journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 4, Article 110, 11 pp.

REVERSE TRIANGLE INEQUALITY IN HILBERT C^*-MODULES

MARYAM KHOSRAVI, HAKIMEH MAHYAR, AND MOHAMMAD SAL MOSLEHIAN

DEPARTMENT OF MATHEMATICS
TARBIAT MOALLEM UNIVERSITY
TAHRAN, IRAN.
khosravi_m@saba.tmu.ac.ir

mahyar@saba.tmu.ac.ir

DEPARTMENT OF PURE MATHEMATICS
CENTRE OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS)
FERDOWSI UNIVERSITY OF MASHHAD
P.O. BOX 1159, MASHHAD 91775, IRAN.
moslehian@ferdowsi.um.ac.ir
URL: http://profsite.um.ac.ir/~moslehian/

Received 16 October, 2009; accepted 13 November, 2009
Communicated by S.S. Dragomir

ABSTRACT. We prove several versions of reverse triangle inequality in Hilbert C^*-modules. We show that if $e_1, \dots, e_m$ are vectors in a Hilbert module $\mathfrak{X}$ over a C^*-algebra $\mathfrak{A}$ with unit 1 such that $\langle e_i, e_j\rangle = 0$ $(1 \le i \ne j \le m)$ and $\|e_i\| = 1$ $(1 \le i \le m)$, and also $r_k, \rho_k \in \mathbb{R}$ $(1 \le k \le m)$ and $x_1, \dots, x_n \in \mathfrak{X}$ satisfy

$$0 \le r_k^2\|x_j\| \le \mathrm{Re}\langle r_k e_k, x_j\rangle\,, \quad 0 \le \rho_k^2\|x_j\| \le \mathrm{Im}\langle \rho_k e_k, x_j\rangle\,,$$

then

$$\left[\sum_{k=1}^{m}\left(r_k^2+\rho_k^2\right)\right]^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\| \le \left\|\sum_{j=1}^{n}x_j\right\|,$$

and the equality holds if and only if

$$\sum_{j=1}^{n}x_j = \sum_{j=1}^{n}\|x_j\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\,.$$

Key words and phrases: Triangle inequality, Reverse inequality, Hilbert C^*-module, C^*-algebra.

2000 *Mathematics Subject Classification.* Primary 46L08; Secondary 15A39, 26D15, 46L05, 51M16.

1. INTRODUCTION AND PRELIMINARIES

The triangle inequality is one of the most fundamental inequalities in mathematics. Several mathematicians have investigated its generalizations and its reverses.

264-09

In 1917, Petrovitch [17] proved that for complex numbers $z_1, \dots, z_n$,

$$\left|\sum_{j=1}^{n} z_j\right| \geq \cos\theta \sum_{j=1}^{n} |z_j|\,, \tag{1.1}$$

where $0 < \theta < \frac{\pi}{2}$ and $\alpha - \theta < \arg\ z_j < \alpha + \theta\ (1 \leq j \leq n)$ for a given real number α.

The first generalization of the reverse triangle inequality in Hilbert spaces was given by Diaz and Metcalf [5]. They proved that for $x_1, \dots, x_n$ in a Hilbert space H, if e is a unit vector of H such that $0 \leq r \leq \frac{\mathrm{Re}\langle x_j, e\rangle}{\|x_j\|}$ for some $r \in \mathbb{R}$ and each $1 \leq j \leq n$, then

$$r\sum_{j=1}^{n} \|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|\,. \tag{1.2}$$

Moreover, the equality holds if and only if $\sum_{j=1}^{n} x_j = r\sum_{j=1}^{n} \|x_j\| e$.

Recently, a number of mathematicians have presented several refinements of the reverse triangle inequality in Hilbert spaces and normed spaces (see [1, 2, 4, 7, 8, 10, 13, 16]). Recently a discussion of C^*-valued triangle inequalities in Hilbert C^*-modules was given in [3]. Our aim is to generalize some of the results of Dragomir in Hilbert spaces to the framework of Hilbert C^*-modules. For this purpose, we first recall some fundamental definitions in the theory of Hilbert C^*-modules.

Suppose that $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{X}$ is a linear space, which is an algebraic right $\mathfrak{A}$-module. The space $\mathfrak{X}$ is called a pre-Hilbert $\mathfrak{A}$-module (or an inner product $\mathfrak{A}$-module) if there exists an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ with the following properties:

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;
(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$;
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$;
(iv) $\langle x, y\rangle^* = \langle y, x\rangle$

for all $x, y, z \in \mathfrak{X},\ a \in \mathfrak{A},\ \lambda \in \mathbb{C}$. By (ii) and (iv), $\langle\cdot,\cdot\rangle$ is conjugate linear in the first variable. Using the Cauchy–Schwartz inequality $\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle$ [11, Page 5] (see also [14]), it follows that $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is a norm on $\mathfrak{X}$ making it a right normed module. The pre-Hilbert module $\mathfrak{X}$ is called a Hilbert $\mathfrak{A}$-module if it is complete with respect to this norm. Notice that the inner structure of a C^*-algebra is essentially more complicated than that for complex numbers. For instance, properties such as orthogonality and theorems such as Riesz' representation in complex Hilbert space theory cannot simply be generalized or transferred to the theory of Hilbert C^*-modules.

One may define an "$\mathfrak{A}$-valued norm" $|\cdot|$ by $|x| = \langle x, x\rangle^{1/2}$. Clearly, $\|\,|x|\,\| = \|x\|$ for each $x \in \mathfrak{X}$. It is known that $|\cdot|$ does not satisfy the triangle inequality in general. See [11, 12] for more information on Hilbert C^*-modules.

We also use elementary C^*-algebra theory, in particular we utilize the property that if $a \leq b$ then $a^{1/2} \leq b^{1/2}$, where a, b are positive elements of a C^*-algebra $\mathfrak{A}$. We also repeatedly apply the following known relation:

$$\frac{1}{2}(aa^* + a^*a) = (\mathrm{Re}\, a)^2 + (\mathrm{Im}\, a)^2\,, \tag{1.3}$$

where a is an arbitrary element of $\mathfrak{A}$. For details on C^*-algebra theory, we refer readers to [15].

Throughout the paper, we assume that $\mathfrak{A}$ is a unital C^*-algebra with unit 1 and for every $\lambda \in \mathbb{C}$, we write λ for $\lambda 1$.

2. MULTIPLICATIVE REVERSE OF THE TRIANGLE INEQUALITY

Utilizing some C^*-algebraic techniques we present our first result as a generalization of [7, Theorem 2.3].

Theorem 2.1. *Let $\mathfrak{A}$ be a C^*-algebra, let $\mathfrak{X}$ be a Hilbert $\mathfrak{A}$-module and let $x_1, \dots, x_n \in \mathfrak{X}$. If there exist real numbers $k_1, k_2 \geq 0$ with*

$$0 \leq k_1\|x_j\| \leq \mathrm{Re}\langle e, x_j\rangle\,, \quad 0 \leq k_2\|x_j\| \leq \mathrm{Im}\langle e, x_j\rangle\,,$$

for some $e \in \mathfrak{X}$ with $|e| \leq 1$ and all $1 \leq j \leq n$, then

$$(k_1^2 + k_2^2)^{\frac{1}{2}} \sum_{j=1}^n \|x_j\| \leq \left\| \sum_{j=1}^n x_j \right\|. \tag{2.1}$$

Proof. Applying the Cauchy–Schwarz inequality, we get

$$\left| \left\langle e, \sum_{j=1}^n x_j \right\rangle \right|^2 \leq \|e\|^2 \left| \sum_{j=1}^n x_j \right|^2 \leq \left\| \sum_{j=1}^n x_j \right\|^2,$$

and

$$\left| \left\langle \sum_{j=1}^n x_j, e \right\rangle \right|^2 \leq \left\| \sum_{j=1}^n x_j \right\|^2 |e|^2 \leq \left\| \sum_{j=1}^n x_j \right\|^2,$$

whence

$$\begin{aligned}
\left\| \sum_{j=1}^n x_j \right\|^2 &\geq \frac{1}{2}\left(\left| \left\langle e, \sum_{j=1}^n x_j \right\rangle \right|^2 + \left| \left\langle \sum_{j=1}^n x_j, e \right\rangle \right|^2 \right) \\
&= \frac{1}{2}\left(\left\langle e, \sum_{j=1}^n x_j \right\rangle^* \left\langle e, \sum_{j=1}^n x_j \right\rangle + \left\langle \sum_{j=1}^n x_j, e \right\rangle^* \left\langle \sum_{j=1}^n x_j, e \right\rangle \right) \\
&= \left(\mathrm{Re} \left\langle e, \sum_{j=1}^n x_j \right\rangle \right)^2 + \left(\mathrm{Im} \left\langle e, \sum_{j=1}^n x_j \right\rangle \right)^2 \qquad \big(\text{by } (1.3)\big) \\
&= \left(\mathrm{Re} \sum_{j=1}^n \langle e, x_j\rangle \right)^2 + \left(\mathrm{Im} \sum_{j=1}^n \langle e, x_j\rangle \right)^2 \\
&\geq k_1^2 \left(\sum_{j=1}^n \|x_j\| \right)^2 + k_2^2 \left(\sum_{j=1}^n \|x_j\| \right)^2 \\
&= (k_1^2 + k_2^2) \left(\sum_{j=1}^n \|x_j\| \right)^2.
\end{aligned}$$

□

Using the same argument as in the proof of Theorem 2.1, one can obtain the following result, where k_1, k_2 are hermitian elements of $\mathfrak{A}$.

Theorem 2.2. *If the vectors $x_1, \dots, x_n \in \mathfrak{X}$ satisfy the conditions*

$$0 \leq k_1^2\|x_j\|^2 \leq (\mathrm{Re}\langle e, x_j\rangle)^2\,, \quad 0 \leq k_2^2\|x_j\|^2 \leq (\mathrm{Im}\langle e, x_j\rangle)^2\,,$$

for some hermitian elements k_1, k_2 in $\mathfrak{A}$, some $e \in \mathfrak{X}$ with $|e| \leq 1$ and all $1 \leq j \leq n$ then the inequality (2.1) *holds.*

One may observe an integral version of inequality (2.1) as follows:

Corollary 2.3. *Suppose that $\mathfrak{X}$ is a Hilbert $\mathfrak{A}$-module and $f : [a,b] \to \mathfrak{X}$ is strongly measurable such that the Lebesgue integral $\int_a^b \|f(t)\|dt$ exists and is finite. If there exist self-adjoint elements a_1, a_2 in $\mathfrak{A}$ with*

$$a_1^2\|f(t)\|^2 \leq \mathrm{Re}\langle f(t), e\rangle^2 , \quad a_2^2\|f(t)\|^2 \leq \mathrm{Im}\langle f(t), e\rangle^2 \quad (\text{a.e. } t \in [a,b]) ,$$

where $e \in \mathfrak{X}$ with $|e| \leq 1$, then

$$(a_1^2 + a_2^2)^{\frac{1}{2}} \int_a^b \|f(t)\|dt \leq \left\| \int_a^b f(t)dt \right\| .$$

Now we prove a useful lemma, which is frequently applied in the next theorems (see also [3]).

Lemma 2.4. *Let $\mathfrak{X}$ be a Hilbert $\mathfrak{A}$-module and let $x, y \in \mathfrak{X}$. If $|\langle x, y\rangle| = \|x\|\|y\|$, then*

$$y = \frac{x\langle x, y\rangle}{\|x\|^2} .$$

Proof. For $x, y \in \mathfrak{X}$ we have

$$\begin{aligned} 0 &\leq \left| y - \frac{x\langle x, y\rangle}{\|x\|^2} \right|^2 = \left\langle y - \frac{x\langle x, y\rangle}{\|x\|^2}, y - \frac{x\langle x, y\rangle}{\|x\|^2} \right\rangle \\ &= \langle y, y\rangle - \frac{1}{\|x\|^2}\langle y, x\rangle\langle x, y\rangle + \frac{1}{\|x\|^4}\langle y, x\rangle\langle x, x\rangle\langle x, y\rangle - \frac{1}{\|x\|^2}\langle y, x\rangle\langle x, y\rangle \\ &\leq |y|^2 - \frac{1}{\|x\|^2}|\langle x, y\rangle|^2 = |y|^2 - \frac{1}{\|x\|^2}\|x\|^2\|y\|^2 \\ &= |y|^2 - \|y\|^2 \leq 0 , \end{aligned}$$

whence $\left| y - \frac{x\langle x,y\rangle}{\|x\|^2} \right| = 0$. Hence $y = \frac{x\langle x,y\rangle}{\|x\|^2}$. □

Using the Cauchy–Schwarz inequality, we have the following theorem for Hilbert modules, which is similar to [1, Theorem 2.5].

Theorem 2.5. *Let $e_1, \dots, e_m$ be a family of vectors in a Hilbert module $\mathfrak{X}$ over a C^*-algebra $\mathfrak{A}$ such that $\langle e_i, e_j\rangle = 0$ $(1 \leq i \neq j \leq m)$ and $\|e_i\| = 1$ $(1 \leq i \leq m)$. Suppose that $r_k, \rho_k \in \mathbb{R}$ $(1 \leq k \leq m)$ and that the vectors $x_1, \dots, x_n \in \mathfrak{X}$ satisfy*

$$0 \leq r_k^2\|x_j\| \leq \mathrm{Re}\langle r_k e_k, x_j\rangle , \quad 0 \leq \rho_k^2\|x_j\| \leq \mathrm{Im}\langle \rho_k e_k, x_j\rangle ,$$

Then

$$\left[\sum_{k=1}^m (r_k^2 + \rho_k^2) \right]^{\frac{1}{2}} \sum_{j=1}^n \|x_j\| \leq \left\| \sum_{j=1}^n x_j \right\| , \tag{2.2}$$

and the equality holds if and only if

$$\sum_{j=1}^n x_j = \sum_{j=1}^n \|x_j\| \sum_{k=1}^m (r_k + i\rho_k)e_k . \tag{2.3}$$

Proof. There is nothing to prove if $\sum_{k=1}^{m}(r_k^2+\rho_k^2)=0$. Assume that $\sum_{k=1}^{m}(r_k^2+\rho_k^2)\neq 0$. From the hypothesis, by $\mathrm{Im}(a)=\mathrm{Re}(ia^*), \mathrm{Re}(a^*)=\mathrm{Re}(a) \quad (a\in\mathfrak{A})$, we have

$$\left(\sum_{k=1}^{m}(r_k^2+\rho_k^2)\right)^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2 \leq \left(\mathrm{Re}\left\langle\sum_{k=1}^{m} r_k e_k, \sum_{j=1}^{n} x_j\right\rangle + \mathrm{Im}\left\langle\sum_{k=1}^{m}\rho_k e_k, \sum_{j=1}^{n} x_j\right\rangle\right)^2$$
$$= \left(\mathrm{Re}\left\langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\rangle\right)^2.$$

By (1.3),

$$\left(\mathrm{Re}\left\langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\rangle\right)^2$$
$$\leq \frac{1}{2}\left(\left|\left\langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\rangle\right|^2 + \left|\left\langle\sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{j=1}^{n} x_j\right\rangle\right|^2\right)$$
$$\leq \frac{1}{2}\left(\left\|\sum_{j=1}^{n} x_j\right\|^2\left|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\right|^2 + \left\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\|^2\left|\sum_{j=1}^{n} x_j\right|^2\right)$$
$$\leq \left\|\sum_{j=1}^{n} x_j\right\|^2\left\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\|^2$$

and since $|a|\leq\|a\| \quad (a\in\mathfrak{A})$,

$$\left\|\sum_{j=1}^{n} x_j\right\|^2\left\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\|^2 \leq \left\|\sum_{j=1}^{n} x_j\right\|^2\left\|\left\langle\sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\rangle\right\|$$
$$= \left\|\sum_{j=1}^{n} x_j\right\|^2\sum_{k=1}^{m}|r_k+i\rho_k|^2\|e_k\|^2$$
$$= \left\|\sum_{j=1}^{n} x_j\right\|^2\sum_{k=1}^{m}(r_k^2+\rho_k^2).$$

Hence

$$\left[\sum_{k=1}^{m}(r_k^2+\rho_k^2)\right]\left(\sum_{j=1}^{n}\|x_j\|\right)^2 \leq \left\|\sum_{j=1}^{n} x_j\right\|^2.$$

By taking square roots the desired result follows.

Clearly we have equality in (2.2) if condition (2.3) holds. To see the converse, first note that if equality holds in (2.2), then all inequalities in the relations above should be equality. Therefore

$$r_k^2\|x_j\| = \mathrm{Re}\langle r_k e_k, x_j\rangle, \quad \rho_k^2\|x_j\| = \mathrm{Im}\langle\rho_k e_k, x_j\rangle,$$
$$\mathrm{Re}\left\langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\rangle = \left\langle\sum_{j=1}^{n} x_j, \sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\rangle,$$

and

$$\left|\left\langle\sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{j=1}^{n} x_j\right\rangle\right| = \left\|\sum_{j=1}^{n} x_j\right\|\left\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\right\|.$$

From Lemma 2.4 and the above equalities we have

$$\begin{aligned}
\sum_{j=1}^{n} x_j &= \frac{\sum_{k=1}^{m}(r_k+i\rho_k)e_k}{\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\|^2}\left\langle \sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{j=1}^{n} x_j\right\rangle \\
&= \frac{\sum_{k=1}^{m}(r_k+i\rho_k)e_k}{\sum_{k=1}^{m}(r_k^2+\rho_k^2)}\ \mathrm{Re}\left\langle \sum_{k=1}^{m}(r_k+i\rho_k)e_k, \sum_{j=1}^{n} x_j\right\rangle \\
&= \frac{\sum_{k=1}^{m}(r_k+i\rho_k)e_k}{\sum_{k=1}^{m}(r_k^2+\rho_k^2)} \sum_{k=1}^{m}\sum_{j=1}^{n}(r_k^2\|x_j\|+\rho_k^2\|x_j\|) \\
&= \sum_{j=1}^{n}\|x_j\|\sum_{k=1}^{m}(r_k+i\rho_k)e_k\,,
\end{aligned}$$

which is the desired result. □

In the next results of this section, we assume that $\mathfrak{X}$ is a right Hilbert $\mathfrak{A}$-module, which is an algebraic left A-module subject to

$$\langle x, ay\rangle = a\langle x, y\rangle \quad (x, y \in \mathfrak{X}, a \in \mathfrak{A})\,. \tag{†}$$

For example if $\mathfrak{A}$ is a unital C^*-algebra and $\mathfrak{I}$ is a commutative right ideal of $\mathfrak{A}$, then $\mathfrak{I}$ is a right Hilbert module over $\mathfrak{A}$ and

$$\langle x, ay\rangle = x^*(ay) = ax^*y = a\langle x, y\rangle \quad (x, y \in \mathfrak{I},\ a \in \mathfrak{A})\,.$$

The next theorem is a refinement of [7, Theorem 2.1]. To prove it we need the following lemma.

Lemma 2.6. *Let $\mathfrak{X}$ be a Hilbert $\mathfrak{A}$-module and $e_1, \ldots, e_n \in \mathfrak{X}$ be a family of vectors such that $\langle e_i, e_j\rangle = 0$ $(i \neq j)$ and $\|e_i\| = 1$. If $x \in \mathfrak{X}$, then*

$$|x|^2 \geq \sum_{k=1}^{n} |\langle e_k, x\rangle|^2 \quad \textit{and} \quad |x|^2 \geq \sum_{k=1}^{n} |\langle x, e_k\rangle|^2\,.$$

Proof. The first result follows from the following inequality:

$$\begin{aligned}
0 \leq \left|x - \sum_{k=1}^{n} e_k\langle e_k, x\rangle\right|^2 &= \left\langle x - \sum_{k=1}^{n} e_k\langle e_k, x\rangle, x - \sum_{j=1}^{n} e_j\langle e_j, x\rangle\right\rangle \\
&= \langle x, x\rangle + \sum_{k=1}^{n}\sum_{j=1}^{n}\langle e_k, x\rangle^*\langle e_k, e_j\rangle\langle e_j, x\rangle - 2\sum_{k=1}^{n}|\langle e_k, x\rangle|^2 \\
&= \langle x, x\rangle + \sum_{k=1}^{n}\langle e_k, x\rangle^*\langle e_k, e_k\rangle\langle e_k, x\rangle - 2\sum_{k=1}^{n}|\langle e_k, x\rangle|^2 \\
&\leq |x|^2 + \sum_{k=1}^{n}\langle e_k, x\rangle^*\langle e_k, x\rangle - 2\sum_{k=1}^{n}|\langle e_k, x\rangle|^2 \\
&= |x|^2 - \sum_{k=1}^{n}|\langle e_k, x\rangle|^2\,.
\end{aligned}$$

By considering $|x - \sum_{k=1}^{n}\langle e_k, x\rangle e_k|^2$, we similarly obtain the second one. □

Now we will prove the next theorem without using the Cauchy–Schwarz inequality.

Theorem 2.7. *Let* $e_1, \dots, e_m \in \mathfrak{X}$ *be a family of vectors with* $\langle e_i, e_j\rangle = 0$ $(1 \le i \ne j \le m)$ *and* $\|e_i\| = 1$ $(1 \le i \le m)$. *If the vectors* $x_1, \dots, x_n \in \mathfrak{X}$ *satisfy the conditions*

$$0 \le r_k\|x_j\| \le \operatorname{Re}\langle e_k, x_j\rangle\,, \quad 0 \le \rho_k\|x_j\| \le \operatorname{Im}\langle e_k, x_j\rangle\,, \tag{2.4}$$

for $1 \le j \le n, 1 \le k \le m$, *where* $r_k, \rho_k \in [0,\infty)$ $(1 \le k \le m)$, *then*

$$\left[\sum_{k=1}^{m}(r_k^2+\rho_k^2)\right]^{\frac{1}{2}} \sum_{j=1}^{n}\|x_j\| \le \left\|\sum_{j=1}^{n} x_j\right\|. \tag{2.5}$$

Proof. Applying the previous lemma for $x = \sum_{j=1}^n x_j$, we obtain

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\|^2 &\ge \frac{1}{2}\left(\sum_{k=1}^{m}\left|\left\langle e_k, \sum_{j=1}^{n} x_j\right\rangle\right|^2 + \sum_{k=1}^{m}\left|\left\langle \sum_{j=1}^{n} x_j, e_k\right\rangle\right|^2\right)\\
&= \sum_{k=1}^{m}\frac{1}{2}\left(\left\langle e_k, \sum_{j=1}^{n} x_j\right\rangle^{*}\left\langle e_k, \sum_{j=1}^{n} x_j\right\rangle + \left\langle \sum_{j=1}^{n} x_j, e_k\right\rangle^{*}\left\langle \sum_{j=1}^{n} x_j, e_k\right\rangle\right)\\
&= \sum_{k=1}^{m}\left(\operatorname{Re}\left\langle e_k, \sum_{j=1}^{n} x_j\right\rangle\right)^2 + \left(\operatorname{Im}\left\langle e_k, \sum_{j=1}^{n} x_j\right\rangle\right)^2 \qquad \text{(by (1.3))}\\
&= \sum_{k=1}^{m}\left(\operatorname{Re}\sum_{j=1}^{n}\langle e_k, x_j\rangle\right)^2 + \left(\operatorname{Im}\sum_{j=1}^{n}\langle e_k, x_j\rangle\right)^2\\
&\ge \sum_{k=1}^{m}\left(r_k^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2 + \rho_k^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2\right) \qquad \text{(by (2.4))}\\
&= \sum_{k=1}^{m}(r_k^2+\rho_k^2)\left(\sum_{j=1}^{n}\|x_j\|\right)^2.
\end{aligned}$$

□

Proposition 2.8. *In Theorem 2.7, if* $\langle e_k, e_k\rangle = 1$, *then the equality holds in (2.5) if and only if*

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}(r_k + i\rho_k)e_k\,. \tag{2.6}$$

Proof. If (2.6) holds, then the inequality in (2.5) turns trivially into equality.

Next, assume that equality holds in (2.5). Then the two inequalities in the proof of Theorem 2.7 should be equalities. Hence

$$\left\|\sum_{j=1}^{n} x_j\right\|^2 = \sum_{k=1}^{m}\left|\left\langle e_k, \sum_{j=1}^{n} x_j\right\rangle\right|^2 \quad \text{and} \quad \left\|\sum_{j=1}^{n} x_j\right\|^2 = \sum_{k=1}^{m}\left|\left\langle \sum_{j=1}^{n} x_j, e_k\right\rangle\right|^2,$$

which is equivalent to

$$\sum_{j=1}^{n} x_j = \sum_{k=1}^{m}\sum_{j=1}^{n} e_k\langle e_k, x_j\rangle = \sum_{k=1}^{m}\sum_{j=1}^{n}\langle e_k, x_j\rangle e_k\,,$$

and

$$r_k\|x_j\| = \operatorname{Re}\langle e_k, x_j\rangle\,, \quad \rho_k\|x_j\| = \operatorname{Im}\langle e_k, x_j\rangle\,.$$

So

$$\begin{aligned}\sum_{j=1}^{n} x_j &= \sum_{k=1}^{m}\sum_{j=1}^{n} e_k\langle e_k, x_j\rangle \\ &= \sum_{k=1}^{m}\sum_{j=1}^{n} e_k(r_k + i\rho_k)\|x_j\| \\ &= \left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}(r_k + i\rho_k)e_k\,.\end{aligned}$$

□

3. ADDITIVE REVERSE OF THE TRIANGLE INEQUALITY

We now present some versions of the additive reverse of the triangle inequality. In [6], Dragomir established the following theorem:

Theorem 3.1. *Let* $\{e_k\}_{k=1}^{m}$ *be a family of orthonormal vectors in a Hilbert space* H *and* $M_{jk} \geq 0$ $(1 \leq j \leq n, 1 \leq k \leq m)$ *such that*

$$\|x_j\| - \mathrm{Re}\langle e_k, x_j\rangle \leq M_{jk}\,,$$

for each $1 \leq j \leq n$ *and* $1 \leq k \leq m$*. Then*

$$\sum_{j=1}^{n}\|x_j\| \leq \frac{1}{\sqrt{m}}\left\|\sum_{j=1}^{n} x_j\right\| + \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,;$$

and the equality holds if and only if

$$\sum_{j=1}^{n}\|x_j\| \geq \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,,$$

and

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n}\|x_j\| - \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\right)\sum_{k=1}^{m} e_k\,.$$

We can prove this theorem for Hilbert C^*-modules using some different techniques.

Theorem 3.2. *Let* $\{e_k\}_{k=1}^{m}$ *be a family of vectors in a Hilbert module* $\mathfrak{X}$ *over a* C^**-algebra* $\mathfrak{A}$ *with unit* 1, $|e_k| \leq 1$ $(1 \leq k \leq m)$, $\langle e_i, e_j\rangle = 0$ $(1 \leq i \neq j \leq m)$ *and* $x_j \in \mathfrak{X}$ $(1 \leq j \leq n)$*. If for some scalars* $M_{jk} \geq 0$ $(1 \leq j \leq n$, $1 \leq k \leq m)$*,*

$$\|x_j\| - \mathrm{Re}\langle e_k, x_j\rangle \leq M_{jk} \qquad (1 \leq j \leq n, 1 \leq k \leq m)\,, \tag{3.1}$$

then

$$\sum_{j=1}^{n}\|x_j\| \leq \frac{1}{\sqrt{m}}\left\|\sum_{j=1}^{n} x_j\right\| + \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,. \tag{3.2}$$

Moreover, if $|e_k| = 1$ $(1 \leq k \leq m)$*, then the equality in (3.2) holds if and only if*

$$\sum_{j=1}^{n}\|x_j\| \geq \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} M_{jk}\,, \tag{3.3}$$

and

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| - \frac{1}{m} \sum_{j=1}^{n} \sum_{k=1}^{m} M_{jk} \right) \sum_{k=1}^{m} e_k \,. \tag{3.4}$$

Proof. Taking the summation in (3.1) over j from 1 to n, we obtain

$$\sum_{j=1}^{n} \|x_j\| \le \operatorname{Re} \left\langle e_k, \sum_{j=1}^{n} x_j \right\rangle + \sum_{j=1}^{n} M_{jk} \,,$$

for each $k \in \{1, \dots, m\}$. Summing these inequalities over k from 1 to m, we deduce

$$\sum_{j=1}^{n} \|x_j\| \le \frac{1}{m} \operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk} \,. \tag{3.5}$$

Using the Cauchy–Schwarz we obtain

$$\begin{aligned} \left(\operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle \right)^2 &\le \frac{1}{2} \left(\left| \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle \right|^2 + \left| \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle^* \right|^2 \right) \\ &\le \frac{1}{2} \left(\left\| \sum_{k=1}^{m} e_k \right\|^2 \left| \sum_{j=1}^{n} x_j \right|^2 + \left| \sum_{k=1}^{m} e_k \right|^2 \left\| \sum_{j=1}^{n} x_j \right\|^2 \right) \\ &\le \left\| \sum_{k=1}^{m} e_k \right\|^2 \left\| \sum_{j=1}^{n} x_j \right\|^2 \le m \left\| \sum_{j=1}^{n} x_j \right\|^2 , \end{aligned} \tag{3.6}$$

since

$$\begin{aligned} \left\| \sum_{k=1}^{m} e_k \right\|^2 &= \left\| \left\langle \sum_{k=1}^{m} e_k, \sum_{k=1}^{m} e_k \right\rangle \right\| \\ &= \left\| \sum_{k=1}^{m} \sum_{l=1}^{m} \langle e_k, e_l \rangle \right\| = \left\| \sum_{k=1}^{m} |e_k|^2 \right\| \le m \,. \end{aligned}$$

Using (3.6) in (3.5), we deduce the desired inequality.

If (3.3) and (3.4) hold, then

$$\begin{aligned} \frac{1}{\sqrt{m}} \left\| \sum_{j=1}^{n} x_j \right\| &= \frac{1}{\sqrt{m}} \left(\sum_{j=1}^{n} \|x_j\| - \frac{1}{m} \sum_{j=1}^{n} \sum_{k=1}^{m} M_{jk} \right) \left\| \sum_{k=1}^{m} e_k \right\| \\ &= \sum_{j=1}^{n} \|x_j\| - \frac{1}{m} \sum_{j=1}^{n} \sum_{k=1}^{m} M_{jk} \,, \end{aligned}$$

and the equality in (3.2) holds true.

Conversely, if the equality holds in (3.2), then obviously (3.3) is valid and we have equalities throughout the proof above. This means that

$$\|x_j\| - \operatorname{Re}\langle e_k, x_j \rangle = M_{jk} \,,$$

$$\operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle = \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle \,,$$

and

$$\left|\left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle\right| = \left\|\sum_{k=1}^{m} e_k\right\| \left\|\sum_{j=1}^{n} x_j\right\| .$$

It follows from Lemma 2.4 and the previous relations that

$$\begin{aligned}
\sum_{j=1}^{n} x_j &= \frac{\sum_{k=1}^{m} e_k}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle \\
&= \frac{\sum_{k=1}^{m} e_k}{m} \operatorname{Re}\left\langle \sum_{k=1}^{m} e_k, \sum_{j=1}^{n} x_j \right\rangle \\
&= \frac{\sum_{k=1}^{m} e_k}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} (\|x_j\| - M_{jk}) \\
&= \left(\sum_{j=1}^{n} \|x_j\| - \frac{1}{m} \sum_{j=1}^{n} \sum_{k=1}^{m} M_{jk} \right) \sum_{k=1}^{m} e_k \, .
\end{aligned}$$

□

REFERENCES

[1] A.H. ANSARI AND M.S. MOSLEHIAN, Refinements of reverse triangle inequality in inner product spaces, *J. Inequal. Pure Appl. Math.*, **6**(3) (2005), Art. 64. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=537`].

[2] A.H. ANSARI AND M.S. MOSLEHIAN, More on reverse triangle inequality in inner product spaces, *Inter. J. Math. Math. Sci.*, **18** (2005), 2883–2893.

[3] Lj. ARAMBAŠIĆ AND R. RAJIĆ, On the C^*-valued triangle equality and inequality in Hilbert C^*-modules, *Acta Math. Hungar.*, **119**(4) (2008), 373–380.

[4] I. BRNETIC, S.S. DRAGOMIR, R. HOXHA AND J. PEČARIĆ, A reverse of the triangle inequality in inner product spaces and applications for polynomials, *Aust. J. Math. Anal. Appl.*, **3**(2) (2006), Art. 9.

[5] J.B. DIAZ AND F.T. METCALF, A complementary triangle inequality in Hilbert and Banach spaces, *Proc. Amer. Math. Soc.*, **17**(1) (1966), 88–97.

[6] S.S. DRAGOMIR, Reverses of the triangle inequality in inner product spaces, *Linear Algebra Appl.*, **402** (2005), 245–254.

[7] S.S. DRAGOMIR, Some reverses of the generalized triangle inequality in complex inner product spaces, *RGMIA Res. Rep. Coll.*, **7**(E) (2004), Art. 7.

[8] S.S. DRAGOMIR, Reverse of the continuous triangle inequality for Bochner integral in complex Hilbert spaces, *J. Math. Anal. Appl.*, **329** (2007), 65–76.

[9] J. KARAMATA, *Teorijia i Praksa Stieltjesova Integrala (Sebro-Coratian)(Stieltjes Integral, Theory and Practice)*, SANU, Posebna izdanja, **154**, Beograd, 1949.

[10] M. KATO, K.S. SAITO AND T. TAMURA, Sharp triangle inequality and its reverse in Banach spaces, *Math. Inequal. Appl.*, **10** (2007), 451–460.

[11] E.C. LANCE, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995.

[12] V.M. MANUILOV AND E.V. TROITSKY, *Hilbert C^*-Modules*, Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005.

[13] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[14] M.S. MOSLEHIAN AND L.-E. PERSSON, Reverse Cauchy–Schwarz inequalities for positive C*-valued sesquilinear forms, *Math. Inequal. Appl.* (to appear).

[15] J.G. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, Boston, 1990.

[16] M. NAKAI AND T. TADA, The reverse triangle inequality in normed spaces, *New Zealand J. Math.*, **25**(2) (1996), 181–193.

[17] M. PETROVICH, Module d'une somme, *L' Ensignement Math.*, **19** (1917), 53–56.