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Definite Quaternion Orders of Class Number One

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The purpose of the paper is to show how to determine all definite quaternion orders of class number one over the integers. First of all, let us recall that a quaternion order is a ring Λ containing the ring of integers $\mathbf{Z}$ as a subring, finitely generated as a $\mathbf{Z}$-module and such that $A = \Lambda \otimes \mathbf{Q}$ is a central simple four dimensional $\mathbf{Q}$-algebra. By the class number H_Λ of Λ, we mean the number of isomorphism classes of locally free left (or right-both numbers are equal) Λ-ideals in A. Recall that a left Λ-ideal I in A is locally free if for each prime number $p, I_p = I \otimes \mathbf{Z}_p$ is a principal left $\Lambda_p = \Lambda \otimes \mathbf{Z}_p$-ideal, where $\mathbf{Z}_p$ denotes the p-adic integers. Two locally free left Λ-ideals I and I' define the same isomorphism class if $I' = I\,\alpha$, where $\alpha \in A$.

A quaternion order is called definite if $\Lambda \otimes \mathbf{R}$ is the algebra of the Hamiltonian quaternions over the real numbers $\mathbf{R}$. We want to show that there are exactly 25 isomorphism classes of definite quaternion orders of class number one over the integers (an analoguous result, which is much more difficult to prove, says that there are 13 $\mathbf{Z}$-orders of class number one in imaginery quadratic fields over the rational numbers).

First of all, we want to explicity describe all quaternion orders over the integers. This can be done by means of integral ternary quadratic forms

$$f = \sum_{1 \le i \le j \le 3} a_{i,j} X_i X_j,$$

where $a_{i,j} \in \mathbf{Z}$, which will be denoted by

$$f = \begin{pmatrix} a_{11} & a_{22} & a_{33} \\ \\ a_{23} & a_{13} & a_{12} \end{pmatrix}.$$

It is well known that each Λ can be given as $C_0(f)$, where f is a suitable integral ternary quadratic form and $C_0(f)$ is the even Clifford algebra of f.

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The following formulae can be considered as a definition of $C_0(f)$: $C_0(f)$ is a $\mathbf{Z}$-order with a basis 1, e_1, e_2, e_3 such that:

$$e_i^2 = a_{jk}e_i - a_{jj}a_{kk},$$

$$e_ie_j = a_{kk}(a_{ij} - e_k) \quad \text{and} \quad e_je_i = a_{1k}e_1 + a_{2k}e_2 + a_{3k}e_3 - a_{ik}a_{jk},$$

where i, j, k is an even permutation of 1, 2, 3.

Moreover, the above construction defines a one-to-one correspondence between the isomorphism classes of quaternion orders and the equivalence classes of positive definite integral ternary quadratic forms.

Two orders Λ and Λ' are in the same genus of orders (that is, for each prime number p the orders Λ_p and Λ'_p are isomorphic over $\mathbf{Z}_p$) if and only if the corresponding ternary quadratic forms are in the same genus (which means equivalence of them over $\mathbf{Z}_p$ for each prime number p). It is well known that the number of non-isomorphic orders in a genus is finite. The number of isomorphism classes of orders in the genus of Λ will be denoted by T_Λ. T_Λ is called the type number of Λ (two isomorphic orders are said to have the same type).

The discriminant of $\Lambda = C_0(f)$ is $d(C_0(f)) = \frac{1}{2}det(M(f))$, where

$$M(f) = \begin{bmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{12} & 2a_{22} & a_{23} \\ a_{13} & a_{23} & 2a_{33} \end{bmatrix}.$$

$C_0(f)$ is a Gorenstein order if and only if f is primitive, that is, $SGD(a_{ij}) = 1$. Recall that Λ is called Gorenstein if $\Lambda^{\#} = \mathrm{Hom}(\Lambda, \mathbf{Z})$ is projective as left (or right) Λ-module (see [CR], p. 778).

If $C_0(f)$ is not isomorphic to the matrix ring $M_2(\mathbf{Z})$, which happens exactly when $d(C_0(f)) \neq \pm 1$, then define

$$e_{\mathbf{p}}(C_0(f)) = \begin{cases} -1 & \text{if} \quad f \quad \text{modulo} \quad p \quad \text{is irreducible,} \\ 0 & \text{if} \quad f \quad \text{modulo} \quad p \quad \text{is a square of a linear factor,} \\ 1 & \text{if} \quad f \quad \text{modulo} \quad p \quad \text{is a product of two different linear factors.} \end{cases}$$

Using the description of quaternion orders by means of ternary quadratic forms, a formula for H_Λ proved in [B2], (4.5) and some results on the structure of quaternion orders proved in [B1], we get the following list:

THEOREM. *There are 25 isomorphism classes of* $\mathbf{Z}$*-orders with class number 1 in definite quaternion* $\mathbf{Q}$*-algebras. These classes are represented by the orders* $C_0(f)$*, where* f *is one of the following forms (the index of the matrix corresponding to a quadratic form* f *is the discriminant of the order* $C_0(f)$*):*

$$\begin{pmatrix}1&1&1\\0&1&1\end{pmatrix}_2, \begin{pmatrix}1&1&1\\0&0&1\end{pmatrix}_3, \begin{pmatrix}1&1&1\\0&0&0\end{pmatrix}_4, \begin{pmatrix}1&1&2\\0&1&1\end{pmatrix}_5, \begin{pmatrix}1&1&2\\1&1&0\end{pmatrix}_6,$$

$$\begin{pmatrix}1&1&2\\0&0&1\end{pmatrix}_6, \begin{pmatrix}1&1&2\\0&1&0\end{pmatrix}_7, \begin{pmatrix}1&1&3\\0&1&1\end{pmatrix}_8, \begin{pmatrix}1&1&2\\0&0&0\end{pmatrix}_8, \begin{pmatrix}1&1&3\\0&0&1\end{pmatrix}_9,$$

$$\begin{pmatrix}1&1&3\\1&1&0\end{pmatrix}_{10}, \begin{pmatrix}1&2&2\\2&0&1\end{pmatrix}_{10}, \begin{pmatrix}1&1&4\\0&0&1\end{pmatrix}_{12}, \begin{pmatrix}1&2&2\\0&1&1\end{pmatrix}_{12}, \begin{pmatrix}1&1&3\\0&0&0\end{pmatrix}_{12},$$

$$\begin{pmatrix}1&2&2\\2&0&0\end{pmatrix}_{12}, \begin{pmatrix}1&2&2\\1&0&1\end{pmatrix}_{13}, \begin{pmatrix}1&2&2\\0&0&0\end{pmatrix}_{16}, \begin{pmatrix}1&1&5\\1&1&0\end{pmatrix}_{18}, \begin{pmatrix}1&2&3\\2&1&0\end{pmatrix}_{18},$$

$$\begin{pmatrix}1&2&3\\2&0&0\end{pmatrix}_{20}, \begin{pmatrix}1&2&3\\0&1&0\end{pmatrix}_{22}, \begin{pmatrix}1&3&3\\-1&1&1\end{pmatrix}_{28}, \begin{pmatrix}2&2&2\\0&2&2\end{pmatrix}_{16}, \begin{pmatrix}2&2&2\\0&0&2\end{pmatrix}_{24}.$$

Proof. Let Λ be a quaternion $\mathbf{Z}$-order with class number $H_\Lambda = 1$. Then

$$M_\Lambda = \frac{d(\Lambda)}{12} \prod_{p|d(\Lambda} \frac{1-p^{-2}}{1-e_p(\Lambda)p^{-2}} \le 1,$$

(see [K], Thm. 1 or [B2], (4.6)). Denoting by ϕ the Euler totient function, we have

$$(*) \quad \phi(d(\Lambda))(1+p_1)\cdots(1+p_r)(1+p'_1)\cdots(1+p'_s) \le 12(p_1-1)\cdots (p_r-1)p'_1\cdots p'_s,$$

where p_i and p'_s are all prime factors of $d(\Lambda$ such that $e_{p_i}(\Lambda) = 1$ and $e_{p'_i}(\Lambda) = 0$. This inequality implies that $\phi(d(\Lambda)) \le 12$ and if $\phi(d(\Lambda)) = 12$, then for each prime factor p of $d(\Lambda)$, $e_p(\Lambda) = -1$. The condition $\phi(d(\Lambda)) \le 12$ says that $2 \le d(\Lambda) \le 16$ or $d(\Lambda) = 18, 20, 21, 22, 24, 26, 28, 30, 36, 42$.

Assume now that Λ is a Gorenstein $\mathbf{Z}$-order. Then $\Lambda = C_0(f)$, where f is a primitive integral ternary quadratic form with only one class in its genus, since $T_\Lambda \le H_\Lambda$ (see [V], p. 88). Thus, using the tables [BI], we can first of all eliminate all classes with $\phi(d(\Lambda)) \le 12$ for which $T_\Lambda \ge 2$. The next test is given by the inequality (*), which can be checked for the genera with $T_\Lambda = 1$ using the reduction modulo the primes $p|d(\Lambda)$. In that way, we get $e_p(\Lambda)$ and choose those classes for which (*) is valid. Finally, using a general formula for H_Λ proved in [B2], (4.5), we compute H_Λ for the remaining classes and obtain our list of 23 isomorphism classes of orders having class number 1.

The case of non-Gorenstein orders unexpecteddly gives two more classes. Assume that Λ is a non-Gorenstein order with class number 1. Then $d(\Lambda)$ must be divisible by a third power of an integer > 1, since $\Lambda = \mathbf{Z} + d\Lambda'$ for an integer $d > 1$ according to [B1], (1.4). The only possibility is $d = 2$ by $\phi(d(\Lambda)) \leq 12$. Thus $d(\Lambda) \in \{8, 16, 24\}$. But $d(\Lambda) = 8$ is impossible, since then $d(\Lambda') = 1$, which can not happen for an order in a definite algebra. If $d(\Lambda) = 24$, then $d(\Lambda') = 3$, which gives $e_3(\Lambda) = -1$ (and, of course, $e_2(\Lambda) = 0$). Using the formula of [B2], (4.5), an easy computation shows that $H_\Lambda = 1$. If $d(\Lambda) = 16$, then $\Lambda = \mathbf{Z} + 2\Lambda'$, where $d(\Lambda') = 2$ and, in a similar way, we get $H_\Lambda = 1$.

Finally notice that 10 isomorphism classes with class number 1 corresponding to Eichler orders (that is, those orders whose discriminant is square-free) were determined by M.-F. Vigneras (see [V], p. 155).

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