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Mean Square of the Remainder Term in the Dirichlet Divisor Problem

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1. Introduction and Main Results

Let $d(n)$ denote the divisor function. In this paper we shall consider a remainder term associated with the mean square of the error term $\triangle(x)$ in the Dirichlet divisor problem, which is defined as

$$\triangle(x) = \sum_{n \le x} d(n) - x(\log x + 2\gamma - 1) \ .$$

Here γ is Euler's constant. The upper bound $\triangle(x) \ll x^{1/2}$ was first obtained by Dirichlet in 1838. This was gradually sharpened by many authors in the ensuing one and a half century. Iwaniec and Mozzochi [5] proved in 1988 that $\triangle(x) \ll x^{7/22+\varepsilon}$ for any $\varepsilon > 0$, by employing intricated techniques for the estimation of certain exponential sums. Such methods, however, do not seem capable of proving the conjectured best bound: $\triangle(x) \ll x^{1/4+\varepsilon}$.

Besides this problem, there are plenty of papers written on other interesting properties of $\triangle(x)$. For instance, Tong [9] showed that $\triangle(x)$ changes sign at least once in every interval of the form $[X, X + c_0\sqrt{X}]$ where c_0 is a certain positive constant. Recently Heath-Brown and Tsang [2] showed that this is essentially best possible: – the length of the intervals cannot be reduced to $o(\sqrt{X}\log^{-5} X)$. In contrast to this erratic behaviour, $\triangle(x)$, when considered in the mean, has very nice asymptotic formula. A classical result of Tong [10] says that

$$\int_2^X \triangle(x)^2 dx = \Big((6\pi^2)^{-1} \sum_{m=1}^{\infty} d(m)^2 m^{-3/2}\Big) X^{3/2} + F(X) \tag{1.1}$$

with $F(X) \ll X \log^5 X$. The order of the remainder term $F(X)$ has significant connection with that of $\triangle(x)$. Indeed, Ivić's argument in Theorem

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3.8 of [4] shows that $\triangle(x) \ll (U \log x)^{1/3}$ for any upper bound U of $F(X)$. Thus from the result $\triangle(x) = \Omega(x^{1/4})$ we infer that

$$F(X) = \Omega(X^{3/4} / \log X) \ . \tag{1.2}$$

Ivić conjectured that $F(X) \ll X^{3/4+\varepsilon}$ is true for any $\varepsilon > 0$. This is a very strong bound since it implies $\triangle(x) \ll x^{1/4+\varepsilon}$. There are not many results on $F(X)$ in the literature. Tong's bound was slightly improved to $F(X) \ll X \log^4 X$ by Preissmann [7] in 1988. However, the gap between this and the Ω-result (1.2) is still very wide.

In this paper we shall prove the following.

Theorem 1. *We have*

$$F(X) = \Omega_{-}(X \log^2 X) \ .$$

Theorem 2. *For $X \geq 2$ we have*

$$\int_2^X F(x)dx = -(8\pi^2)^{-1} X^2 \log^2 X + c_1 X^2 \log X + \mathcal{O}(X^2)$$

for a certain constant c_1.

Theorem 1, which is a direct consequence of Theorem 2, disproves the above conjecture of Ivić. Unfortunately we are still unable to obtain a comparable Ω_{+}-result for $F(x)$. In fact we believe that there is an asymptotic formula for $F(x)$ of the form

$$F(x) = -(4\pi^2)^{-1} x \log^2 x + c_2 x \log x + \mathcal{O}(x) \tag{1.3}$$

with a certain constant c_2. In a forthcoming paper, the second author [11] proves that

$$\int_X^{2X} \big(F(x+\sqrt{X}) - F(x)\big)^2 dx \asymp X^3 \ .$$

Using Preissmann's bound we see easily that

$$\int_X^{2X} \big(F(x+\sqrt{X}) - F(x)\big) dx = \int_{2X}^{2X+\sqrt{X}} - \int_X^{X+\sqrt{X}} F(x)dx$$
$$\ll X^{3/2} \log^4 X \ .$$

These two results together shows that $F(x+\sqrt{X})-F(x)$ changes signs in $[X,2X]$ and

$$F(x+\sqrt{X})-F(x)=\Omega(X)\ .$$

Consequently, if (1.3) is true the $\mathcal{O}$-term on the right hand side is oscillatory and cannot be reduced.

One of the key ingredients in our argument is an asymptotic formula for the sum

$$\sum_{m\le x} d(m)d(m+h)\ .$$

Such a sum has been investigated by several authors in connection with other problems in analytic number theory. In our proof we use a result of Heath-Brown [1] which is quite sufficient for our purpose. (see (2.12)-(2.15) below)

2. Notations and some Preparation

Throughout the paper, ε denotes an arbitrary small positive number which need not be the same at each occurrence. The symbols $c_0, c_1, c_2, \ldots$ etc. denote certain constants. We shall also use the well-known inequality $d(n)\ll n^\varepsilon$ from time to time without explicit reference. The constants implicit in the symbols $\mathcal{O}$ and $\ll$ depend at most on ε.

A useful formula for studying problems concerning $\triangle(x)$ was obtained by Voronoi [12] at the beginning of this century. The formula expresses $\triangle(x)$ as an infinite series involving the Bessel functions. In practice, the following truncated form of the formula

$$\begin{aligned}\triangle(x) &= (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)\\ &\quad+\mathcal{O}(x^\varepsilon + x^{1/2+\varepsilon}N^{-1/2})\end{aligned}$$

for $1\le N\ll x$ is quite sufficient. However, for our present problem, the above $\mathcal{O}$-term is far too large and we shall use instead the following approximation to $\triangle(x)$ given by Meurman [6, Lemma 3].

Lemma 1. *For $x\ge 1$ and $M\gg x$, let*

$$\delta_M(x)=(\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le M} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)\ .$$

Then $\triangle(x) = \delta_M(x) + R(x)$ *where* $R(x) \ll x^{-1/4}$ *if* $\|x\| \gg x^{5/2}M^{-1/2}$ *and* $R(x) \ll x^{\varepsilon}$ *otherwise.*

Using this we obtain

LEMMA 2. *Let* $x \geq 2$ *and* $x^7 \ll M \ll x^{100}$. *Then*

$$\int_2^x \triangle(u)^2 du = \int_2^x \delta_M(u)^2 du + \mathcal{O}(x) \ .$$

Proof. Firstly,

$$\int_2^x \triangle(u)^2 du = \int_2^x \delta_M(u)^2 du + 2\int_2^x \delta_M(u)R(u)du + \int_2^x R(u)^2 du \ .$$

Next, by Lemma 1, we have

$$\int_2^x R(u)^2 du \ll \sum_{n=2}^{[x]+1} n^{\varepsilon} n^{5/2} M^{-1/2} + \int_2^x (u^{-1/4})^2 du \ll \sqrt{x} \ . \tag{2.1}$$

Moreover, following the argument of [3, Theorem 13.5] we show that

$$\int_2^x \delta_M(u)^2 du \asymp x^{3/2}$$

for $M \ll x^{100}$. Thus, by Cauchy-Schwarz's inequality and (2.1) we have

$$\int_2^x \delta_M(u)R(u)du \ll x$$

and hence our lemma.

Square out $\delta_M(u)$ and then integrate term by term, we get

$$\begin{aligned}
&\int_2^x \delta_M(u)^2 du \\
&= (4\pi^2)^{-1} \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u}\cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)du \\
&+ (4\pi^2)^{-1} \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u}\sin\left(4\pi(\sqrt{n}+\sqrt{m})\sqrt{u}\right)du.
\end{aligned}$$

In the first double sum the diagonal terms yield a total contribution of

$$\begin{aligned}&(4\pi^2)^{-1}\sum_{m\le M} d(m)^2 m^{-3/2}\frac{2}{3}(x^{3/2}-2^{3/2})\\ =\ &(6\pi^2)^{-1}\sum_{m=1}^{\infty} d(m)^2 m^{-3/2}x^{3/2}+\mathcal{O}(x^{3/2}M^{\varepsilon-1/2}\ +\ 1)\ .\end{aligned}$$

Here the main term is the same as that in (1.1). Hence by Lemma 2, we can write

$$F(x)=S_1(x)+S_2(x)+\mathcal{O}(x)\ , \tag{2.2}$$

where for any $y\ge 2$,

$$S_1(y)=(2\pi^2)^{-1}\sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}\times \int_2^y \sqrt{u}\cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)du\ , \tag{2.3}$$

and

$$S_2(y)=(4\pi^2)^{-1}\sum_{m,n\le M} d(m)d(n)(mn)^{-3/4}\times \int_2^y \sqrt{u}\sin\left(4\pi(\sqrt{n}+\sqrt{m})\sqrt{u}\right)du\ . \tag{2.4}$$

From now on, we let X to be a sufficiently large number, $M=X^7$ and $L=\log X$. For any $\nu\ge 0$, let

$$g(\nu)=\nu^{-3/2}J_{3/2}(\nu)-4\nu^{-5/2}J_{5/2}(\nu)\ , \tag{2.5}$$

where J_k denotes the Bessel function of order k. It is well-known that [13, §§3.3, 3.4]

$$J_k(z)\ll \min(|z|^k,|z|^{-1/2})$$

for any real z. Hence,

$$g(\nu)\ll\min(1,\nu^{-2}) \tag{2.6}$$

for any $\nu\ge 0$.

LEMMA 3. *We have*

(2.7)
$$\int_0^X F(x)dx = \sqrt{2}\pi^{-3/2}X^{5/2} \sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}g(\theta_{m,n}) + \mathcal{O}(X^2)$$

where $\theta_{m,n} = 4\pi(\sqrt{n}-\sqrt{m})\sqrt{X}$.

Proof. By [8, Lemma 4.2], for any real α and y, $y \ge 2$ we have

$$\int_2^y \sqrt{u}e^{i\alpha\sqrt{u}}du \ll y|\alpha|^{-1} \ . \tag{2.8}$$

We first obtain some preliminary bounds for $S_1(y)$ and $S_2(y)$. According to (2.3) and on applying (2.8), we have

$$S_1(y) \ll y \sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}(\sqrt{n}-\sqrt{m})^{-1} \tag{2.9}$$

$$S_1(y) \ll y \sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}(\sqrt{n}-\sqrt{m})^{-1} \tag{2.9}$$

$$\begin{aligned}
&\ll yM^\varepsilon\Big\{ \sum_{m<n\le 2m\le M} (mn)^{-3/4}(\sqrt{n}+\sqrt{m})(n-m)^{-1} \\
&\qquad + \sum_{2m<n\le M} m^{-3/4}n^{-5/4}\Big\} \\
&\ll yM^\varepsilon\Big\{ \sum_{m\le M/2} m^{-1} \sum_{m<n\le M}(n-m)^{-1} + \log M\Big\} \ll yM^\varepsilon \ .
\end{aligned}$$

Similarly,

$$S_2(y) \ll y \sum_{m,n\le M} d(m)d(n)(mn)^{-3/4}(\sqrt{n}+\sqrt{m})^{-1} \ll yM^\varepsilon \ . \tag{2.10}$$

Next, for $x \in [\sqrt{X}, X]$ we have $x^7 \ll M \ll x^{14}$ so that, by (2.2)

(2.11)
$$\begin{aligned}
\int_{\sqrt{X}}^X F(x)dx &= \int_{\sqrt{X}}^X S_1(x)dx + \int_{\sqrt{X}}^X S_2(x)dx + \mathcal{O}(X^2) \\
&= \int_2^X S_1(x)dx + \int_2^X S_2(x)dx + \mathcal{O}(XM^\varepsilon + X^2) \ ,
\end{aligned}$$

since, by (2.9) and (2.10), $\int_2^{\sqrt{X}} S_i(x)dx \ll XM^\varepsilon$. The main term on the right hand side of (2.7) arises from $\int_2^X S_1(x)dx$. Indeed, by (2.3),

$$\int_2^X S_1(x)dx = (2\pi^2)^{-1} \sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}\times$$
$$\int_2^X \int_2^x \sqrt{u}\cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)dudx \ .$$

Write $\theta = 4\pi(\sqrt{n}-\sqrt{m})\sqrt{X}$ for short. Then the above double integral is equal to

$$\begin{aligned}
&\int_2^X (X-u)\sqrt{u}\cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)du \\
&= 2X^{5/2}\int_{\sqrt{2/X}}^1 (1-v^2)v^2\cos(\theta v)dv \\
&= 2X^{5/2}\Big\{\int_0^1 (1-v^2)\cos(\theta v)dv - \int_0^1 (1-v^2)^2\cos(\theta v)dv \\
&\quad - \int_0^{\sqrt{2/X}} (1-v^2)v^2\cos(\theta v)dv\Big\} \ .
\end{aligned}$$

By the well-known integral representation

$$J_{k+\frac12}(z) = \frac{2}{\sqrt{\pi}}\Big(\frac{z}{2}\Big)^{k+\frac12}\frac{1}{k!}\int_0^1 (1-v^2)^k\cos(zv)dv \ , \quad k=0,1,2,\ldots$$

for the Bessel functions [13, §3.3], the first two integrals on the right hand side is equal to

$$\sqrt{2\pi}\left(\theta^{-3/2}J_{3/2}(\theta) - 4\theta^{-5/2}J_{5/2}(\theta)\right) = \sqrt{2\pi}g(\theta) \ ,$$

by (2.5). Moreover, using integration by parts we find that

$$\int_0^{\sqrt{2/X}} (1-v^2)v^2\cos(\theta v)dv \ll X^{-1}\theta^{-1} \ .$$

Hence

$$\int_2^X (X-u)\sqrt{u}\cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)du = 2\sqrt{2\pi}X^{5/2}g(\theta) + \mathcal{O}(X^{3/2}\theta^{-1}),$$

and then

$$\int_2^X S_1(x)dx = \sqrt{2}\pi^{-3/2}X^{5/2}\sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}g(\theta)$$
$$+\,\mathcal{O}\Big(X\sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}(\sqrt{n}-\sqrt{m})^{-1}\Big)\ .$$

The sum inside the $\mathcal{O}$-term can be treated by the argument in (2.9), and we then find that the $\mathcal{O}$-term is bounded by XM^{ε}, which is smaller than that on the right hand side of (2.7).

In view of (2.11) and (2.7), it remains to bound the two integrals $\int_0^{\sqrt{X}} F(x)dx$ and $\int_2^X S_2(x)dx$ by X^2. By Preissmann's bound, we have

$$\int_0^{\sqrt{X}} F(x)dx \ll X\log^4 X$$

which is acceptable. Next, by (2.4),

$$\int_2^X S_2(x)dx = (4\pi^2)^{-1}\sum_{m,n\le M} d(m)d(n)(mn)^{-3/4}\times$$
$$\int_2^X (X-u)\sqrt{u}\sin\big(4\pi(\sqrt{n}+\sqrt{m})\sqrt{u}\big)du$$
$$= (2\pi^2)^{-1}X^{5/2}\sum_{m,n\le M} d(m)d(n)(mn)^{-3/4}\times$$
$$\int_{\sqrt{2/X}}^1 (1-v^2)v^2\sin\big(4\pi(\sqrt{n}+\sqrt{m})\sqrt{X}v\big)dv\ .$$

The inner integral, on applying integration by parts twice, is found to be

$$\ll X^{-3/2}(\sqrt{n}+\sqrt{m})^{-1}+X^{-1}(\sqrt{n}+\sqrt{m})^{-2}.$$

Thus,

$$\int_2^X S_2(x)dx \ll X\sum_{m\le n\le M} d(m)d(n)(mn)^{-3/4}n^{-1/2}+$$
$$X^{3/2}\sum_{m\le n\le M} d(m)d(n)(mn)^{-3/4}n^{-1}$$
$$\ll XM^{\varepsilon}+X^{3/2}\ll X^{3/2}\ .$$

This completes the proof of Lemma 3.

For any $y > 0$, let

$$\psi_h(y) = \sum_{m \le y} d(m)d(m+h) \ . \tag{2.12}$$

In his work on the fourth power moment of the Riemann zeta-function on the critical line, Heath-Brown [1] proved that

$$\psi_h(y) = I_h(y) + E_h(y), \tag{2.13}$$

where the main term $I_h(y)$ is of the form

$$I_h(y) = y \sum_{i=0}^{2} \log^i y \sum_{d|h} d^{-1}(\alpha_{i0} + \alpha_{i1} \log d + \alpha_{i2} \log^2 d) \tag{2.14}$$

for certain constants α_{ij}, and the remainder $E_h(y)$ satisfies

$$E_h(y) \ll y^{5/6+\varepsilon} \tag{2.15}$$

uniformly for $1 \le h \le y^{5/6}$. In particular $\alpha_{20} = 6\pi^{-2}$, $\alpha_{21} = \alpha_{22} = 0$. We note that $I_h(y)$ is roughly of order $y \log^2 y$. In our proof of Theorem 2 in §3 we shall need $I_h'(y)$, the derivative of $I_h(y)$. By (2.14)

$$I_h'(y) = a_2(h) \log^2 y + a_1(h) \log y + a_0(h) \tag{2.16}$$

where

$$\begin{aligned} a_2(h) &= 6\pi^{-2} \sum_{d|h} d^{-1} \ , \\ a_1(h) &= \sum_{d|h} d^{-1}(12\pi^{-2} + \alpha_{10} + \alpha_{11} \log d + \alpha_{12} \log^2 d) \ , \\ a_0(h) &= \sum_{d|h} d^{-1} \sum_{j=0}^{2} (\alpha_{0j} + \alpha_{1j}) \log^j d \ . \end{aligned} \tag{2.17}$$

For any $y > 0$, $Q > 3$ let

$$\xi(y, Q) = \sum_{h \le y} h^{-1} \big(4a_2(h) \log^2 Qh + 2a_1(h) \log Qh + a_0(h)\big) \ . \tag{2.18}$$

Lemma 4. *We have*

$$\xi(y,Q)=\frac{4}{3}\log^3 Qy+c_3\log^2 Qy-\frac{4}{3}\log^3 Q+c_4\log^2 Q+c_5\log Q$$
$$+c_6\log y+c_7+\mathcal{O}(y^{-1}\log^3 y\log^2 Qy)\ .$$

Proof. In the argument below we use the symbol c to denote a certain constant which may not be the same at each occurrence.

Firstly, for $j=0,1,2$ there are constants β_0,β_1,β_2 such that

$$\sum_{h\le y}a_j(h)=\beta_j y+B_j(y) \tag{2.19}$$

with $B_j(y)\ll\log^3 y$. (Note $B_j(1^-)=-\beta_j$). Indeed, by (2.17),

$$\begin{aligned}\sum_{h\le y}a_0(h)&=\sum_{d\le y}d^{-1}\sum_{j=0}^{2}(\alpha_{0j}+\alpha_{1j})(\log^j d)(yd^{-1}+\mathcal{O}(1))\\&=y\sum_{d\le y}d^{-2}\sum_{j=0}^{2}(\alpha_{0j}+\alpha_{1j})\log^j d+\mathcal{O}\Big(\sum_{d\le y}d^{-1}\log^2 d\Big)\\&=y\sum_{d=1}^{\infty}d^{-2}\sum_{j=0}^{2}(\alpha_{0j}+\alpha_{1j})\log^j d+\mathcal{O}\Big(y\sum_{d>y}d^{-2}\log^2 d\Big)+\\&\quad+\mathcal{O}(\log^3 y)\\&=\beta_0 y+\mathcal{O}(\log^3 y)\end{aligned}$$

with

$$\beta_0=\sum_{d=1}^{\infty}d^{-2}\sum_{j=0}^{2}(\alpha_{0j}+\alpha_{1j})\log^j d\ .$$

Similar argument establishes (2.19) for $j=1$ and 2. Further we find that $\beta_2=1$.

Next by Riemann Stieltjes integration and (2.19), we have

$$\begin{aligned}
&\sum_{h\le y} a_2(h)h^{-1}\log^2 Qh \\
&= \int_1^y t^{-1}\log^2 Qt dt + \big[t^{-1}\log^2 Qt\ B_2(t)\big]_{1-}^{y} \\
&\quad - \int_1^y B_2(t)t^{-2}(2\log Qt - \log^2 Qt)dt \\
&= \frac{1}{3}(\log^3 Qy - \log^3 Q) + \log^2 Q + \mathcal{O}(y^{-1}\log^3 y\log^2 Qy) \\
&\quad - \int_1^y B_2(t)t^{-2}(-\log^2 Q + 2(1-\log t)\log Q + 2\log t - \log^2 t)dt \\
&= \frac{1}{3}(\log^3 Qy - \log^3 Q) + \log^2 Q + c\log^2 Q + c\log Q + c \\
&\quad + \mathcal{O}\Big(\int_y^\infty (\log^3 t)t^{-2}(\log^2 Q + \log^2 t)dt\Big) + \mathcal{O}(y^{-1}\log^3 y\log^2 Qy) \\
&= \frac{1}{3}(\log^3 Qy - \log^3 Q) + c\log^2 Q + c\log Q + c + \mathcal{O}(y^{-1}\log^3 y\log^2 Qy)\ .
\end{aligned}$$

In the same way, we find that

$$\sum_{h\le y} a_1(h)h^{-1}\log Qh = \frac{1}{2}\beta_1(\log^2 Qy - \log^2 Q) + c\log Q + c + \\ + \mathcal{O}(y^{-1}\log^3 y\log Qy)$$

and

$$\sum_{h\le y} a_0(h)h^{-1} = \beta_0\log y + c + \mathcal{O}(y^{-1}\log^3 y)\ .$$

Collecting all these in (2.18) our lemma follows.

Lastly we evaluate some integrals involving the function $g(\nu)$.

Lemma 5. *We have*

$$\int_0^\infty g(\nu)d\nu = 0\ ,$$
$$\int_0^\infty g(\nu)\log\nu d\nu = -\sqrt{\pi}2^{-7/2}\ .$$

Proof. It is known that [13, §13.24]

$$\int_0^\infty J_k(\nu)\nu^{s-k-1}d\nu = \Gamma\big(\frac{s}{2}\big)2^{s-k-1}/\Gamma\big(k-\frac{s}{2}+1\big)$$

for $0 < \operatorname{Re} s < \operatorname{Re} k + 1/2$. Hence

$$\begin{aligned}\int_0^\infty &\big(\nu^{-k}J_k(\nu) - (2k+1)\nu^{-k-1}J_{k+1}(\nu)\big)\nu^s d\nu \\ &= -s2^{s-k-1}\Gamma\big(\frac{s+1}{2}\big)/\Gamma\big(k-\frac{s}{2}+\frac{3}{2}\big)\end{aligned}$$

for $-1 < \operatorname{Re} s < \operatorname{Re} k - 1/2$. Setting $k = 3/2$ and in view of (2.5) we have

$$\int_0^\infty g(\nu)\nu^s d\nu = -s2^{s-5/2}\Gamma\big(\frac{s+1}{2}\big)/\Gamma\big(3-\frac{s}{2}\big) \tag{2.20}$$

for $-1 < \operatorname{Re} s < 1$. On putting $s = 0$ we get $\int_0^\infty g(\nu)d\nu = 0$. The remaining integral is equal to

$$\frac{d}{ds}\Big(\int_0^\infty g(\nu)\nu^s d\nu\Big)\Big|_{s=0}$$

which can be evaluated by differentiating the right hand side of (2.20).

3. Proof of Theorem 2

We shall now complete the proof of Theorem 2 by evaluating the double sum

$$T = \sum_{m<n\le M} u_{m,n} \tag{3.1}$$

in Lemma 3, where

$$u_{m,n} = d(m)d(n)(mn)^{-3/4}g\big(4\pi(\sqrt{n}-\sqrt{m})\sqrt{X}\big)\ .$$

In view of Lemma 3, we can allow errors of order up to $X^{-1/2}$ in the course of our analysis.

First of all, we consider those terms $u_{m,n}$ for which $m < n/2$. In this case $\sqrt{n} - \sqrt{m} \asymp \sqrt{n}$ so that, by (2.6)

$$g\big(4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}\big) \ll (nX)^{-1} \ .$$

The contribution to T from these $u_{m,n}$ is therefore

$$\ll X^{-1} \sum_{m<n\le M} d(m)d(n)(mn)^{-3/4}n^{-1} \ll X^{-1},$$

which is acceptable.

For the remaining terms $u_{m,n}$ in T, we write $n = m+h$ with $1 \le h \le m$. Then

$$T = \sum_{h\le M/2} \ \sum_{h\le m\le M-h} u_{m,m+h} + \mathcal{O}(X^{-1}) \ .$$

For $h \le m$, we have $4\pi\big(\sqrt{m+h} - \sqrt{m}\big)\sqrt{X} \asymp 2\pi h\sqrt{X/m}$ so that, by (2.6) again

$$g\big(4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X}\big) \ll mh^{-2}X^{-1} \tag{3.2}$$

and each term $u_{m,m+h}$ satisfies

$$u_{m,m+h} \ll M^{\varepsilon}m^{-3/2}mh^{-2}X^{-1} \ .$$

Thus, the contribution to T from those $u_{m,m+h}$ with $h > \sqrt{M}$ is $\ll X^{-1}M^{\varepsilon}$ and the error caused by extending the upper limit for the summation on m to M is $\mathcal{O}(X^{-1}M^{-1/2+\varepsilon})$. Hence we have

$$T = \sum_{h\le\sqrt{M}} \ \sum_{h\le m\le M} u_{m,m+h} + \mathcal{O}(X^{-1}M^{\varepsilon}) \ .$$

For simplicity let

$$D_h = h^2XL^{-8} \ .$$

Then we can further write

$$\begin{aligned} T &= \sum_{h\le\sqrt{M}} \ \sum_{h\le m\le\min(D_h,M)} + \sum_{h\le X^3L^4} \ \sum_{D_h<m\le M} + \mathcal{O}(X^{-1}M^{\varepsilon}) \\ &= \sum\nolimits_1 + \sum\nolimits_2 + \mathcal{O}(X^{-1}M^{\varepsilon}) \ , \end{aligned} \tag{3.3}$$

say. Using the same bound (3.2), each term $u_{m,m+h}$ in $\sum_1$ is

$$\begin{aligned}&\ll \big(d(m)^2+d(m+h)^2\big)\big(m(m+h)\big)^{-3/4}mh^{-2}X^{-1}\\ &\ll d(m)^2m^{-1/2}h^{-2}X^{-1}+d(m+h)^2(m+h)^{-1/2}h^{-2}X^{-1}\ ,\end{aligned}$$

since $m+h\asymp m$ for $1\le h\le m$. An application of the well-known estimate

$$\sum_{m\le y} d(m)^2m^{-1/2}\ll \sqrt{y}\log^3 y \qquad \text{for}\quad y>1\ ,$$

then yields

$$\sum\nolimits_1 \ll X^{-1}\sum_{h\le\sqrt{M}} h^{-2}\sqrt{\min(D_h,M)}\log^3 M\ll X^{-1/2}\ .$$

Putting this into (3.3), we have

$$\begin{aligned}T=&\sum_{h\le X^3L^4}\ \sum_{D_h<m\le M} d(m)d(m+h)\big(m(m+h)\big)^{-3/4}g(\theta_{m,m+h})\\ &+\mathcal{O}(X^{-1/2})\end{aligned} \tag{3.4}$$

with $\theta_{m,m+h}=4\pi\big(\sqrt{m+h}-\sqrt{m}\big)\sqrt{X}$.

Next, we transform the above inner sum over m into an integral. By (2.12), (2.13) and Riemann Stieltjes integration we have

$$\begin{aligned}\sum_{D_h<m\le M} &=\int_{D_h}^{M}\big(y(y+h)\big)^{-3/4}g(\theta_{y,y+h})d\psi_h(y)\\ &=\int_{D_h}^{M}\big(y(y+h)\big)^{-3/4}g(\theta_{y,y+h})I_h'(y)dy\\ &\quad+\Big[\big(y(y+h)\big)^{-3/4}g(\theta_{y,y+h})E_h(y)\Big]_{D_h}^{M}\\ &\quad-\int_{D_h}^{M}E_h(y)\frac{d}{dy}\big\{\big(y(y+h)\big)^{-3/4}g(\theta_{y,y+h})\big\}dy\\ &=W_1(h)+W_2(h)+W_3(h)\ ,\end{aligned} \tag{3.5}$$

say. We bound $W_2(h)$ by using (2.15) and the trivial estimate $g(\nu)\ll 1$. Whence

$$W_2(h)\ll M^{-3/2}M^{5/6+\varepsilon}+D_h^{-3/2}D_h^{5/6+\varepsilon}\ll D_h^{-2/3+\varepsilon}\ll h^{-4/3}X^{-2/3+\varepsilon}\ .$$

For $W_3(h)$, by [13, §3.2] we have

$$g'(\nu) = -\nu^{-3/2}J_{5/2}(\nu) + 4\nu^{-5/2}J_{7/2}(\nu) \ll \nu \quad \text{for} \quad \nu \geq 0 \ ,$$

since $J_k(\nu) \ll \nu^k$. Hence, by $g(\nu) \ll 1$ and (2.15) we have

$$\begin{aligned} W_3(h) &\ll \int_{D_h}^{M} y^{5/6+\varepsilon}\Big\{y^{-5/2} + y^{-3/2}\big|\theta_{y,y+h}\big|\Big|\frac{d}{dy}\theta_{y,y+h}\Big|\Big\}dy \\ &\ll \int_{D_h}^{M} \big\{y^{-5/3+\varepsilon} + y^{-2/3+\varepsilon}hy^{-1/2}X^{1/2}hy^{-3/2}X^{1/2}\big\}dy \\ &\ll h^{-4/3}X^{-2/3+\varepsilon} \ . \end{aligned}$$

In view of (3.4) and (3.5), the contribution to T from $W_2(h)$ and $W_3(h)$ is therefore

$$\ll \sum_{h \leq X^3L^4} h^{-4/3}X^{-2/3+\varepsilon} \ll X^{-2/3+\varepsilon} \ ,$$

which is again acceptable. Thus,

$$T = \sum_{h \leq X^3L^4} \int_{D_h}^{M} \big(y(y+h)\big)^{-3/4} g(\theta_{y,y+h})I_h'(y)dy + \mathcal{O}(X^{-1/2}) \ . \tag{3.6}$$

To evaluate the inner integral, we begin by making the change of variable

$$\omega = \theta_{y,y+h} = 4\pi\big(\sqrt{y+h} - \sqrt{y}\big)\sqrt{X} \ .$$

Then

$$y = 4\pi^2X\omega^{-2}h^2 - \frac{1}{2}h + (64\pi^2X)^{-1}\omega^2 = 4\pi^2X\omega^{-2}h^2\big(1 + \mathcal{O}(\omega^2X^{-1}h^{-1})\big) \ ,$$

so that

$$\begin{aligned} \big(y(y+h)\big)^{-3/4} &= \big(4\pi^2X\omega^{-2}h^2 - (64\pi^2X)^{-1}\omega^2\big)^{-3/2} \\ &= (2\pi h)^{-3}X^{-3/2}\omega^3\big(1 + \mathcal{O}(\omega^4X^{-2}h^{-2})\big) \end{aligned}$$

and

$$\frac{dy}{d\omega} = -8\pi^2X\omega^{-3}h^2\big(1 + \mathcal{O}(\omega^4X^{-2}h^{-2})\big) \ .$$

Moreover, by (2.16)

$$I_h'(y) = 4a_2(h)\log^2\big(2\pi\sqrt{X}\omega^{-1}h\big) + 2a_1(h)\log(2\pi\sqrt{X}\omega^{-1}h) + a_0(h) + \mathcal{O}\big(\omega^2X^{-1}h^{-1}(|a_2(h)|L + |a_1(h)|)\big)\ .$$

Set

$$u_1 = 4\pi\big(\sqrt{M+h} - \sqrt{M}\big)\sqrt{X} = 2\pi hX^{-3} + \mathcal{O}(h^2X^{-10}) \tag{3.7}$$

and

$$u_2 = 4\pi\big(\sqrt{D_h+h} - \sqrt{D_h}\big)\sqrt{X} = 2\pi L^4 + \mathcal{O}(h^{-1}X^{-1}L^{12})\ . \tag{3.8}$$

Then with the help of all these estimates we find that

$$\begin{aligned}&\int_{D_h}^{M}\big(y(y+h)\big)^{-3/4}g(\theta_{y,y+h})I_h'(y)dy\\ &= \frac{1}{\pi\sqrt{X}}\int_{u_1}^{u_2} g(\omega)h^{-1}\big\{4a_2(h)\log^2\big(2\pi\sqrt{X}\omega^{-1}h\big)\\ &\qquad + 2a_1(h)\log\big(2\pi\sqrt{X}\omega^{-1}h\big) + a_0(h)\big\}d\omega + \mathcal{O}\big(h^{-2}X^{-3/2+\varepsilon}\big)\ .\end{aligned}$$

In obtaining the above $\mathcal{O}$-term, we have used $g(\omega) \ll 1$, (3.8) and the observation that $a_j(h) \ll \log^3 h \ll L^3$. The integration limits u_1 and u_2 can be replaced by $2\pi hX^{-3}$ and $2\pi L^4$ respectively, since the error thus caused is

$$\begin{aligned}\ll X^{-1/2}h^{-1}L^5(h^2X^{-10} + h^{-1}X^{-1}L^{12}) &\ll hX^{-21/2}L^5 + h^{-2}X^{-3/2}L^{17}\\ &\ll h^{-2}X^{-3/2+\varepsilon}\ ,\end{aligned}$$

by (3.7) and (3.8). Collecting these into (3.6), we get

$$\begin{aligned}T = \frac{1}{\pi\sqrt{X}}\sum_{h\le X^3L^4}\int_{2\pi hX^{-3}}^{2\pi L^4} & g(\omega)h^{-1}\big\{4a_2(h)\log^2\big(2\pi\sqrt{X}\omega^{-1}h\big)\\ &+ 2a_1(h)\log\big(2\pi\sqrt{X}\omega^{-1}h\big) + a_0(h)\big\}d\omega + \mathcal{O}(X^{-1/2})\ .\end{aligned}$$

Next we interchange the summation and integration. In view of (2.18) we have

$$T = \frac{1}{\pi\sqrt{X}}\int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\xi\big((2\pi)^{-1}\omega X^3, 2\pi\sqrt{X}\omega^{-1}\big)d\omega + \mathcal{O}(X^{-1/2})\ . \tag{3.9}$$

By Lemma 4, and after some simplifications, we have

$$\xi\big((2\pi)^{-1}\omega X^3, 2\pi\sqrt{X}\omega^{-1}\big) = \log\omega\log^2 X + (c_8\log\omega + c_9\log^2\omega)\log X + \\ + c_{10}\log\omega + c_{11}\log^2\omega + c_{12}\log^3\omega + \Phi(X) + \mathcal{O}(\omega^{-1}X^{-3}L^5)\ ,$$

where $\Phi(X) = c_{13}\log^3 X + c_{14}\log^2 X + c_{15}\log X + c_{16}$ and $c_8, c_9, \dots, c_{16}$ are certain constants. Finally inserting this into (3.9) we get

$$T = \frac{1}{\pi\sqrt{X}}\int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\big\{\log\omega\log^2 X + (c_8\log\omega + c_9\log^2\omega)\log X \\ + c_{10}\log\omega + c_{11}\log^2\omega + c_{12}\log^3\omega + \Phi(X)\big\}d\omega + \mathcal{O}(X^{-1/2})\ . \tag{3.10}$$

It remains to evaluate the integrals

$$K_j = \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\log^j\omega d\omega$$

for $j = 0, 1, 2, 3$. Writing

$$K_j = \int_0^\infty g(\omega)\log^j\omega d\omega - \int_0^{2\pi X^{-3}} g(\omega)\log^j\omega d\omega - \int_{2\pi L^4}^\infty g(\omega)\log^j\omega d\omega\ ,$$

we see, by (2.6), that the last two integrals are bounded by $X^{-3}L^j$ and L^{-4+j} respectively. Hence, by Lemma 5 we have

$$K_0 \ll L^{-4},\ K_1 = -\sqrt{\pi}2^{-7/2} + \mathcal{O}(L^{-3})$$

and by (2.6),

$$K_2, K_3 = \text{ constant } + \mathcal{O}(L^{-1})\ .$$

When these are inserted into (3.10) we obtain

$$T = -2^{-3}(2\pi X)^{-1/2}\log^2 X + c_{17}X^{-1/2}\log X + \mathcal{O}(X^{-1/2})\ ,$$

and Theorem 2 now follows from (3.1) and Lemma 3.

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