ON THE DEGREE OF STRONG APPROXIMATION OF CONTINUOUS FUNCTIONS BY SPECIAL MATRIX

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Abstract: In the presented paper we will generalize the result of L. Leindler [3] to the class $MRBVS$ and extend it to the strong summability with a mediate function satisfying the standard conditions.

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1. Introduction

Let f be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $S_n(x) = S_n(f, x)$ the n-th partial sum of (1.1) and by $\omega(f, \delta)$ the modulus of continuity of $f \in C_{2\pi}$. The usual supremum norm will be denoted by $\|\cdot\|$.

Let $A := (a_{nk})$ $(k, n = 0, 1, \ldots)$ be a lower triangular infinite matrix of real numbers satisfying the following conditions:

$$a_{nk} \geq 0 \ (0 \leq k \leq n), \quad a_{nk} = 0, \ (k > n) \quad \text{and} \quad \sum_{k=0}^{n} a_{nk} = 1, \tag{1.2}$$

where $k, n = 0, 1, 2, \ldots$.

Let the $A-$transformation of $(S_n(f; x))$ be given by

$$t_n(f) := t_n(f; x) := \sum_{k=0}^{n} a_{nk} S_k(f; x) \qquad (n = 0, 1, \ldots) \tag{1.3}$$

and the strong A_r-transformation of $(S_n(f; x))$ for $r > 0$ be given by

$$T_n(f, r) := T_n(f, r; x) := \left\{ \sum_{k=0}^{n} a_{nk} \left| S_k(f; x) - f(x) \right|^r \right\}^{\frac{1}{r}} (n = 0, 1, \ldots).$$

Now we define two classes of sequences.

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A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)\, c_m \tag{1.4}$$

for $m = 0, 1, 2, ...$, where $K(c)$ is a constant depending only on c (see [3]).

A null sequence $c := (c_n)$ of positive numbers is called of Mean Rest Bounded Variation, or briefly $c \in MRBVS$, if it has the property

$$\sum_{n=2m}^{\infty} |c_n - c_{n+1}| \leq K(c) \frac{1}{m+1} \sum_{n=m}^{2m} c_n \tag{1.5}$$

for $m = 0, 1, 2, ...$ (see [5]).

Therefore we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all n, where $K(\alpha_n)$ denotes the sequence of constants appearing in the inequalities (1.4) or (1.5) for the sequence $\alpha_n := (a_{nk})_{k=0}^{\infty}$. Now we can give some conditions to be used later on. We assume that for all n

$$\sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{nm} \qquad (0 \leq m \leq n) \tag{1.6}$$

and

$$\sum_{k=2m}^{\infty} |a_{nk} - a_{nk+1}| \leq K \frac{1}{m+1} \sum_{k=m}^{2m} a_{nk} \qquad (0 \leq 2m \leq n) \tag{1.7}$$

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hold if $\alpha_n := (a_{nk})_{k=0}^{\infty}$ belongs to $RBVS$ or $MRBVS$, respectively.

In [1] and [2] P. Chandra obtained some results on the degree of approximation for the means (1.3) with a mediate function H such that:

$$\int_u^\pi \frac{\omega(f;t)}{t^2}dt = O(H(u)) \qquad (u \to 0_+), \; H(t) \geq 0 \tag{1.8}$$

and

$$\int_0^t H(u)\,du = O(tH(t)) \qquad (t \to O_+). \tag{1.9}$$

In [3], L. Leindler generalized this result to the class $RBVS$. Namely, he proved the following theorem:

Theorem 1.1. *Let (1.2), (1.6), (1.8) and (1.9) hold. Then for* $f \in C_{2\pi}$

$$\|t_n(f) - f\| = O(a_{n0}H(a_{n0})).$$

It is clear that

$$RBVS \subseteq MRBVS. \tag{1.10}$$

In [7], we proved that $RBVS \neq MRBVS$. Namely, we showed that the sequence

$$d_n := \begin{cases} 1 & \text{if } n = 1, \\ \frac{1+m+(-1)^n m}{(2^{\mu_m})^2 m} & \text{if } \mu_m \leq n < \mu_{m+1}, \end{cases}$$

where $\mu_m = 2^m$ for $m = 1, 2, 3, \ldots$, belongs to the class $MRBVS$ but it does not belong to the class $RBVS$.

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In the present paper we will generalize the mentioned result of L. Leindler [3] to the class $MRBVS$ and extend it to strong summability with a mediate function H defined by the following conditions:

$$\int_u^\pi \frac{\omega^r(f;t)}{t^2}dt = O(H(r;u)) \qquad (u \to 0_+), \; H(t) \geq 0 \text{ and } r > 0, \tag{1.11}$$

and

$$\int_0^t H(r;u)\,du = O(tH(r;t)) \qquad (t \to O_+). \tag{1.12}$$

By $K_1, K_2, \ldots$ we shall denote either an absolute constant or a constant depending on the indicated parameters, not necessarily the same in each occurrence.

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2. Main Results

Our main results are the following.

Theorem 2.1. *Let (1.2), (1.7) and (1.11) hold. Then for* $f \in C_{2\pi}$ *and* $r > 0$

$$\|T_n(f,r)\| = O\left(\left\{a_{n0} H\left(r; \frac{\pi}{n}\right)\right\}^{\frac{1}{r}}\right). \tag{2.1}$$

If, in addition (1.12) holds, then

$$\|T_n(f,r)\| = O\left(\{a_{n0} H(r; a_{n0})\}^{\frac{1}{r}}\right). \tag{2.2}$$

Using the inequality

$$\|t_n(f) - f\| \le \|T_n(f,1)\|,$$

we can formulate the following corollary.

Corollary 2.2. *Let (1.2), (1.7) and (1.11) hold. Then for* $f \in C_{2\pi}$

$$\|t_n(f) - f\| = O\left(a_{n0} H\left(1; \frac{\pi}{n}\right)\right).$$

If, in addition (1.12) holds, then

$$\|t_n(f) - f\| = O\left(a_{n0} H(1; a_{n0})\right).$$

Remark 1. By the embedding relation (1.7) we can observe that Theorem 1.1 follows from Corollary 2.2.

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For special cases, putting

$$H(r;t) = \begin{cases} t^{r\alpha-1} & \text{if } \alpha r < 1, \\ \ln \frac{\pi}{t} & \text{if } \alpha r = 1, \\ K_1 & \text{if } \alpha r > 1, \end{cases}$$

where $r > 0$ and $0 < \alpha \leq 1$, we can derive from Theorem 2.1 the next corollary.

Corollary 2.3. *Under the conditions (1.2) and (1.7) we have, for $f \in C_{2\pi}$ and $r > 0$,*

$$\|T_n(f,r)\| = \begin{cases} O\left(\{a_{n0}\}^{\alpha}\right) & \text{if } \alpha r < 1, \\ O\left(\left\{\ln\left(\frac{\pi}{a_{n0}}\right) a_{n0}\right\}^{\alpha}\right) & \text{if } \alpha r = 1, \\ O\left(\{a_{n0}\}^{\frac{1}{r}}\right) & \text{if } \alpha r > 1. \end{cases}$$

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3. Lemmas

To prove our main result we need the following lemmas.

Lemma 3.1 ([6]). *If (1.11) and (1.12) hold, then for $r > 0$*

$$\int_0^s \frac{\omega^r (f;t)}{t} dt = O\left(sH\left(r;s\right)\right) \qquad \left(s \to 0_+\right).$$

Lemma 3.2. *If (1.2) and (1.7) hold, then for $f \in C_{2\pi}$ and $r > 0$*

$$\left\| T_n \left(f, r\right) \right\|_C \leq O\left(\left\{ \sum_{k=0}^{n} a_{nk} E_k^r \left(f\right) \right\}^{\frac{1}{r}}\right), \tag{3.1}$$

where $E_n(f)$ denotes the best approximation of the function f by trigonometric polynomials of order at most n.

Proof. It is clear that (3.1) holds for $n = 0, 1, ..., 5$. Namely, by the well known inequality [8]

$$\left\| \sigma_{n,m} - f \right\| \leq 2 \frac{n+1}{m+1} E_n\left(f\right) \qquad \left(0 \leq m \leq n\right), \tag{3.2}$$

where

$$\sigma_{n,m}\left(f;x\right) = \frac{1}{m+1} \sum_{k=n-m}^{n} S_k\left(f;x\right),$$

for $m = 0$, we obtain

$$\left\{ T_n\left(f, r; x\right) \right\}^r \leq 12^r \sum_{k=0}^{n} a_{nk} E_k^r\left(f\right)$$

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and (3.1) is obviously valid, for $n \leq 5$.

Let $n \geq 6$ and let $m = m_n$ be such that

$$2^{m+1} + 4 \leq n < 2^{m+2} + 4.$$

Hence

$$\{T_n(f, r; x)\}^r \leq \sum_{k=0}^{3} a_{nk} |S_k(f; x) - f(x)|^r$$
$$+ \sum_{k=1}^{m-1} \sum_{i=2^k+2}^{2^{k+1}+4} a_{ni} |S_i(f; x) - f(x)|^r + \sum_{k=2^m+5}^{n} a_{nk} |S_k(f; x) - f(x)|^r.$$

Applying the Abel transformation and (3.2) to the first sum we obtain

$$\{T_n(f, r; x)\}^r$$
$$\leq 8^r \sum_{k=0}^{3} a_{nk} E_k^r(f) + \sum_{k=1}^{m-1} \left(\sum_{i=2^k+2}^{2^{k+1}+3} (a_{ni} - a_{n,i+1}) \sum_{l=2^k+2}^{i} |S_l(f; x) - f(x)|^r \right.$$
$$\left. + a_{n,2^{k+1}+4} \sum_{i=2^k+2}^{2^{k+1}+4} |S_i(f; x) - f(x)|^r \right)$$
$$+ \sum_{k=2^m+2}^{n-1} (a_{nk} - a_{n,k+1}) \sum_{l=2^{m-1}}^{k} |S_l(f; x) - f(x)|^r$$
$$+ a_{nn} \sum_{k=2^m+2}^{n} |S_k(f; x) - f(x)|^r$$

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$$\le 8^r \sum_{k=0}^{3} a_{nk} E_k^r(f) + \sum_{k=1}^{m-1} \left(\sum_{i=2^k+2}^{2^{k+1}+3} |a_{ni} - a_{n,i+1}| \sum_{l=2^k+2}^{2^{k+1}+3} |S_l(f;x) - f(x)|^r \right.$$
$$\left. + a_{n,2^{k+1}+4} \sum_{i=2^k+2}^{2^{k+1}+4} |S_i(f;x) - f(x)|^r \right)$$
$$+ \sum_{k=2^m+2}^{n-1} |a_{nk} - a_{n,k+1}| \sum_{l=2^m+2}^{2^{m+2}+3} |S_l(f;x) - f(x)|^r$$
$$+ a_{nn} \sum_{k=2^m+2}^{2^{m+2}+4} |S_k(f;x) - f(x)|^r .$$

Using the well-known Leindler's inequality [4]

$$\left\{ \frac{1}{m+1} \sum_{k=n-m}^{n} |S_k(f;x) - f(x)|^s \right\}^{\frac{1}{s}} \le K_1 E_{n-m}(f)$$

for $0 \le m \le n$, $m = O(n)$ and $s > 0$, we obtain

$$\{T_n(f,r;x)\}^r \le 8^r \sum_{k=0}^{3} a_{nk} E_k^r(f)$$
$$+ K_2 \left\{ \sum_{k=1}^{m-1} \left(\left(2^k + 3\right) E_{2^k+2}^r(f) \left(\sum_{i=2^k+2}^{2^{k+1}+3} |a_{ni} - a_{n,i+1}| + a_{n,2^{k+1}+4} \right) \right) \right.$$
$$\left. 3\left(2^m + 1\right) E_{2^m+2}^r \left(\sum_{k=2^m+2}^{n-1} |a_{nk} - a_{n,k+1}| + a_{nn} \right) \right\} .$$

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Using (1.7) we get

$$\{T_n(f,r;x)\}^r \le 8^r \sum_{k=0}^{3} a_{nk} E_k^r(f)$$
$$+ K_2 \left\{ \sum_{k=1}^{m-1} \left(\left(2^k+3\right) E_{2^k+2}^r(f) \left(K \frac{1}{2^{k-1}+2} \sum_{i=2^{k-1}+1}^{2^k+2} a_{ni} + a_{n,2^{k+1}+4} \right) \right) \right.$$
$$\left. 3\left(2^m+1\right) E_{2^m+2}^r(f) \left(K \frac{1}{2^{m-1}+2} \sum_{i=2^{m-1}+1}^{2^m+2} a_{ni} + a_{nn} \right) \right\}.$$

In view of (1.7), we also obtain for $1 \le k \le m-1$,

$$a_{n,2^{k+1}+4} = \sum_{i=2^{k+1}+4}^{\infty} (a_{ni} - a_{ni+1}) \le \sum_{i=2^{k+1}+4}^{\infty} |a_{ni} - a_{ni+1}|$$
$$\le \sum_{i=2^k+2}^{\infty} |a_{ni} - a_{ni+1}| \le K \frac{1}{2^{k-1}+2} \sum_{i=2^{k-1}+1}^{2^k+2} a_{ni}$$

and

$$a_{nn} = \sum_{i=n}^{\infty} (a_{ni} - a_{ni+1}) \le \sum_{i=n}^{\infty} |a_{ni} - a_{ni+1}|$$
$$\le \sum_{i=2^m+2}^{\infty} |a_{ni} - a_{ni+1}| \le K \frac{1}{2^{m-1}+2} \sum_{i=2^{m-1}+1}^{2^m+2} a_{ni}.$$

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Hence

$$\{T_n(f,r;x)\}^r \le 8^r \sum_{k=0}^{3} a_{nk} E_k^r(f)$$
$$+ K_3 \left\{ \sum_{k=1}^{m-1} E_{2^k+2}^r(f) \sum_{i=2^{k-1}+1}^{2^k+2} a_{ni} + E_{2^m+2}^r(f) \sum_{i=2^{m-1}+1}^{2^m+2} a_{ni} \right\}$$
$$\le 8^r \sum_{k=0}^{3} a_{nk} E_k^r(f) + 2K_3 \sum_{k=3}^{2^m+2} a_{nk} E_k^r(f)$$
$$\le K_4 \sum_{k=0}^{n} a_{nk} E_k^r(f).$$

This ends our proof. □

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4. Proof of Theorem 2.1

Using Lemma 3.2 we have

$$\left|T_n(f,r;x)\right| \le K_1 \left\{ \sum_{k=0}^{n} a_{nk} E_k^r(f) \right\}^{\frac{1}{r}} \le K_2 \left\{ \sum_{k=0}^{n} a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) \right\}^{\frac{1}{r}}. \tag{4.1}$$

If (1.7) holds, then, for any $m = 1, 2, \dots, n$,

$$a_{nm} - a_{n0} \le |a_{nm} - a_{n0}| = |a_{n0} - a_{nm}| = \left| \sum_{k=0}^{m-1} (a_{nk} - a_{nk+1}) \right|$$
$$\le \sum_{k=0}^{m-1} |a_{nk} - a_{nk+1}| \le \sum_{k=0}^{\infty} |a_{nk} - a_{nk+1}| \le K a_{n0},$$

whence

$$a_{nm} \le (K+1)\, a_{n0}. \tag{4.2}$$

Therefore, by (1.2),

$$(K+1)(n+1)\, a_{n0} \ge \sum_{k=0}^{n} a_{nk} = 1. \tag{4.3}$$

First we prove (2.1). Using (4.2), we get

$$\sum_{k=0}^{n} a_{nk} \omega^r \left(f; \frac{\pi}{k+1} \right) \le (K+1)\, a_{n0} \sum_{k=0}^{n} \omega^r \left(f; \frac{\pi}{k+1} \right)$$

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$$\leq K_3 a_{n0} \int_1^{n+1} \omega^r \left(f; \frac{\pi}{t}\right) dt$$
$$= \pi K_3 a_{n0} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega^r (f; u)}{u^2} du$$

and by (4.1), (1.11) we obtain that (2.1) holds.

Now, we prove (2.2). From (4.3) we obtain

$$\sum_{k=0}^{n} a_{nk}\omega^r \left(f; \frac{\pi}{k+1}\right)$$
$$\leq \sum_{k=0}^{\left[\frac{1}{(K+1)a_{n0}}\right]-1} a_{nk}\omega^r \left(f; \frac{\pi}{k+1}\right) + \sum_{k=\left[\frac{1}{(K+1)a_{n0}}\right]-1}^{n} a_{nk}\omega^r \left(f; \frac{\pi}{k+1}\right).$$

Again using (1.2), (4.2) and the monotonicity of the modulus of continuity, we get

$$\sum_{k=0}^{n} a_{nk}\omega^r \left(f; \frac{\pi}{k+1}\right)$$
$$\leq (K+1)\, a_{n0} \sum_{k=0}^{\left[\frac{1}{(K+1)a_{n0}}\right]-1} \omega^r \left(f; \frac{\pi}{k+1}\right)$$
$$+ K_4 \omega^r \left(f; \pi \left(K+1\right) a_{no}\right) \sum_{k=\left[\frac{1}{(K+1)a_{n0}}\right]-1}^{n} a_{nk}$$

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$$\leq K_5 a_{n0} \int_1^{\frac{1}{(K+1)a_{n0}}} \omega^r\left(f;\frac{\pi}{t}\right)dt + K_4\omega^r\left(f;\pi\left(K+1\right)a_{no}\right)$$

$$\leq K_6\left(a_{n0}\int_{a_{n0}}^{\pi}\frac{\omega^r\left(f;u\right)}{u^2}du + \omega^r\left(f;a_{n0}\right)\right). \tag{4.4}$$

Moreover

$$\begin{aligned}\omega^r\left(f;a_{n0}\right) &\leq 4^r\omega^r\left(f;\frac{a_{n0}}{2}\right) \\ &\leq 2\cdot 4^r\int_{\frac{a_{n0}}{2}}^{a_{n0}}\frac{\omega^r\left(f;t\right)}{t}dt \\ &\leq 2\cdot 4^r\int_0^{a_{n0}}\frac{\omega^r\left(f;t\right)}{t}dt.\end{aligned} \tag{4.5}$$

Thus collecting our partial results (4.1), (4.4), (4.5) and using (1.11) and Lemma 3.1 we can see that (2.2) holds. This completes our proof. □

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