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On Gauss Sum Characters of Finite Groups and Generalized Bernoulli Numbers

par Shoichi NAKAJIMA

1. Introduction

In this paper we give a generalization of a result of Hecke which gives a relation between modular forms and class numbers of imaginary quadratic fields. To begin with, we briefly summarize Hecke's result: Let p be an odd prime and $\Gamma(p)$ the principal congruence subgroup of level p of the modular group $SL_2(\mathbf{Z})$. Then the finite group $G = PSL_2(\mathbf{F}_p) = SL_2(\mathbf{F}_p)/\{\pm 1\}$ acts on the vector space $V = S_2(\Gamma(p))$ of the cusp forms of weight 2 with respect to $\Gamma(p)$ ($\mathbf{F}_p$ is the finite field with p elements). Denote by η the character of G determined from the above action. Hecke called it a "fundamental problem" to decompose η into irreducible characters of G . In treating the problem, the following was the most difficult: When $p \equiv 3 \pmod 4$, G has a pair χ and $\bar{\chi}$ of irreducible characters that are complex conjugate to each other (their values generate the imaginary quadratic field $\mathbf{Q}(\sqrt{-p})$). First Hecke could not determine $m(\chi)$ and $m(\bar{\chi})$ separately, though he computed the sum $m(\chi) + m(\bar{\chi})$ rather easily ([6,No.28] ; $m(\chi)$ and $m(\bar{\chi})$ are the multiplicities of χ and $\bar{\chi}$ in η respectively). Later, Hecke determined them ([6,No.29]) by proving the equality

$$m(\chi) - m(\bar{\chi}) = h(\mathbf{Q}(\sqrt{-p})),$$

where $h(\mathbf{Q}(\sqrt{-p}))$ is the class number of the field $\mathbf{Q}(\sqrt{-p})$. This mysterious relation between η and the class number gives the motivation of this paper.

The result above obtained by Hecke was generalized to modular forms of higher weights (Feldmann[2]) or higher levels(Spies[14], McQuillan[8]), and further to modular forms of several variables(see Saito[11], Hashimoto[4] and the references there). Besides the above it has another direction of generalization, which concerns us here. The space V can be considered as the space of holomorphic differentials on the Riemann surface $X(p)$, the modular curve of level p (i.e. $X(p)$ is the compactification of $\Gamma(p) \setminus \mathcal{H}$,

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where $\mathcal{H}$ is the upper half plane.) Putting $Y(=\mathbf{P}^1)=$ the compactification of $SL_2(\mathbf{Z})\setminus\mathcal{H}$, we have a Galois covering $f: X\to Y$ with Galois group $PSL_2(\mathbf{F}_p)$. Thus we can enlarge the situation as follows: Let $f: X\to Y$ be a Galois covering of compact Riemann surfaces (not necessarily modular curves) with $G=\mathrm{Gal}(X/Y)$ and $V=H^0(X,\Omega_X^1)$, the space of holomorphic differentials on X. The problem is the decomposition of V into irreducible characters of G. When $G=PSL_2(\mathbf{F}_p)$, Hecke's result was generalized to this setting (Shih [13], Weintraub [17]). Further we obtained a generalization when the group G has a pair of characters whose values generate an imaginary quadratic field (Nakajima[10]). In this paper we treat the general case (i.e. no assumption on G) and generalize Hecke's result. Namely, we show (Theorem in §4) that a certain linear combination of multiplicities (in V) of algebraically conjugate characters of G is an (explicitly given) multiple of the generalized Bernoulli number $B_{1,\lambda}$ for a Dirichlet character λ (the "linear combination" above is determined by λ). When λ is the Dirichlet character corresponding to the extension $\mathbf{Q}(\sqrt{-p})/\mathbf{Q}$ ($p\equiv 3 \pmod 4), p>3$), $B_{1,\lambda}=h(\mathbf{Q}(\sqrt{-p}))$ by virtue of the Dirichlet class number formula. Hence our Theorem reduces to Hecke's one if $X=X(p)$ and $G=PSL_2(\mathbf{F}_p)$.

The content of the paper is as follows: In §2 we introduce notation and explain Hasse's formula concerning general (i.e. not necessarily primitive) Gauss sums. In the next §3 we define "Gauss sum character" of a finite group G, which is a linear combination of algebraically conjugate characters of G. The final §4 contains our Theorem mentioned above. It is formulated by using the Gauss sum characters defined in §3.

2. General Gauss Sums

In this section we give Hasse's formula for general Gauss sums after introducing notation used throughout the paper. First we give standard notation: Let $N=\{1,2,3,\ldots\}$ be the set of natural numbers, and let $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{C}$ be the rational integer ring, the rational number field and the complex number field, respectively. For m and n in $\mathbf{N}$, $m\mid n$ means that m divides n. Further, (m,n) denotes the greatest common divisor of m and n. When A is a finite set, $|A|$ denotes the number of elements of A.

For $n\in\mathbf{N}$, we put $\zeta_n=\exp(2\pi\sqrt{-1}/n)$, a fixed primitive n-th root of unity in $\mathbf{C}$. Next we explain notation necessary to define a Gauss sum. For $m\in\mathbf{N}$ we put $J(m)=(\mathbf{Z}/m\mathbf{Z})^\times$, the unit group of the ring $\mathbf{Z}/m\mathbf{Z}$, and $J(m)^\wedge=\mathrm{Hom}(J(m),\mathbf{C}^\times)$. When $m\mid n$, there is a natural surjection $J(n)\to J(m)$ and hence an injection $J(m)^\wedge\to J(n)^\wedge$. For a Dirichlet character λ, we denote its conductor by f_λ. Hence we can regard $\lambda \in J(m)^\wedge$ if and only if $f_\lambda \mid m$.

Now we define a general Gauss sum $\tau(\lambda, m, a)$ as follows, when a Dirichlet character λ, $m \in \mathbf{N}$ satisfying $f_\lambda \mid m$ and $a \in \mathbf{N}$ are given:

$$\tau(\lambda, m, a) = \sum_{t \in J(m)} \lambda(t)\zeta_m^{at}.$$

Note that if $m = 1$ (in this case λ must be the trivial character), we put $\tau(\lambda, m, a) = 1$. Further we put

$$\tau(\lambda) = \tau(\lambda, f_\lambda, 1),$$

the usual primitive Gauss sum. Here we quote a formula for $\tau(\lambda, m, a)$ which was proved by Hasse[5] (see also Joris[7]).

Proposition 1. *Put $m_0 = m/(m, a)$ and $a_0 = a/(m, a)$ (i.e. $(m_0, a_0) = 1$ and $\zeta_{m_0}^{a_0} = \zeta_m^a$).*

(1) *If m_0 is not divisible by f_λ, then $\tau(\lambda, m, a) = 0$.*

(2) *When $f_\lambda \mid m_0$,*

$$\tau(\lambda, m, a) = \frac{\varphi(m)}{\varphi(m_0)}\mu(\frac{m_0}{f_\lambda})\lambda(\frac{m_0}{f_\lambda})\bar{\lambda}(a_0)\tau(\lambda),$$

where μ and φ mean the Möbius function and the Euler totient function, respectively.

3. Gauss Sum Characters

In this section we define Gauss sum characters of a finite group and give their properties. Hereafter let G be a finite group with exponent N. We denote by $R(G)$ the ring of virtual (ordinary) characters of G. (For the character theory of finite groups, we refer to Serre[12] .) When a character χ of G is given, we put $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g) \mid g \in G) \subset \mathbf{C}$, the value field of χ, and $\Gamma_\chi = \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$. Then $\mathbf{Q}(\chi)$ is a subfield of $\mathbf{Q}(\zeta_N)$, N being the exponent of G. We have a natural isomorphism $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \cong J(N)(\sigma(t) \leftrightarrow t)$ which is defined by $\zeta_N^{\sigma(t)} = \zeta_N^t$ (note that $\sigma(-1)$ is nothing but the complex conjugation). Hence we can regard Γ_χ as a quotient of $J(N)$ and at the same time regard an element of $(\Gamma_\chi)^\wedge = \mathrm{Hom}(\Gamma_\chi, \mathbf{C}^\times)$ as a Dirichlet character (i.e. an element of $J(N)^\wedge$). Thus we have $\lambda(\sigma(t)) =$

$\lambda(t)$ with this identification. For $\gamma \in \Gamma_\chi$, we have an algebraically conjugate character $\chi^\gamma \in R(G)$ defined by $\chi^\gamma(g) = \chi(g)^\gamma$.

Now we define Gauss sum characters of G. For a character χ of G and $\lambda \in (\Gamma_\chi)^\wedge$, we put

$$\alpha(\chi, \lambda) = \sum_{\gamma \in \Gamma_\chi} \lambda(\gamma)\chi^\gamma,$$

and call it a Gauss sum character. Here $\alpha(\chi, \lambda)$ is an element of $R(G) \otimes_{\mathbf{Z}} R_\lambda$, where R_λ is the integer ring of the field $K_\lambda = \mathbf{Q}(\lambda(\gamma) \mid \gamma \in \Gamma_\chi)$. Note that when $\mathbf{Q}(\chi)$ is an imaginary quadratic field (as in the case $G = PSL_2(\mathbf{F}_p)$ and $p \equiv 3 \pmod 4$), $\alpha(\chi, \lambda)$ equals $\chi + \bar{\chi}$ or $\chi - \bar{\chi}$ according as λ is the trivial or the non-trivial character of Γ_χ ($|\Gamma_\chi| = 2$).

We call $\alpha(\chi, \lambda)$ "Gauss sum character" because its values are expressed in terms of Gauss sums (Proposition 4 below). Before showing it, we give elementary properties of $\alpha(\chi, \lambda)$. The module $R(G)$ has the inner product $< \chi_1, \chi_2 >_G$ given by

$$< \chi_1, \chi_2 >_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}$$

($\chi_1, \chi_2 \in R(G)$; cf. Serre[12]). We note here that an equality $< \chi_1^\gamma, \chi_2^\gamma >_G$ $= < \chi_1, \chi_2 >_G$ holds for any $\gamma \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$. The inner product is extended to $R(G) \otimes_{\mathbf{Z}} R_\lambda$ by $< \chi_1 \otimes a_1, \chi_2 \otimes a_2 >_G = a_1 \bar{a}_2 < \chi_1, \chi_2 >_G$, where $\bar{\ }$ denotes the complex conjugation.

Proposition 2. *With the notation above, the following hold.*

(1) *For $\gamma \in \Gamma_\chi$, we have*

$$\alpha(\chi^\gamma, \lambda) = \bar{\lambda}(\gamma)\alpha(\chi, \lambda).$$

In particular, when $\gamma = \sigma(-1)$ we obtain

$$\alpha(\bar{\chi}, \lambda) = \lambda(-1)\alpha(\chi, \lambda).$$

(2) *When λ varies we can recover χ from $\alpha(\chi, \lambda)$. Namely, we have for any $\gamma \in \Gamma_\chi$,*

$$\chi^\gamma = \frac{1}{|\Gamma_\chi|} \sum_{\lambda \in (\Gamma_\chi)^\wedge} \bar{\lambda}(\gamma)\alpha(\chi, \lambda).$$

(3) *If χ is an irreducible character,*

$$< \alpha(\chi, \lambda_1), \alpha(\chi, \lambda_2) >_G = \begin{cases} |\Gamma_\chi| & (\lambda_1 = \lambda_2), \\ & (\lambda_1 \neq \lambda_2), \end{cases}$$

for $\lambda_1, \lambda_2 \in (\Gamma_\chi)^\wedge$.

Proof. (1) follows easily from $\lambda(\gamma^{-1}) = \bar{\lambda}(\gamma)$. We obtain (2) from the relation

$$\sum_{\lambda\in(\Gamma_\chi)^\wedge} \lambda(\gamma\prime\gamma^{-1}) = \begin{cases} |\Gamma_\chi| & \gamma\prime = \gamma, \\ 0 & \gamma\prime \neq \gamma. \end{cases}$$

In (3) , we note that

$$< \chi^{\gamma\prime}, \chi^{\gamma} >_G = \begin{cases} 1 & \gamma\prime = \gamma, \\ 0 & \gamma\prime \neq \gamma \end{cases}$$

holds because χ is assumed to be irreducible. Then we obtain (3) by the orthogonality relation of the characters of Γ_χ.

Let $\alpha = \alpha(\chi, \lambda)$ be as above and take an element $g \in G$. We now describe the restriction $\alpha\,|_H$ of α to $H =< g >$, the cyclic group generated by g. We put $n = |H|$ and define $\theta_g \in \mathrm{Hom}(H, \mathbf{C}^\times)$ by $\theta_g(g) = \zeta_n$. Then θ_g generates the character group $\mathrm{Hom}(H, \mathbf{C}^\times)$ of H. For a divisor r of n, put

$$\begin{aligned} M_r = M_r(\chi, \lambda, g) &= < \alpha\,|_H, \theta_g^r >_H \\ &= \sum_{\gamma\in\Gamma_\chi} \lambda(\gamma)< \chi^\gamma\,|_H, \theta_g^r >_H , \end{aligned}$$

where $< , >_H$ is the inner product of the character ring of H. The number M_r belongs to the integer ring R_λ.

Proposition 3. *Let the notation be as above.*

(1) *If f_λ does not divide $\frac{n}{r}$, then $M_r = 0$.*
(2) *The following decomposition of $\alpha\,|_H$ holds.*

$$\alpha\,|_H = \sum_{r|(n/f_\lambda)} M_r \left(\sum_{t\in J(n/r)} \lambda(t)\theta_g^{rt} \right) .$$

When f_λ does not divide n, this formula means $\alpha\,|_H = 0$.

Proof. For $t \in J(N)$, we let $\sigma(t) \in \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ act on the characters of

H. Then we have the following equality:

$$\begin{aligned}
<\alpha\,|_H, \theta_g^{rt}>_H &= \sum_{\gamma\in\Gamma_\chi} \lambda(\gamma)<\chi^\gamma\,|_H, \theta_g^{r\sigma(t)}>_H \\
&= \sum_{\gamma\in\Gamma_\chi} \lambda(\gamma)<\chi^{\gamma\sigma(t)^{-1}}\,|_H, \theta_g^{r}>_H \\
&= \sum_{\gamma\in\Gamma_\chi} \lambda(\gamma\sigma(t))<\chi^{\gamma}\,|_H, \theta_g^{r}>_H \\
&= \lambda(t)M_r
\end{aligned}$$

If f_λ does not divide n/r, then we can take a $t \in J(N)$ which satisfies $t \equiv 1 \pmod{n/r}$ and $\lambda(t) \neq 1$. For such t, we have $\theta_g^{rt} = \theta_g^r$. Therefore the above formula shows $M_r = 0$. Further, since every character of H is uniquely written in the form θ_g^{rt} for a divisor r of n and $t \in J(\frac{n}{r})$,

that formula also proves the decomposition of $\alpha\,|_H$ given in (2).

Here we can give a proposition referring to the values of α. It states that the values of α are always multiples of the Gauss sum $\tau(\lambda)$.

PROPOSITION 4. *Notation is as above.*

(1) *If f_λ does not divide $n(=$ the order of $g)$, then $\alpha(g) = 0$.*
(2) *When $f_\lambda \mid n$, the number $\alpha(g)/\tau(\lambda)$ belongs to R_λ. More precisely,*

$$\begin{aligned}
\alpha(g) &= \sum_{r|(n/f_\lambda)} M_r \tau(\lambda, \frac{n}{r}, 1) \\
&= \left(\sum_{r|(n/f_\lambda)} M_r \mu(\frac{n}{rf_\lambda})\lambda(\frac{n}{rf_\lambda}) \right) \tau(\lambda),
\end{aligned}$$

with the notation in §2.
(3) *If $f_\lambda \mid n$ and $k \in \mathbf{Z}$ is prime to n, then*

$$\alpha(g^k) = \bar{\lambda}(k)\alpha(g).$$

Proof. (1) follows from (2) of Proposition 3. When $f_\lambda \mid n$, Proposition 3 also shows, for $k \in \mathbf{N}$, the formula

$$\alpha(g^k) = \sum_{r|(n/f_\lambda)} M_r \tau(\lambda, \frac{n}{r}, k),$$

with the notation in §2. Thus Hasse's formula (Proposition 1) completes the proof.

4. Connection with generalized Bernoulli numbers

In this section we give a relation between Gauss sum characters and the generalized Bernoulli numbers which was mentioned in the introduction.

Let $f : X \to Y$ be a finite Galois covering of connected compact Riemann surfaces with Galois group G. When $\mathcal{F}$ is a locally free $\mathcal{O}_Y$ - module of finite rank on Y, $f^*\mathcal{F}$ denotes its pull-back to X. The group G naturally acts on the sheaf $f^*\mathcal{F}$ ($f^*\mathcal{F}$ is a G - sheaf), and the cohomology groups $H^i(X, f^*\mathcal{F})$ $(i = 0, 1)$ are G - modules. Here we are concerned with the character

$$\mu = ch(H^0(X, f^*\mathcal{F})) - ch(H^1(X, f^*\mathcal{F})) \in R(G),$$

where, for a G - module V, $ch(V)$ denotes the character of G determined by V.

Remark. The sheaf Ω_X^1 of holomorphic differentials on X is not necessarily of the form $f^*\mathcal{F}$ for a locally free sheaf $\mathcal{F}$ on X. However, by virtue of the Serre duality, we can obtain a statement about $H^0(X, \Omega_X^1)$ (see Corollary below).

The character μ can be described by using the genus of Y, the rank and degree of $\mathcal{F}$ and the ramification of the covering $f : X \to Y$ (Proposition 5 below). That result was first obtained by Chevalley and Weil[1] (see also Weil[16]). Here we adopt a formulation which uses induced characters ([9]). In order to give the result we introduce some notation. For a point $P \in X$ put $G_P = \{g \in G \mid g \cdot P = P\}$, the stabilizer of P. The group G_P acts on the cotangent space $T_P^*(X)$ of X at P (i.e.$T_P^*(X) = m_P/m_P^2$ where m_P is the maximal ideal of the local ring at P), which determines an element $\theta_P \in \mathrm{Hom}(G_P, \mathbf{C}^\times)$. (As a consequence, we see that G_P is a cyclic group.) Denoting by $\mathrm{Ind}_{G_P}^G$ the induction of characters from G_P to G, we define a character $\nu_P \in R(G)$ by

$$\nu_P = \sum_{d=1}^{n_P - 1} d \cdot \mathrm{Ind}_{G_P}^G(\theta_P^d),$$

where $n_P = |G_P|$. Note that we have $\nu_P = \nu_{g \cdot P}$ for $P \in X$ and $g \in G$, because $G_{g \cdot P} = gG_Pg^{-1}$ and $\theta_{g \cdot P}(h) = \theta_P(g^{-1}hg)$ hold $(h \in G_P)$. We denote the regular character of G by reg_G, i.e. $\mathrm{reg}_G = ch(\mathbf{C}[G])$, the character of the group ring $\mathbf{C}[G]$. Then we have

Proposition 5. *Notation is the same as above.*

(1) *The sum* $\sum_{P \in X} \nu_P$ *is divisible by* $|G|$*, i.e.* $\nu = \frac{1}{|G|} \sum_{P \in X} \nu_P$ *belongs to* $R(G)$.

(2) *We have*

$$\mu = (\deg(\mathcal{F}) - \mathrm{rank}(\mathcal{F})(g_Y - 1)) \cdot \mathrm{reg}_G - \mathrm{rank}(\mathcal{F}) \cdot \nu,$$

where $\deg(\mathcal{F})$ *and* $\mathrm{rank}(\mathcal{F})$ *are the degree and rank of* $\mathcal{F}$*, respectively, and* g_Y *is the genus of* Y.

Proof. (1) and the fact that $\mu = m \cdot \mathrm{reg}_G - \mathrm{rank}(\mathcal{F}) \cdot \nu$ holds for an integer m are shown in [9,Theorem 2]. Comparing degrees, we obtain $m = \deg(\mathcal{F}) - \mathrm{rank}(\mathcal{F})(g_Y - 1)$ by virtue of the Riemann-Roch theorem and the Riemann-Hurwitz formula for genera applied to the covering $f : X \to Y$ (see e.g. [3,Chapter IV], [16]).

Before stating our Theorem, we introduce the generalized Bernoulli number $B_{1,\lambda}$. For a non-trivial Dirichlet character λ, $B_{1,\lambda}$ is defined by

$$B_{1,\lambda} = \frac{1}{f_\lambda} \sum_{a=1}^{f_\lambda - 1} a \, \lambda(a)$$

(see e.g. [15, Chapter 4]). It is well-known that $B_{1,\lambda} = 0$ holds when λ is even (i.e. $\lambda(-1) = 1$). Further we have a relation between $B_{1,\lambda}$ and L - functions. Namely, denoting by $L(s, \lambda)$ the Dirichlet L - function associated with λ, we have the equalities $L(0, \lambda) = -B_{1,\lambda}$ and $L(1, \lambda) = \frac{\pi\sqrt{-1}\tau(\lambda)}{f_\lambda} B_{1,\bar{\lambda}}$. When an (irreducible) character of G is given, we consider, following Hecke, a certain linear combination of the inner products $< \mu, \chi^\gamma >_G$ instead of $< \mu, \chi >_G$ itself. Thus we have

Theorem. *Let the notation be as above. For a character* χ *of* G *and* $\lambda \in (\Gamma_\chi)^\wedge$*, let* $\alpha(\chi, \lambda)$ *be the Gauss sum character defined in §3. Then for the inner product (a linear combination of the "multiplicities" in* μ *of the conjugates of* χ *)*

$$\begin{aligned} m(\chi, \lambda) &= < \alpha(\chi, \lambda), \mu >_G \\ &= \sum_{\gamma \in \Gamma_\chi} \lambda(\gamma) < \chi^\gamma, \mu >_G , \end{aligned}$$

we have the following result.

(1) *If λ is the trivial character, then*

$$\frac{1}{|\Gamma_\chi|}m(\chi,\lambda) = (\deg(\mathcal{F}) - \operatorname{rank}(\mathcal{F})(g_Y - 1))\deg(\chi) \\ - \frac{1}{2}\operatorname{rank}(\mathcal{F}) \sum_{P\in X} \frac{|G_P|}{|G|}(\deg(\chi) - < \chi\,|_{G_P}, 1_{G_P} >_{G_P}),$$

where G_P is the stabilizer of P defined above and 1_{G_P} denotes the trivial character of G_P. Here recall that there are $|G|/|G_P|$ points on X which are conjugate to P (i.e. of the form $g\cdot P$ with $g \in G$).

(2) *If λ is even ($\lambda(-1) = 1$) and non-trivial, then $m(\chi,\lambda) = 0$.*

(3) *If λ is odd ($\lambda(-1) = -1$), then $m(\chi,\lambda)$ is a multiple of $B_{1,\lambda}$, that is, $m(\chi,\lambda)/B_{1,\lambda}$ belongs to R_λ (= the integer ring of the field $\mathbf{Q}(\lambda(\gamma) \mid \gamma \in \Gamma_\chi)$) . More precisely, for $P \in X$ and a divisor r of $|G_P|$, define $M_{r,P} \in R_\lambda$ by*

$$M_{r,P} = < \alpha\,|_{G_P}, \theta_P^r >_{G_P} = \sum_{\gamma\in\Gamma_\chi} \lambda(\gamma) < \chi^\gamma\,|_{G_P}, \theta_P^r >_{G_P}.$$

Further for $m \in \mathbf{N}$ put $\rho(m) = \prod_p (1 - \lambda(p))$, where p runs over the prime divisors of m. Then we have

$$m(\chi,\lambda)/B_{1,\lambda} = -\operatorname{rank}(\mathcal{F}) \sum_{P\in X} \frac{|G_P|}{|G|}\Big(\sum_{r|(|G_P|/f_\lambda)} M_{r,P}\,\rho(|G_P|/r)\Big),$$

and its right hand side belongs to R_λ.

Proof. Put $\alpha = \alpha(\chi,\lambda)$. In view of Propositon 5, it is sufficient to compute $< \alpha, \operatorname{reg}_G >_G$ and $< \alpha, \nu >_G = \frac{1}{|G|}\sum_{P\in X} < \alpha, \nu_P >_G$. First, we obtain an equality $< \alpha, \operatorname{reg}_G >_G = \alpha(e) = \sum_{\gamma\in\Gamma_\chi} \lambda(\gamma)\chi^\gamma(e) = \deg(\chi)\sum_{\gamma\in\Gamma_\chi}\lambda(\gamma)$, where e denotes the unit element of G . Therefore we have

$$< \alpha, \operatorname{reg}_G >_G = \begin{cases} |\Gamma_\chi|\deg(\chi) & (\lambda \text{ is the trivial character}), \\ 0 & (\lambda \text{ is not trivial}). \end{cases}$$

Take a point $P \in X$ and define G_P and θ_P as above. Then we have $\nu_P = \sum_{d=1}^{n_P-1} d\cdot \operatorname{Ind}_{G_P}^G(\theta_P^d)$ where $n_P = |G_P|$. In computing $< \alpha, \nu_P >_G$, we first assume that λ is even. In this case $< \alpha, \nu_P >_G = < \alpha, \bar{\nu}_P >_G$ holds because of (1) in Proposition 2 (note that we have $< \bar{\chi}^\gamma, \nu_P >_G = < \chi^\gamma, \bar{\nu}_P >_G$). Since the complex conjugate of θ_P^d is $\theta_P^{n_P-d}$, we obtain

$\nu_P + \bar{\nu}_P = n_P(\sum_{d=1}^{n_P-1} \mathrm{Ind}_{G_P}^G(\theta_P^d)) = n_P(\mathrm{reg}_G - \mathrm{Ind}_{G_P}^G(1_{G_P}))$. Therefore, by the Frobenius reciprocity for induced characters,

$$\begin{aligned} < \alpha, \nu_P >_G &= \frac{1}{2} < \alpha, \nu_P + \bar{\nu}_P >_G \\ &= \begin{cases} \frac{1}{2} n_P |\Gamma_\chi| (\deg(\chi) - < \chi\,|_{G_P}, 1_{G_P} >_{G_P}) & (\lambda \text{ is trivial}), \\ 0 & (\lambda \text{ is even, non-trivial}). \end{cases} \end{aligned}$$

Combinig these, we obtain (1) and (2) in view of Proposition 5. Hereafter we assume that λ is odd. By Proposition 3 we have the following decomposition of $\alpha\,|_{G_P}$:

$$\alpha\,|_{G_P} = \sum_r M_{r,P} \left(\sum_{t \in J(n_P/r)} \lambda(t) \theta_P^{rt} \right),$$

where, in the summation, r runs through the divisors of n_P/f_λ. Accordingly, again by the Frobenius reciprocity,

$$\begin{aligned} < \alpha, \nu_P >_G &= \sum_r \sum_{t \in J(n_P/r)} \sum_{d=1}^{n_P-1} M_{r,P} \lambda(t) d < \theta_P^{rt}, \theta_P^d >_{G_P} \\ &= \sum_r \sum_{t \in J(n_P/r)} M_{r,P} \lambda(t) r \tilde{t} \\ &= \sum_r r M_{r,P} (\sum_{t \in J(n_P/r)} \tilde{t} \lambda(t)), \end{aligned}$$

where for $t \in J(m)$, $\tilde{t}$ is the integer satisfying $0 < \tilde{t} < m$ and $\tilde{t} \equiv t \pmod{m}$. For $m \in \mathbf{N}$ satisfying $f_\lambda \mid m$, elementary calculation shows

$$\sum_{t \in J(m)} \tilde{t} \lambda(t) = m \rho(m) B_{1,\lambda}.$$

Consequently, $< \alpha, \nu_P >_G = n_P(\sum_r M_{r,P} \rho(n_P/r)) B_{1,\lambda}$. Summing over $P \in X$ we obtain

$$< \alpha, \nu >_G / B_{1,\lambda} = \sum_{P \in X} (n_P/|G|)(\sum_r M_{r,P} \rho(n_P/r)).$$

Since $M_{r,P} = M_{r,g \cdot P}$ holds for $g \in G$ and the set $\{g \cdot P \in X \mid g \in G\}$ consists of $|G|/n_P$ points, we see that the sum above belongs to R_λ (recall

$M_{r,P} \in R_\lambda$). Thus, in view of Proposition 5, we have completed the proof of (3) .

Finally we refer to the case of the sheaf Ω^1_X of holomorphic differentials on X. Since $H^1(X, \Omega^1_X) = \mathbf{C}$, the trivial module, we are concerned with the G - module $V = H^0(X, \Omega^1_X)$. When $f : X \to Y$ is ramified, Ω^1_X is not of the form $f^*\mathcal{F}$, and hence we can not apply Theorem directly to Ω^1_X. However, by virtue of the Serre duality (see e.g. [3,Chapter III]) , V is the G - module dual to $H^1(X, \mathcal{O}_X)$. Because $\mathcal{O}_X = f^*\mathcal{O}_Y$, we can apply the Theorem to $\mathcal{O}_X$ ($H^0(X, \mathcal{O}_X) = \mathbf{C}$), and consequently obtain a result about V. We indicate in the Corollary below that generalized Bernoulli numbers appear in the decomposition of the G - module V, omitting to state the detailed decomposition.

Corollary. *Let V be as above and put $\eta = ch(V) \in R(G)$. For a character χ of G and an odd element $\lambda \in (\Gamma_\chi)^\wedge$, the inner product $< \alpha(\chi, \lambda), \eta >_G$ is a multiple of the generalized Bernoulli number $B_{1,\lambda}$.*

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Shoichi Nakajima
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153, Japon
e-mail : Shoichi@tansei.cc.u-tokyo.ac.jp