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ON CERTAIN INEQUALITIES INVOLVING THE LAMBERT W FUNCTION

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ABSTRACT. In this note we obtain certain inequalities involving the Lambert W function $\mathrm{W}_0(-x\mathrm{e}^{-x})$ which has recently been found to arise in the classic problem of a projectile moving through a linearly resisting medium.

Key words and phrases: Lambert W function, special function, linearly resisted projectile motion.

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1. INTRODUCTION

The Lambert W function $\mathrm{W}(x)$ is defined by the equation $\mathrm{W}(x)\mathrm{e}^{\mathrm{W}(x)} = x$ for $x \geq -\mathrm{e}^{-1}$. When $-\mathrm{e}^{-1} \leq x < 0$ the function takes on two real branches. By convention, the branch satisfying $\mathrm{W}(x) \geq -1$ is taken to be the principal branch, denoted by $\mathrm{W}_0(x)$, while that satisfying $\mathrm{W}(x) < -1$ is known as the secondary real branch and is denoted by $\mathrm{W}_{-1}(x)$. The history of the function dates back to the mid-eighteenth century and is named in honour of J. H. Lambert (1728–1777) who in 1758 first considered a problem requiring $\mathrm{W}(x)$ for its solution. For a brief historical survey, a detailed definition of the function when its argument is complex, important properties of the function, together with an overview of some of the areas where the function has been found to arise, see [1]. More recently, sharp bounds for the function have been considered in [2].

In this note, motivated by the appearance of the Lambert W function in the classic problem of a projectile moving through a linearly resisting medium [6], [4], [3], [5], we consider a number of inequalities involving $\mathrm{W}_0(-x\mathrm{e}^{-x})$ for $x > 1$.

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2. Preliminaries

From the defining equation for the Lambert W function it is readily seen that $\mathrm{W}_0(-\mathrm{e}^{-1}) = -1$, $\mathrm{W}_0(0) = 0$, and $\mathrm{W}_0(\mathrm{e}) = 1$. Also, implicit differentiation of the defining equation yields

$$\frac{d}{dx}\mathrm{W}(x) = \frac{\mathrm{W}(x)}{x(1+\mathrm{W}(x))} = \frac{\mathrm{e}^{-\mathrm{W}(x)}}{1+\mathrm{W}(x)},$$

where we note that the singularity at the origin is removable. For the principal branch, as $\mathrm{W}_0(x) > -1$ and $\mathrm{e}^{-\mathrm{W}_0(x)} > 0$ for $x > -\mathrm{e}^{-1}$, $\frac{d}{dx}\mathrm{W}_0(x) > 0$ for $x > -\mathrm{e}^{-1}$ and consequently $\mathrm{W}_0(x)$ is strictly increasing for $x > -\mathrm{e}^{-1}$.

3. Main Results

Lemma 3.1. *For $x \geq 1$, we have*

$$-1 \leq \mathrm{W}_0(-x\mathrm{e}^{-x}) < 0, \tag{3.1}$$

with equality holding for $x = 1$.

Proof. For $x > 1$, let $g(x) = -x\mathrm{e}^{-x}$. Since $\mathrm{e}^{-x} > 0$ one has $g(x) < 0$ for $x > 1$ and we need only show $g(x)$ increases for $x > 1$ such that $g(x) > -\mathrm{e}^{-1}$ as $\mathrm{W}_0(x)$ is strictly increasing for $x > -\mathrm{e}^{-1}$. Since $\frac{d}{dx}g(x) = (x-1)\mathrm{e}^{-x} > 0$ for $x > 1$, $g(x)$ clearly increases and consequently $g(x) > g(1) = -\mathrm{e}^{-1}$. Thus $-\mathrm{e}^{-1} < -x\mathrm{e}^{-x} < 0$ from which (3.1) follows. Trivially, equality on the left hand side holds only for $x = 1$. □

Theorem 3.2. *For $x \geq 1$, we have*

$$x - 2 \geq \mathrm{W}_0(-x\mathrm{e}^{-x}), \tag{3.2}$$

with equality holding only for $x = 1$.

Proof. For $x > 1$, let $U(x) = (x-2)\mathrm{e}^{x-2} + x\mathrm{e}^{-x}$ and let $t = x - 1$ so that $t > 0$. Then $U(t) = \mathrm{e}^{-1}h(t)$ where $h(t) = (t-1)\mathrm{e}^{t} + (t+1)\mathrm{e}^{-t}$. Since $\frac{d}{dt}h(t) = 2t\sinh t > 0$ for $t > 0$, one has $h(t) > h(0) = 0$, or, equivalently $(x-2)\mathrm{e}^{x-2} > -x\mathrm{e}^{-x}$ for $x > 1$. Since $\mathrm{W}_0(x)$ is strictly increasing for $x > -\mathrm{e}^{-1}$ and as $-x\mathrm{e}^{-x} > -\mathrm{e}^{-1}$ for $x > 1$ (see Lemma 3.1), it is immediate that $\mathrm{W}_0((x-2)\mathrm{e}^{x-2}) > \mathrm{W}_0(-x\mathrm{e}^{-x})$. Finally, since $x - 2 > -1$, the desired result follows on recognising the simplification $\mathrm{W}_0((x-2)\mathrm{e}^{x-2}) = x - 2$, with equality at $x = 1$. □

Theorem 3.3. *For $x > 1$, we have*

$$1 < \frac{x + \mathrm{W}_0(-x\mathrm{e}^{-x})}{x-1} < 2. \tag{3.3}$$

Proof. For $x > 1$, combining the left hand side of inequality (3.1) with (3.2) gives $-1 < \mathrm{W}_0(-x\mathrm{e}^{-x}) < x - 2$. Adding x to each term appearing in the inequality before dividing throughout by $x - 1 > 0$ for $x > 1$, yields the desired result. □

Lemma 3.4. *For $x \geq 1$, we have*

$$x\mathrm{W}_0(-x\mathrm{e}^{-x}) + 1 \geq 0, \tag{3.4}$$

with equality holding only for $x = 1$.

Proof. For $x > 1$, let $g(x) = x\mathrm{W}_0(-x\mathrm{e}^{-x}) + 1$. As

$$\frac{d}{dx}g(x) = -\frac{\mathrm{W}_0(-x\mathrm{e}^{-x})(x - 2 - \mathrm{W}_0(-x\mathrm{e}^{-x}))}{1 + \mathrm{W}_0(-x\mathrm{e}^{-x})},$$

from (3.1) and (3.2), we have $\frac{d}{dx}g(x) > 0$ for $x > 1$ and consequently $g(x) > g(1) = 0$, with equality holding only at $x = 1$. This completes the proof. □

Theorem 3.5. *For $x \geq 1$, we have*

$$2\ln x - x \leq \mathrm{W}_0(-x\mathrm{e}^{-x}) \leq 2\ln x - 1, \tag{3.5}$$

with equality holding only for $x = 1$.

Proof. Consider the function $f(x) = 2\ln x - x - \mathrm{W}_0(-x\mathrm{e}^{-x})$ for $x \geq 1$. As

$$\frac{d}{dx}f(x) = -\frac{x - 2 - \mathrm{W}_0(-x\mathrm{e}^{-x})}{x(1 + \mathrm{W}_0(-x\mathrm{e}^{-x}))},$$

from (3.1) and (3.2) it is clear that $\frac{d}{dx}f(x) < 0$ for $x > 1$ and consequently $f(x) < f(1) = 0$, which gives the left hand side of (3.5). Trivially, equality only holds for $x = 1$.

For the right hand side, proceeding in a similar manner, we let $g(x) = 2\ln x - 1 - \mathrm{W}_0(-x\mathrm{e}^{-x})$ for $x \geq 1$. Again, since

$$\frac{d}{dx}g(x) = \frac{(x\mathrm{W}_0(-x\mathrm{e}^{-x}) + 1) + (\mathrm{W}_0(-x\mathrm{e}^{-x}) + 1)}{x(1 + \mathrm{W}_0(-x\mathrm{e}^{-x}))},$$

from (3.1) and (3.4), it is immediate that $\frac{d}{dx}g(x) > 0$ for $x > 1$. Thus $g(x) > g(1) = 0$ with equality holding only for $x = 1$, giving the right hand side of (3.5). This completes the proof of the theorem. □

Corollary 3.6. *For $x > 1$, we have*

$$0 < \frac{x + \mathrm{W}_0(-x\mathrm{e}^{-x}) - 2\ln x}{x - 1} < 1. \tag{3.6}$$

Proof. Rearranging terms in (3.5) followed by dividing throughout by $x - 1 > 0$ for $x > 1$, the result follows. □

Theorem 3.7. *For $x > 1$, we have*

$$\frac{(x + \mathrm{W}_0(-x\mathrm{e}^{-x}))^2}{x - 1 - \ln x} > 8. \tag{3.7}$$

Proof. Consider the function $L(x) = \frac{(x+\mathrm{W}_0(-x\mathrm{e}^{-x}))^2}{x-1-\ln x}$ for $x > 1$. Differentiating and simplifying gives

$$\frac{d}{dx}L(x) = -\frac{1 + x\mathrm{W}_0(-x\mathrm{e}^{-x}) + 2\ln x - x - \mathrm{W}_0(-x\mathrm{e}^{-x})}{x(1 + \mathrm{W}_0(-x\mathrm{e}^{-x}))}\left(\frac{x + \mathrm{W}_0(-x\mathrm{e}^{-x})}{x - 1 - \ln x}\right)^2.$$

From the left hand side of (3.1), since $x > 1$ it is clear that $-1 < \mathrm{W}_0(-x\mathrm{e}^{-x})/x$, or $x + \mathrm{W}_0(-x\mathrm{e}^{-x}) > 0$ for $x > 1$. Also, trivially, $x - 1 > \ln x$ for $x > 1$. Hence the squared term appearing in $\frac{d}{dx}L(x)$ is non-zero and therefore always positive. Its denominator is also positive since from (3.1) one has $1 + \mathrm{W}_0(-x\mathrm{e}^{-x}) > 0$ for $x > 1$. To show that the numerator is negative for $x > 1$, let $g(x) = 1 + x\mathrm{W}_0(-x\mathrm{e}^{-x}) + 2\ln x - x - \mathrm{W}_0(-x\mathrm{e}^{-x})$. As

$$\frac{d}{dx}g(x) = -\frac{(x - 2 - \mathrm{W}_0(-x\mathrm{e}^{-x}))(x\mathrm{W}_0(-x\mathrm{e}^{-x}) + 1)}{x(1 + \mathrm{W}_0(-x\mathrm{e}^{-x}))},$$

from (3.1), (3.2), and (3.4) it is clear that $\frac{d}{dx}g(x) < 0$ for $x > 1$ and consequently $g(x) < g(1) = 0$ as required. Thus $\frac{d}{dx}L(x) > 0$. It follows that $L(x) > \lim_{x\to 1^+} L(x) = 8$ for $x > 1$. This completes the proof. □

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