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On the average number of direct factors of a finite Abelian Group

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1. Introduction

Let $a(n)$ denote the number of non-isomorphic Abelian groups with n elements. This is a well-known multiplicative function such that $a(p^\alpha) = P(\alpha)$. Historically, the summatory function $A(x) := \sum_{n \leq x} a(n)$ was first investigated by Erdös-Szekeres [2] in 1935, and from that time much research was done on this subject (for an account the reader is referred to Chapter 14 of A.Ivić [3] or Chapter 7 of E. Krätzel [5]).We remark, that the best published estimate of $A(x)$ is due to H.-Q. LIU [8]. He obtained the result

$$A(x) = \sum_{j=1}^{3} Res_{s=1/j} F(s) x^s/s + \Delta(x)$$

with $\Delta(x) \ll_\epsilon x^{40/159+\epsilon}$ and $40/159 = 0.251572\ldots$. For $\sigma = Re(s) > 1$ we define the Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} a(n) n^{-s}, \tag{1}$$

and

$$G(s) := F^2(s) \quad , \quad H(s) := F^3(s). \tag{2}$$

It is easy to show that

$$F(s) = \prod_{\nu=1}^{\infty} \zeta(\nu s), \tag{3}$$

$$G(s) = \prod_{\nu=1}^{\infty} \zeta^2(\nu s), \;\; H(s) = \prod_{\nu=1}^{\infty} \zeta^3(\nu s), \tag{4}$$

where $\zeta(s)$ as usual denotes the Riemann zeta-function.

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Furthermore, we define the multiplicative functions $t(n)$ and $w(n)$ by the Dirichlet convolution in the following way $t(n) := \sum_{uv=n} a(u)a(v)$ and $w(n) := \sum_{uv=n} a(u)t(v)$.

Thus for any prime p and integer $\alpha \geq 1$ one has

$$t(p^\alpha) = 2P(\alpha) + \sum_{j=1}^{\alpha-1} P(j)P(\alpha - j), \tag{5}$$

$$w(p^\alpha) = 3P(\alpha) + \sum_{j=1}^{\alpha-1} (P(j) + t(p^j))P(\alpha - j). \tag{6}$$

In particular, since $P(1) = 1, P(2) = 2, P(3) = 3, P(4) = 5$, we have $t(p) = 2$, $t(p^2) = 5, t(p^3) = 10, t(p^4) = 20$ and $w(p) = 3, w(p^2) = 9, w(p^3) = 22, w(p^4) = 68$.

In this paper we shall be concerned with obtaining estimates for the sums $T(x) = \sum_{n \leq x} t(n)$ and $W(x) = \sum_{n \leq x} w(n)$. The asymptotic behaviour of $T(x)$ was first studied by E. Cohen [1] and Krätzel [6]. It is known that $T(x) = \sum_{ord G \leq x} \tau(G)$, where $\tau(G)$ denotes the number of direct factors of an Abelian group G.

In [10] Seibold and the author proved the representation

$$T(x) = c_1 + (\log x + 2C - 1) + c_2 x + c_3 x^{1/2}(0.5 \log x + 2C - 1) + c_4 x^{1/2} + \Delta_1(x)$$

with $\Delta_1(x) \ll_\epsilon x^{45/109+\epsilon}$ and $45/109 = 0.412844\ldots$. Furthermore A. Ivić [4] proved several estimates for the sums $\sum_{n \leq x} t^k(n)$ and $\sum_{n \leq x} t(n^l)$ with $k = 2, 3, 4$ and $l = 2$.

The aim of this paper is to establish new asymptotic results for $T(x)$ and $W(x)$. First, we prove a sharper result for $\Delta_1(x)$ and get the estimate $\Delta_1(x) \ll x^{9/22} \log^4 x$, $(9/22 = 0.4\overline{09})$. Second, we establish a representation for $W(x)$. We prove the following result

$$W(x) = \sum_{i=1}^{6} K_i(x) + \Delta_2(x)$$

with the error term $\Delta_2(x) \ll x^{76/153} \log^6 x$, $(76/153 = 0.496732\ldots)$. Here $K_i(x)$ are well-known functions which will be defined by (32) to (37) later.

The paper has the following structure. In section two we formulate four preliminary lemmas. In section three we prove four theorems. Theorem 1 and Theorem 2 describe results for two special divisor problems with the dimensions four and six, respectively. In Theorems 3 and 4 we obtain two results based on the first and second theorem.

2. Preliminary Results

Throughout the paper Landau's O-symbol and Vinogradoff's $\ll$-notation (all constants involved are absolute ones) and the well known function $\psi(t) = t - [t] - 1/2$ are used. Furthermore, we denote the Euler's constant by C,

$$C = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{1}{n} - \log N \right\},$$

and the constant C_1 by

$$C_1 = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{\log n}{n} - \frac{1}{2}\log^2 N \right\}.$$

Lemma 1.

(1) a)" *Let* $d(1,1,2,2;n)$ *denote the divisor function* $d(1,1,2,2;n) = \#\{(n_1,\dots,n_4) : n_1 n_2 n_3^2 n_4^2 = n\}$ *and let* $\tau_3(n)$ *be defined by*

$$\sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^s} = \prod_{\nu=3}^{\infty} \zeta^2(\nu s) \quad , \quad (\sigma > \frac{1}{3})$$

Then

$$T(x) = \sum_{mn\le x} d(1,1,2,2;m)\tau_3(n). \tag{7}$$

b) *Let* $d(1,1,1,2,2,2;n)$ *denote the divisor function*

$$d(1,1,1,2,2,2;n) = \#\{(n_1,\dots,n_6) : n_1 n_2 n_3 n_4^2 n_5^2 n_6^2 = n\}$$

and let $t_3(n)$ *be defined by*

$$\sum_{n=1}^{\infty} \frac{t_3(n)}{n^s} = \prod_{\nu=3}^{\infty} \zeta^3(\nu s) \quad , \quad (\sigma > \frac{1}{3}).$$

Then

$$W(x) = \sum_{mn\le x} d(1,1,1,2,2,2;m)t_3(n). \tag{8}$$

Proof. It is known that

$$G(s) = \zeta^2(s)\zeta^2(2s)\sum_{n=1}^{\infty}\frac{\tau_3(n)}{n^s} \quad \text{and} \quad H(s) = \zeta^3(s)\zeta^3(2s)\sum_{n=1}^{\infty}\frac{t_3(n)}{n^s},$$

hence (7) and (8) follows at once.

LEMMA 2 (Vogts [11]). *Let $r \geq 2$ and $a_1, \dots, a_r$ be real numbers with $1 \leq a_1 \leq \dots \leq a_r$. We put $a := (a_1, \dots, a_r)$ and $A_r := a_1 + \dots + a_r$. Let*

$$D(a;x) = \#\{(n_1, \dots, n_r) : n_1^{a_1} \dots n_r^{a_r} \leq x\}.$$

Then

$$D(a;x) = H(a;x) + \Delta(a;x),$$

where the main term $H(a;x)$ is given by

$$H(a;x) = \sum_{i=1}^{r} \alpha_i x^{1/a_i}, \ \alpha_i = \prod_{\substack{j=1 \\ j\neq i}}^{r} \zeta\left(\frac{a_j}{a_i}\right). \tag{9}$$

For the error term $\Delta(a;x)$ we have the representation

$$\Delta(a;x) = -\sum_{u\in\pi(a)} S(u;x) + O\left(x^{(r-2)/A_r}\right), \tag{10}$$

where $S(u;x)$ is defined by

$$S(u;x) = \sum \psi\left(\left(\frac{x}{n_1^{u_1}\dots n_{r-1}^{u_{r-1}}}\right)^{1/u_r}\right), \tag{11}$$

with the condition of summation $SC(\sum) : n_1^{u_1} \dots n_{r-2}^{u_{r-2}} n_{r-1}^{u_{r-1}+u_r} \leq x, 1 \leq n_1 (\leq) n_2 (\leq) \dots (\leq) n_{r-1}$ and $u := (u_1, \dots, u_r)$.

In (10) the notations $u \in \pi(a)$ means that u runs over all permutations $\pi(a)$ of a. In the condition of summation $S(\sum)$ the notation $n_i(\leq)n_{i+1}$ means that $n_i \leq n_{i+1}$ for $u_i = a_i$, $u_{i+1} = a_j$ and $i < j$ and $n_i < n_{i+1}$ otherwise.

Remark. The representation (9) for the main term holds if $a_1 < a_2 < \dots < a_r$. However, in cases of some equalities we can take the limit values (see Krätzel [5]).

LEMMA 3 (Krétzel [7]). *Let* $N := (N_1, N_2, \dots, N_{r-1}), N_i \geq 1 (i = 1, \dots, r-1)$ *and* $s_1, s_2, \dots, s_{r-1}, R$ *be positive numbers. Let* $S(u, N; x)$ *be defined by*

$$S(u,N;x) = \sum \psi\left(\left(\frac{x}{n_1^{u_1}\dots n_{r-1}^{u_{r-1}}}\right)^{1/u_r}\right) , \tag{12}$$

with the condition of summation

$$SC(\textstyle\sum):\ n_1^{u_1}\dots n_{r-2}^{u_{r-2}}\ n_{r-1}^{u_{r-1}+u_r} \leq x\ ,\ n_1(\leq)n_2(\leq)\dots(\leq)n_{r-1}\ ,$$
$$N_i \leq n_i \leq 2N_i \qquad\qquad (i = 1, 2, \dots, r-1\ ).$$

For the numbers N_i are valid the inequalities

$$N_1 \ll N_2 \ll \dots \ll N_{r-1}\ , \qquad N_1^{u_1} N_2^{u_2} \dots N_{r-2}^{u_{r-2}} N_{r-1}^{u_{r-1}+u_r} \leq x\ .$$

Assume that

$$S(u,N;x) \ll \left(x N_1^{s_1 u_r - u_1} N_2^{s_2 u_r - u_2} \dots N_{r-1}^{s_{r-1} u_r - u_{r-1}}\right)^{R/u_r}$$

for each permutations u of $(a_1, \dots, a_r)$. Then the estimate

$$S(u,N;x) \ll \left(x N_1^{s_1 a_1 - a_r} N_2^{s_2 a_1 - a_{r-1}} \dots N_{r-1}^{s_{r-1} a_1 - a_2}\right)^{R/a_1}$$

holds with $N_1^{a_r} N_2^{a_{r-1}} \dots N_{r-2}^{a_3} N_{r-1}^{a_2+a_1} \leq x$ for all permutations u.

LEMMA 4 (Menzer, Krétzel [5]). *Let* $S(u, N; x)$ *be defined by (12), then we have*

$$S(u,N;x) \ll \left(x^3 N_1^{6u_4 - 3u_1} N_2^{5u_4 - 3u_2} N_3^{3(u_4 - u_3)}\right)^{1/7u_4} \log x, \tag{13}$$

$$S(u,N;x) \ll \left(x^6 N_1^{15u_4 - 6u_1} N_2^{13u_4 - 6u_2} N_3^{6(u_4 - u_3)}\right)^{1/16u_4} \log x, \tag{14}$$

$$S(u,N;x) \ll \left(x^7 N_1^{17u_4 - 7u_1} N_3^{10u_4 - 7u_2} N_3^{8u_4 - 7u_3}\right)^{1/17u_4} \log x. \tag{15}$$

Remark. The first both formulas (13) and (14) was proved by the author by applying two results of three-dimensional sums in [9]. The third formula (15) was proved by Krétzel [5] by applying a result of two-dimensional exponential sums.

3. Estimates for $T(x)$ and $W(x)$.

According to the formulas (7) and (8) of Lemma 1 one can see that in order to establish estimates for $T(x)$ and $W(x)$ it is necessary to know the asymptotic behaviour of the counting functions $D(1,1,2,2;x)$ and $D(1,1,1,2,2,2;x)$, respectively. Hence, we prove first two theorems including asymptotic results for these functions.

THEOREM 1. *For the remainder term* $\Delta(1,1,2,2;x)$ *holds the estimate*

$$\Delta(1,1,2,2;x) \ll x^{9/22}\log^4 x. \tag{16}$$

Proof. We start with the formula (10) and put $a = (1,1,2,2)$. This yields

$$\Delta(1,1,2,2;x) = -\sum_{u\in\pi(1,1,2,2)} S(u;x) + O(x^{1/3}) \ ,$$

where $S(u;x)$ is defined in the sense of (11). Now instead of function $S(u;x)$ we take the special function $S(u,N;x)$ which is defined by formula (12). All sums $S(u,N;x)$, where $u \in \pi(1,1,2,2)$ are divided into two subsums corresponding to $n_2 \le z$ and $z < n_2$, where z is a suitable value, which is defined later. In the first case $n_2 > z$ we take formula (13) and obtain by using

$$N_1 \ll N_2 \ll N_3 \ll (xN_1^{-u_1}N_2^{-u_2})^{1/(u_3+u_4)}, z \ll N_2$$

and Lemma 3 the estimate

$$S(u;N;x) \ll (xN_1^0 N_2^{-1/3} N_3^0)^{3/7}\log x \ll (xz^{-1/3})^{3/7}\log x \ , \tag{17}$$

for all permutations $u \in \pi(1,1,2,2)$.

In the second case $n_2 \le z$ we use the formula (14). By applying

$$N_1 \ll N_2 \ll N_3 \ll \left(xN_1^{-u_1}N_2^{-u_2}\right)^{1/(u_3+u_4)}, N_2 \ll z$$

and Lemma 3 we get the estimate

$$S(u;N;x) \ll (xN_1^{1/2}N_2^{1/6}N_3^0)^{3/8}\log x \ll (xz^{2/3})^{3/8}\log x \ , \tag{18}$$

for all permutations $u \in \pi(1,1,2,2)$.

Now we compare the estimates (17) and (18) and get for z the value

$$z = x^{3/22} \ .$$

This completes the proof of (16).

Theorem 2. *Let $a=(1,1,1,2,2,2)$ and let $H_i(x)$ $(i=1,\dots 6)$ be define by*

$$H_1(x)=\zeta^3(2)x\left(\frac{1}{2}\log^2 x+(3C-1)(\log x-1)+3(C^2+C_1)\right), \qquad (19)$$

$$H_2(x)=6\zeta^2(2)\zeta'(2)x(\log x+3C-1), \qquad (20)$$

$$H_3(x)=6\zeta(2)x(\zeta(2)\zeta''(2)+2\zeta'^2(2)), \qquad (21)$$

$$H_4(x)=\zeta^3(1/2)x^{1/2}\left(\frac{1}{8}\log^2 x+(3C-1)\left(\frac{1}{2}\log x-1\right)+3(C^2+C_1)\right), \qquad (22)$$

$$H_5(x)=\frac{3}{2}\zeta^2(1/2)\zeta'(1/2)x^{1/2}\left(\frac{1}{2}\log x+3C-1\right), \qquad (23)$$

$$H_6(x)=\frac{3}{8}\zeta(1/2)x^{1/2}(\zeta(1/2)\zeta''(1/2)+2\zeta'^2(1/2)). \qquad (24)$$

Then the representation

$$D(a;x)=\sum_{i=1}^{6}H_i(x)$$

holds with

$$\Delta(a;x)=-\sum_{u\in\pi(a)}S(u;x)+O(x^{4/9}),$$

where $S(u;x)$ is defined by (11) with $r=6$.

Moreover, we have

$$\Delta(a;x)\ll x^{76/153}\log^6 x\ . \qquad (25)$$

Proof. We use Lemma 2 with $a_1=a_2=a_3=1$, $a_4=a_5=a_6=2$ and $A_6=9$. We obtain by simple calculations for the main term $H(a;x)$ the representation $H(a;x)=\sum\limits_{i=1}^{6}H_i(x)$. Now, we apply the formula (15) of Lemma 4 and substitute n_1,n_2,n_3 by n_3,n_4,n_5. After that we introduce two new variables of summation n_1,n_2 and sum trivially over them. Hence we get the following estimate

$$S(u,N;x)\ll$$
$$\left\{x^7N_1^{17u_6-7u_1}N_2^{17u_6-7u_2}N_3^{17u_6-7u_3}N_4^{10u_6-7u_4}N_5^{8u_6-7u_5}\right)^{1/17u_6}\log x,$$

and we have by using Lemma 3

$$S(u,N;x) \ll \left\{x^7 (N_1 N_2 N_3 N_4)^3 N_5\right\}^{1/17} \log x \ , \tag{26}$$

for all permutations $u \in \pi(1,1,1,2,2,2)$. From this estimate it is easily seen that

$$\begin{aligned} S(u,N;x) \ll \{x^7 ((N_1 N_2 N_3 N_5)^2 N_4)^{y_1} \\ (N_1/N_2)^{y_2} (N_2/N_3)^{y_3} (N_3/N_4)^{y_4} (N_4/N_5)^{y_5}\}^{1/17} \log x, \end{aligned} \tag{27}$$

where $y_1 = 13/9$, $y_2 = 1/9$, $y_3 = 2/9$, $y_4 = 3/9$ and $y_5 = 17/9$.

Therefore, we can use the inequalities

$$(N_1 N_2 N_3 N_5)^2 N_4 \ll x \ , \ N_1 \ll N_2 \ll N_3 \ll N_4 \ll N_5$$

in (27). Then

$$\begin{aligned} S(u,N;x) &\ll x^{76/153} \log x \ , \\ S(u;x) &\ll x^{76/153} \log^6 x \end{aligned}$$

and (25) follows immediately.

Theorem 3. *Let c_1, c_2, c_3, c_4 be defined by*

$$c_1 = \zeta^2(2) \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n} \ , \tag{28}$$

$$c_2 = -\zeta(2) \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n} (\zeta(2) \log n - 4\zeta'(2)) \ , \tag{29}$$

$$c_3 = \zeta^2(1/2) \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^{1/2}} \ , \tag{30}$$

$$c_4 = -\zeta(1/2) \sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^{1/2}} \left(\frac{1}{2}\zeta(1/2) \log n - \zeta'(1/2)\right). \tag{31}$$

Then

$$\begin{aligned} T(x) = c_1 x(\log x + 2C - 1) + c_2 x \\ + c_3 x^{1/2} (0.5 \log x + 2C - 1) + c_4 x^{1/2} + O(x^{9/22} \log^4 x). \end{aligned}$$

Proof. From Theorem 2 of [10] and Theorem 1 our result follows at once.

Theorem 4. *Let $K_i(x)$ $(i=1,\ldots,6)$ be defined by*

$$K_1(x)=\sum_{n=1}^{\infty}\frac{t_3(n)}{n}H_1(x)\ , \tag{32}$$

$$K_2(x)=\sum_{n=1}^{\infty}\frac{t_3(n)}{n}\left\{H_2(x)-\zeta^3(2)x\log xn\right\}\ , \tag{33}$$

$$K_3(x)=\sum_{n=1}^{\infty}\frac{t_3(n)}{n} \left\{H_3(x)-\zeta^3(2)x\log n\left(6\zeta'(2)/\zeta(2)+3C-1+\frac{1}{2}\log n\right)\right\}\ , \tag{34}$$

$$K_4(x)=\sum_{n=1}^{\infty}\frac{t_3(n)}{n^{1/2}}H_4(x)\ , \tag{35}$$

$$K_5(x)=\sum_{n=1}^{\infty}\frac{t_3(n)}{n^{1/2}}\left\{H_5(x)-\frac{1}{4}\zeta^3(1/2)\log xn\right\}\ , \tag{36}$$

$$K_6(x)=\sum_{n=1}^{\infty}\frac{t_3(n)}{n^{1/2}}\left\{H_6(x)-\frac{1}{2}\zeta^3(1/2)x^{1/2}\log n \left(\frac{3}{2}\zeta'(1/2)/\zeta(1/2)+3C-1+\frac{1}{4}\log n\right)\right\}\ , \tag{37}$$

where $H_i(x)$ $(i=1,\ldots,6)$ are given by (19) to (24). Then

$$W(x)=\sum_{i=1}^{6}K_i(x)+O(x^{76/153}\log^6 x). \tag{38}$$

Proof. We apply equation (8) and obtain

$$W(x)=\sum_{n\le x}t_3(n)D(a,x/n)$$

with $a=(1,1,1,2,2,2)$. Now, we use our Theorem 2.

It is easily seen that

$$W(x)=\sum_{n\le x}t_3(n)\left\{\sum_{i=1}^{6}H_i(x/n)+O\left((x/n)^{76/153}\log(x/n)\right)\right\}.$$

Since the Dirichlet series $\sum_{n=1}^{\infty} \frac{t_3(n)}{n^s}$ are absolutely convergent for $\sigma > 1/3$, the estimate (38) follows immediately.

References

[1] E. COHEN, *On the average number of direct factors of a finite Abelian group*, Acta Arith. **6** (1960), 159–173.

[2] P. ERDÖS and G. SZEKERES, *Öber die Anzahl Abelscher Gruppen gegebener Ordnung und Âber ein verwandtes zahlentheoretisches Problem.*, Acta Sci. Math. (Szeged) **7** (1935), 95–102.

[3] A. IVIĆ, *The Riemann zeta-function*, John Wiley & Sons, New York (1985).

[4] A. IVIĆ, *On a multiplicative function connected with the number of direct factors of a finite Abelian group*, (to appear).

[5] E. KRÄTZEL, *Lattice Points*, Kluwer Academic Publishers, Dordrecht Boston London, (1988).

[6] E. KRÄTZEL, *On the average numbers of direct factors of a finite Abelian group*, Acta Arith. **11** (1988), 369–379.

[7] E. KRÄTZEL, *Estimates in the General Divisor Problem*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 191–206.

[8] H.-Q. LIU, *On the number of Abelian groups of a given order*, Acta Arith. **56** (1991), 261–277.

[9] H. MENZER, *Exponentialsummen und verallgemeinerte Teilerprobleme Dissertation*, B. Friedrich-Schiller-Universitaet Jena, (1991).

[10] H. MENZER & R. SEIBOLD, *On the average number of direct factors of a finite Abelian group.*, Monatsh. Math. **110** (1990), 63–72.

[11] M. VOGTS, *Many-dimensional Generalized Divisor Problems*, Math. Nachr. **124** (1985), 103–121.

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