NEW GENERAL INTEGRAL OPERATORS OF $p$-VALENT FUNCTIONS

B.A. FRASIN
Department of Mathematics
Al al-Bayt University
P.O. Box: 130095
Mafraq, Jordan
EMail: bafrasin@yahoo.com

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*Abstract:* In this paper, we introduce new general integral operators. New sufficient conditions for these operators to be $p$-valently starlike, $p$-valently close-to-convex, uniformly $p$-valent close-to-convex and strongly starlike of order $\gamma$ $(0 < \gamma \leq 1)$ in the open unit disk are obtained.



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## 1. Introduction and Definitions

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} \in \{1, 2, \dots\}),$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be $p$-valently starlike of order $\beta$ $(0 \leq \beta < p)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{S}_p^{\star}(\beta)$, the class of all such functions. On the other hand, a function $f \in \mathcal{A}_p$ is said to be $p$-valently convex of order $\beta$ $(0 \leq \beta < p)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \qquad (z \in \mathcal{U}).$$

Let $\mathcal{K}_p(\beta)$ denote the class of all those functions which are $p$-valently convex of order $\beta$ in $\mathcal{U}$. Furthermore, a function $f(z) \in \mathcal{A}_p$ is said to be in the subclass $\mathcal{C}_p(\beta)$ of $p$-valently close-to-convex functions of order $\beta(0 \leq \beta < p)$ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > \beta \qquad (z \in \mathcal{U}).$$

Note that $\mathcal{S}_p^{\star}(0) = \mathcal{S}_p^{\star}$, $\mathcal{K}_p(0) = \mathcal{K}_p$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are, respectively, the classes of $p$-valently starlike, $p$-valently convex and $p$-valently close-to-convex functions in $\mathcal{U}$. Also, we note that $\mathcal{S}_1^{\star} = \mathcal{S}^{\star}$, $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the usual classes of starlike, convex and close-to-convex functions in $\mathcal{U}$.



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A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{UC}_p(\beta)$ of uniformly $p$-valent close-to-convex functions of order $\beta$ $(0 \leq \beta < p)$ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)} - \beta\right) \geq \left|\frac{zf'(z)}{g(z)} - p\right| \qquad (z \in \mathcal{U}),$$

for some $g(z) \in \mathcal{US}_p(\beta)$, where $\mathcal{US}_p(\beta)$ is the class of uniformly $p$-valent starlike functions of order $\beta$ $(-1 \leq \beta < p)$ in $\mathcal{U}$ and satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right) \geq \left|\frac{zf'(z)}{f(z)} - p\right| \qquad (z \in \mathcal{U}). \tag{1.1}$$

Uniformly $p$-valent starlike functions were first introduced in [10].

For $\alpha_i > 0$ and $f_i \in \mathcal{A}_p$, we define the following general integral operators

$$F_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1(t)}{t^p}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p}\right)^{\alpha_n} dt \tag{1.2}$$

and

$$G_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1'(t)}{pt^{p-1}}\right)^{\alpha_1} \cdots \left(\frac{f_n'(t)}{pt^{p-1}}\right)^{\alpha_n} dt. \tag{1.3}$$

If we take $p = 1$, we obtain of the general integral operators $F_1(z) = F_n(z)$ and $G_1(z) = F_{\alpha_1,\dots,\alpha_n}(z)$ introduced and studied by Breaz and Breaz [3] and Breaz et al. [6] (see also [2, 4, 8, 9]). Also for $p = n = 1$, $\alpha_1 = \alpha \in [0,1]$ in (1.2), we obtain the integral operator $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$ studied in [12] and for $p = n = 1$, $\alpha_1 = \delta \in \mathbb{C}$, $|\delta| \leq 1/4$ in (1.3), we obtain the integral operator $\int_0^z (f'(t))^{\alpha} dt$ studied in [11, 15].

There are many papers in which various sufficient conditions for multivalent starlikeness have been obtained. In this paper, we derive new sufficient conditions for the


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operators $F_p(z)$ and $G_p(z)$ to be $p$-valently starlike, $p$-valently close-to-convex and uniformly $p$-valent close-to-convex in $\mathcal{U}$. Furthermore, we give new sufficient conditions for these two general operators to be strongly starlike of order $\gamma$ $(0 < \gamma \leq 1)$ in $\mathcal{U}$.

In order to derive our main results, we have to recall here the following results:

**Lemma 1.1 ([13]).** *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{4} \qquad (z \in \mathcal{U}),$$

*then $f$ is $p$-valently starlike in $\mathcal{U}$.*

**Lemma 1.2 ([7]).** *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)} + 1 - p\right| < p + 1 \qquad (z \in \mathcal{U}),$$

*then $f$ is $p$-valently starlike in $\mathcal{U}$.*

**Lemma 1.3 ([16]).** *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{a+b}{(1+a)(1-b)} \qquad (z \in \mathcal{U}),$$

*where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then $f$ is $p$-valently close-to-convex in $\mathcal{U}$.*

**Lemma 1.4 ([1]).** *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{3} \qquad (z \in \mathcal{U}),$$

*then $f$ is uniformly $p$-valent close-to-convex in $\mathcal{U}$.*


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**Lemma 1.5 ([17]).** *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{p}{4}-1 \qquad (z \in \mathcal{U}),$$

*then*

$$\operatorname{Re}\sqrt{\frac{zf'(z)}{f(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}).$$

**Lemma 1.6 ([14]).** *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > p-\frac{\gamma}{2} \qquad (z \in \mathcal{U}),$$

*then*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\gamma \qquad (0<\gamma\leq 1;\ z \in \mathcal{U}),$$

*or $f$ is strongly starlike of order $\gamma$ in $\mathcal{U}$.*



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## 2. Sufficient Conditions for the Operator $F_p$

We begin by establishing sufficient conditions for the operator $F_p$ to be in $\mathcal{S}_p^\star$.

**Theorem 2.1.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < p + \frac{1}{4\sum_{i=1}^n \alpha_i} \qquad (z \in \mathcal{U}), \tag{2.1}$$

*then $F_p$ is $p$-valently starlike in $\mathcal{U}$.*

*Proof.* From the definition (1.2), we observe that $F_p(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$F_p'(z) = pz^{p-1}\left(\frac{f_1(z)}{z^p}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z^p}\right)^{\alpha_n}. \tag{2.2}$$

Differentiating (2.2) logarithmically and multiplying by $z$, we obtain

$$\frac{zF_p''(z)}{F_p'(z)} = (p-1) + \sum_{i=1}^n \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - p\right).$$

Thus we have

$$1 + \frac{zF_p''(z)}{F_p'(z)} = p\left(1 - \sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i \left(\frac{zf_i'(z)}{f_i(z)}\right). \tag{2.3}$$

Taking the real part of both sides of (2.3), we have

$$\operatorname{Re}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) = p\left(1 - \sum_{i=1}^n \alpha_i\right) + \sum_{i=1}^n \alpha_i \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right). \tag{2.4}$$


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From (2.4) and (2.1), we obtain

$$\text{(2.5)} \qquad \operatorname{Re}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right) < p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\left(p+\frac{1}{4\sum_{i=1}^{n}\alpha_i}\right) = p+\frac{1}{4}.$$

Hence by Lemma 1.1, we get $F_p \in \mathcal{S}_p^\star$. This completes the proof. □

Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.1, we have:

**Corollary 2.2.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1+\frac{1}{4\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0,$ then $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$ is starlike in $\mathcal{U}$.*

In the next theorem, we derive another sufficient condition for $p$-valently starlike functions in $\mathcal{U}$.

**Theorem 2.3.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\text{(2.6)} \qquad \left|\frac{zf_i'(z)}{f_i(z)} - p\right| < \frac{p+1}{\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}),$$

*then $F_p$ is $p$-valently starlike in $\mathcal{U}$.*

*Proof.* From (2.3) and the hypotheses (2.6), we have



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$$\left|1+\frac{zF_p''(z)}{F_p'(z)}-p\right| = \left|\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-p\right)\right|$$
$$< \sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-p\right|$$
$$< \sum_{i=1}^{n}\alpha_i\left(\frac{p+1}{\sum_{i=1}^{n}\alpha_i}\right) = p+1.$$

Now using Lemma 1.2, we immediately get $F_p \in \mathcal{S}_p^\star$. □

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.3, we have:

**Corollary 2.4.** *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \frac{2}{\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is starlike in $\mathcal{U}$.*

Applying Lemmas 1.3 and 1.4, we obtain the following sufficient conditions for $F_p$ to be $p$-valently close-to-convex and uniformly $p$-valent close-to-convex in $\mathcal{U}$.

**Theorem 2.5.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\text{(2.7)} \qquad \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < p + \frac{(a+b)}{(1+a)(1-b)\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}),$$

*where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then $F_p$ is $p$-valently close-to-convex in $\mathcal{U}$.*

*Proof.* From (2.4) and the hypotheses (2.7) and applying Lemma 1.3, we have $F_p \in \mathcal{C}_p(\beta)$. □


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Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.5, we have:

**Corollary 2.6.** *If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{(a+b)}{(1+a)(1-b)\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0, a > 0,\ b \geq 0$ and $a + 2b \leq 1$, then $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is close-to-convex in $\mathcal{U}$.*

**Theorem 2.7.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < p + \frac{1}{3\sum_{i=1}^n \alpha_i} \qquad (z \in \mathcal{U}), \tag{2.8}$$

*then $F_p$ is uniformly $p$-valent close-to-convex in $\mathcal{U}$.*

*Proof.* The proof of the theorem follows by applying Lemma 1.4 and using (2.4), (2.8) to get $F_p \in \mathcal{UC}_p(\beta)$. □

Letting $n = p = 1,\ \ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.7, we have:

**Corollary 2.8.** *If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{1}{3\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ is uniformly close-to-convex in $\mathcal{U}$.*

Using Lemma 1.5, we obtain the next result


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**Theorem 2.9.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) > p - \frac{3p+4}{4\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{2.9}$$

*then*

$$\operatorname{Re}\sqrt{\frac{zF_p'(z)}{F_p(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}).$$

*Proof.* It follows from (2.4) and (2.9) that

$$\operatorname{Re}\left(1 + \frac{zF_p''(z)}{F_p'(z)}\right) > p\left(1 - \sum_{i=1}^{n}\alpha_i\right) + \sum_{i=1}^{n}\alpha_i\left(p - \frac{3p+4}{4\sum_{i=1}^{n}\alpha_i}\right) = \frac{p}{4} - 1.$$

By Lemma 1.5, we conclude that

$$\operatorname{Re}\sqrt{\frac{zF_p'(z)}{F_p(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}).$$

□

Letting $n = p = 1,\ \alpha_1 = 1$ and $f_1 = f$ in Theorem 2.9, we have:

**Corollary 2.10.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) > -\frac{3}{4} \qquad (z \in \mathcal{U}), \tag{2.10}$$

*then*

$$\operatorname{Re}\sqrt{\frac{f(z)}{\int_0^z\left(\frac{f(t)}{t}\right)dt}} > \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{2.11}$$


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## 3. Sufficient Conditions for the Operator $G_p$

The first two theorems in this section give a sufficient condition for the integral operator $G_p$ to be in the class $\mathcal{S}_p^\star$.

**Theorem 3.1.** *Let* $\alpha_i > 0$ *be real numbers for all* $i = 1, 2, \ldots, n$. *If* $f_i \in \mathcal{A}_p$ *for all* $i = 1, 2, \ldots, n$ *satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) < p + \frac{1}{4\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{3.1}$$

*then* $G_p$ *is* $p$*-valently starlike in* $\mathcal{U}$.

*Proof.* From the definition (1.3), we observe that $G_p(z) \in \mathcal{A}_p$ and

$$\frac{zG_p''(z)}{G_p'(z)} = (p-1) + \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)} - (p-1)\right)$$

or

$$1 + \frac{zG_p''(z)}{G_p'(z)} = p\left(1 - \sum_{i=1}^{n}\alpha_i\right) + \sum_{i=1}^{n}\alpha_i\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right). \tag{3.2}$$

Taking the real part of both sides of (3.2), we have

$$\operatorname{Re}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) = p\left(1 - \sum_{i=1}^{n}\alpha_i\right) + \sum_{i=1}^{n}\alpha_i \operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right). \tag{3.3}$$



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From (3.3) and the hypotheses (3.1), we obtain

$$\text{(3.4)} \qquad \operatorname{Re}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right) < p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\left(p+\frac{1}{4\sum_{i=1}^{n}\alpha_i}\right) = p+\frac{1}{4}.$$

Therefore, using Lemma 1.1, it follows that the integral operator $G_p$ belongs to the class $\mathcal{S}_p^{\star}$. □

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.1, we obtain

**Corollary 3.2.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < 1+\frac{1}{4\alpha} \qquad (z\in\mathcal{U}),$$

*where $\alpha > 0$, then $\int_0^z (f'(t))^{\alpha}dt$ is starlike in $\mathcal{U}$.*

**Theorem 3.3.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\text{(3.5)} \qquad \left|\frac{zf_i''(z)}{f_i'(z)}\right| < \frac{p+1}{\sum_{i=1}^{n}\alpha_i} - p + 1 \qquad (z\in\mathcal{U}),$$

*where $\sum_{i=1}^{n}\alpha_i > 1$, then $G_p$ is $p$-valently starlike in $\mathcal{U}$.*

*Proof.* It follows from (3.2) and (3.5) that

$$\left|1+\frac{zG_p''(z)}{G_p'(z)}-p\right| = \left|\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}\right)-(p-1)\sum_{i=1}^{n}\alpha_i\right|$$
$$< (p-1)\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\alpha_i\left(\frac{p+1}{\sum_{i=1}^{n}\alpha_i}-p+1\right) < p+1.$$



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Therefore, it follows from Lemma 1.2 that $G_p$ is in the class $\mathcal{S}_p^\star$. □

Letting $n = p = 1,\ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.3, we obtain:

**Corollary 3.4.** *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{2}{\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0,$ then $\int_0^z (f'(t))^\alpha dt$ is starlike in $\mathcal{U}$.*

Applying Lemmas 1.3 and 1.4, we obtain the following sufficient conditions for $G_p$ to be $p$-valently close-to-convex and uniformly $p$-valent close-to-convex in $\mathcal{U}$.

**Theorem 3.5.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\text{(3.6)} \qquad \operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) < p + \frac{a+b}{(1+a)(1-b)\sum_{i=1}^n \alpha_i} \qquad (z \in \mathcal{U}),$$

*where $a > 0,\ b \geq 0$ and $a + 2b \leq 1,$ then $G_p$ is $p$-valently close-to-convex in $\mathcal{U}$.*

*Proof.* In view of (3.3) and (3.6) and by using Lemma 1.3, we have $G_p \in \mathcal{C}_p(\beta)$. □

Letting $n = p = 1,\ \alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.5, we obtain

**Corollary 3.6.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{a+b}{(1+a)(1-b)\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0,\ \ a > 0,\ b \geq 0$ and $a + 2b \leq 1,$ then $\int_0^z (f'(t))^\alpha dt$ is close-to-convex in $\mathcal{U}$.*



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**Theorem 3.7.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) < p + \frac{1}{3\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{3.7}$$

*then $G_p$ is uniformly $p$-valent close-to-convex in $\mathcal{U}$.*

*Proof.* In view of (3.3) and (3.7) and by using Lemma 1.4, we have $G_p \in \mathcal{UC}_p(\beta)$. □

Letting $n = p = \alpha = 1$ and $f_1 = f$ in Theorem 3.7, we have:

**Corollary 3.8.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{1}{3\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is uniformly close-to-convex in $\mathcal{U}$.*

Using Lemma 1.5, we obtain the next result.

**Theorem 3.9.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) > p - \frac{3p+4}{4\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}), \tag{3.8}$$

*then*

$$\operatorname{Re}\sqrt{\frac{zG_p'(z)}{G_p(z)}} > \frac{\sqrt{p}}{2} \qquad (z \in \mathcal{U}). \tag{3.9}$$


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*Proof.* It follows from (3.3) and (3.8) that

$$\operatorname{Re}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right) > p\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i\left(p-\frac{3p+4}{4\sum_{i=1}^{n}\alpha_i}\right)=\frac{p}{4}-1.$$

By Lemma 1.5, we get the result (3.9). □

Letting $n=p=1,\ \ \alpha_1=1$ and $f_1=f$ in Theorem 3.9, we have

**Corollary 3.10.** *If $f\in\mathcal{A}$ satisfies*

(3.10) $$\operatorname{Re}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right) > -\frac{3}{4} \qquad (z\in\mathcal{U}),$$

*then*

(3.11) $$\operatorname{Re}\sqrt{\frac{zf'(z)}{\int_0^z f'(t)dt}} > \frac{1}{2} \qquad (z\in\mathcal{U}).$$


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## 4. Strong Starlikeness of the Operators $F_p$ and $G_p$

Applying Lemma 1.6 and using (2.4), we obtain the following sufficient condition for the operator $F_p$ to be strongly starlike of order $\gamma$ in $\mathcal{U}$.

**Theorem 4.1.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) > p - \frac{\gamma}{2\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}),$$

*then $F_p$ is strongly starlike of order $\gamma$ $(0 < \gamma \le 1)$ in $\mathcal{U}$.*

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 4.1, we have

**Corollary 4.2.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 1 - \frac{\gamma}{2\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt$ is strongly starlike of order $\gamma$ $(0 < \gamma \le 1)$ in $\mathcal{U}$.*

Applying once again Lemma 1.6 and using (3.3), we obtain the following sufficient condition for the operator $G_p$ to be strongly starlike of order $\gamma$ in $\mathcal{U}$.

**Theorem 4.3.** *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \ldots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) > p - \frac{\gamma}{2\sum_{i=1}^{n}\alpha_i} \qquad (z \in \mathcal{U}),$$

*then $G_p$ is strongly starlike of order $\gamma$ $(0 < \gamma \le 1)$ in $\mathcal{U}$.*

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Letting $n = p = \alpha_1 = 1$ and $f_1 = f$ in Theorem 4.3, we have

**Corollary 4.4.** *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 1 - \frac{\gamma}{2\alpha} \qquad (z \in \mathcal{U}),$$

*where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is strongly starlike of order $\gamma$ $(0 < \gamma \leq 1)$ in $\mathcal{U}$.*


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