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# Bessel Functionals and Siegel Modular Forms

par Rainer Schulze-Pillot

The existence and uniqueness of Whittaker models plays an essential role in the theory of automorphic forms for the group $GL_n$. In contrast, it is well known that holomorphic Siegel modular forms of degree $n \geq 2$ (or the corresponding automorphic representations of the adelic symplectic group $Sp_n$ of rank $n$) do not possess Whittaker models. As a replacement in the case $n = 2$, Novodvorski and Piatetski-Shapiro studied Bessel models (to be defined below) and proved uniqueness [N-PS]. Their existence is trivial in the case of holomorphic modular forms. If an automorphic form for the group $PGSp_2(\mathbb{A})$ possesses such a model of a so called special type it was shown in [PS-S] that it can be related to a generic form (i.e., one having a Whittaker model) for the same group by theta lifting it to the metaplectic cover of $Sp_2$ and back again (using different additive characters). The key ingredient in this is the explicit computation of the Whittaker coefficients of the lifted form in terms of the Bessel functionals of special type of the original automorphic form. The purpose of this note is to show that this procedure can fail in a nontrivial way by exhibiting Siegel modular forms of degree and weight 2 which have no Bessel model of special type (the Bessel functionals of special type vanish trivially for Siegel modular forms of odd weight for $\Gamma_0(N)$-groups). The examples presented here came up in joint work with Böcherer and with Furusawa [B-SP 1, 2, 3, B-F-SP].

Let $F$ be a Siegel modular form of degree 2 for some congruence subgroup $\Gamma$ of level $N$ of $Sp_2(\mathbb{Z})$ and put

$$\Gamma' := \{g \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \in \Gamma\}.$$

If $F$ has Fourier expansion

$$F(Z) = \sum_T A(F,T) \exp(2\pi i \mathrm{tr}(TZ)$$

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we put for a discriminant $\Delta < 0$

$$A(F,\Delta) = \sum_{\{T\}_{\Gamma'}} \frac{A(F,T)}{\epsilon_{\Gamma'}(T)} \tag{1}$$

where the summation is over a set of representatives of the $\Gamma'$-equivalence classes of positive definite half integral symmetric matrices of discriminant $\Delta$ and $\epsilon_{\Gamma'}(T)$ is the number of proper units (integral automorphs) in $\Gamma'$ of $T$. The Koecher-Maaß series of $F$ is then

$$D_{KM}(F,s) := \sum_{\Delta<0} A(F,\Delta)\Delta^{-s}.$$

Let $G_1 = Sp_2 \subseteq GL_4$ be the symplectic group of rank 2, $G = GSp_2 \supseteq G_1$ the group of symplectic similitudes, both groups viewed as algebraic groups over $\mathbb{Q}$. For a Siegel modular form $F$ of degree 2 and weight $r$ with respect to a congruence subgroup $\Gamma_0^{(2)}(N) \subseteq Sp_2(\mathbb{Z})$ for some integer $N$ we denote by $F_s$ the automorphic form on $G(\mathbb{A})$ corresponding in the usual way to $F$. In particular, $F_s$ is left invariant under $G(\mathbb{Q})$ and right invariant under the groups

$$K_p = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_2(\mathbb{Z}_p) \mid C \equiv 0 \bmod N\mathbb{Z}_p\}$$

(embedded into $G(\mathbb{A})$ by putting all other components equal to 1) for all finite primes $p$. The function $F_s$ is invariant under the center of $G(\mathbb{A})$ and hence induces an automorphic form on $PGSp_2(\mathbb{A})$. We use the well known identification of $G/Z(G) = PGSp_2$ with the special orthogonal group of a quadratic form in five variables: Following [Si] we let $V$ be the $\mathbb{Q}$-vector space of all matrices

$$X = \begin{pmatrix} x_{-1}1_2 & M \\ \det(M)M^{-1} & x_1 1_2 \end{pmatrix} \begin{pmatrix} J & \\ & J \end{pmatrix}$$

with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $x_1, x_{-1} \in \mathbb{Q}$, $M \in M_2^{\mathrm{sym}}(\mathbb{Q})$. $V$ is equipped with the quadratic form $q(x) = \det M - x_{-1}x_1$ with associated bilinear form $B(x,y) = q(x+y) - q(x) - q(y)$ and decomposes as $V = (\mathbb{Q}e_1 + \mathbb{Q}e_{-1}) \perp M_2^{\mathrm{sym}}(\mathbb{Q})$, where $\mathbb{Q}e_1 + \mathbb{Q}e_{-1}$ is a hyperbolic plane and the embeddings are the obvious ones. On $V$ the group of symplectic similitudes $G(\mathbb{Q})$ acts through $X \mapsto \lambda(g)(g^t)^{-1}Xg^{-1} =: \iota(g)(X)$ (where $\lambda(g)$ is the similitude norm of $g$) by orthogonal transformations of determinant 1, and the map

$(g \mapsto \iota(g)) : G \to H := SO(V,q)$ gives an isomorphism from $G/Z(G)$ onto $H$ as algebraic groups over $\mathbb{Q}$. We write

$$F_o(\iota(g)) = F_s(g)$$

for the induced automorphic form on $H(\mathbb{A})$ (the index $o$ standing for orthogonal).

We need a few more notations (see [PS-S]). For $T \in M_2^{\mathrm{sym}}(\mathbb{Q}) \subseteq V$ let $D_T = \{h \in H \mid he_1 = e_1, hT = T\}$, let $S$ denote the unipotent radical of the parabolic subgroup $P$ (with Levi decomposition $P = MS$) of $H$ fixing the line through $e_1$, put $R_T = D_T S$. We fix the standard additive character $\psi$ of $\mathbb{A}/\mathbb{Q}$, let $\chi_T$ be the character of $S$ given by $\chi_T(s) = -B(se_{-1}, T)$, extend $\chi_T$ trivially to $R_T$ and consider the Bessel functional of special type

$$l_{T,1,\psi}(F_o, h) = \int\limits_{R_T(\mathbb{Q})\backslash R_T(\mathbb{A})} (\psi \circ \chi_T)^{-1}(r) F_o(rh)\, dr.$$

(More general, the Bessel functional $l_{T,\nu,\psi}(F_o,h)$ is defined for a character $\nu$ of $D_T(k)\backslash D_T(\mathbb{A})$ by inserting $\nu(r) := \nu(d)$ (for $r = ds \in R_T(\mathbb{A}), d \in D_T(\mathbb{A}), s \in S(\mathbb{A})$) into the defining integral above.)

Proposition 1. *Let $F$ be a Siegel modular form of weight $r$ for the group $\Gamma_0^{(2)}(N)$ and $F_o$ the associated automorphic form on $H(\mathbb{A})$. Let $\{\gamma_1, \dots, \gamma_n\}$ be a set of representatives of the double cosets $\Gamma_0^{(2)}(N)\gamma\Gamma_\infty$ in $Sp_2(\mathbb{Z})$ (with $\Gamma_\infty^{(2)} = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp_2(\mathbb{Z})\}$). Then the Bessel functionals of special type $l_{T,1,\psi}(F_o,h)$ are zero for all binary symmetric $T$ if and only if $D_{KM}(F|\gamma_i, s) = 0$ for $i = 1, \dots, n$.*

*Proof.* This is a standard computation: Denote by $\bar{K}_p$ the image of $GSp_2(\mathbb{Z}_p)$ under $\iota$ in $H(\mathbb{Q}_p)$, consider the Iwasawa decomposition $H(\mathbb{Q}_p) = S(\mathbb{Q}_p)M(\mathbb{Q}_p)\bar{K}_p$ and decompose the finite part $h_f$ of $h$ accordingly as $h_f = s_f m_f k_f$. We write $m_f = m m'_f k'_f$ with $m$ in $M(\mathbb{Q})$ commuting with $D_T$, the components $m'_p$ of $m'_f$ of the form $\iota(\begin{pmatrix} g_p & 0 \\ 0 & (g_p^t)^{-1} \end{pmatrix})$ with $g_p \in SL_2(\mathbb{Z}_p)$, and $k'_f \in \bar{K}_p$. Using strong approximation for $SL_2$ and $l_{T,1,\psi}(F_o, mh) = l_{T[g],1,\psi}(F_o,h)$ for $m = \iota(\begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix})$ with $g \in GL_2(\mathbb{Q})$ we see that we have to check whether all $l_{T,1,\psi}(F_o, h_\infty k_f)$ with $k_f \in \bar{K}_f$ are zero. The invariance property of $F_o$ on the right implies that it is sufficient to consider $k_f$ with all components $k_p$ in $\iota(Sp_2(\mathbb{Z}_p))$, and by strong approximation for $Sp_2$ and the right invariance of $F_o$ under the $\iota(K_p)$ we can assume $k_f = (\gamma_i, \gamma_i, \ldots)$ for some $1 \le i \le n$ (absorbing an element of $\Gamma_\infty$ into the integration over $S$). From formula 1-26 of [Su] it follows then as in [Fu] that for $h = (h_\infty, \gamma_i, \gamma_i, \ldots)$ with $h_\infty = \iota(g_\infty)$ and a binary symmetric matrix $T_d = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix}$ of discriminant $\Delta = -4d = 4(t_2^2 - t_1 t_3)$ one has

$$\begin{aligned} &l_{T_d,1,\psi}(F_o, h)\lambda(g_\infty)^{-2}(j(g_\infty, i1_2))^r \\ &\qquad = \exp(2\pi i \mathrm{tr}(T_d g_\infty < i1_2 >))\sqrt{d} \sum_{\{T\}_{\Gamma_i'}} \frac{A(F|\gamma_i, T)}{\epsilon_{\Gamma_i'}(T)} \end{aligned}$$

Here we write $\Gamma_i' = \{g \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \in \gamma_i^{-1}\Gamma_0^{(2)}(N)\gamma_i$ and the summation is as in (1). This proves the assertion.

To come to our examples we have to recall some notations and results from [B-SP 1, 2, 3]. Let $N$ be prime, $f_1, f_2 \in S_2(\Gamma_0(N))$ be two cusp forms of weight 2 for the group $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$, normalized eigenforms of all Hecke operators with the same eigenvalue under the Fricke involution $w_N$, $f_1 \ne f_2$. Let $D$ be the definite quaternion algebra over $\mathbb{Q}$ unramified at all primes $\ne N$, let $R$ be a maximal order in $D$.

With $R_{\mathbb{A}}^\times = (D \otimes \mathbb{R})^\times \times \prod_p R_p^\times$ consider a double coset decomposition

$$D_{\mathbb{A}}^\times = \bigcup_{i=1}^h R_{\mathbb{A}}^\times y_i^{-1} D^\times = \bigcup_{i=1}^h D^\times y_i R_{\mathbb{A}}^\times$$

with $n(y_i) = 1$ $(i = 1, \ldots, h)$. Put $I_{ij} = y_i R y_j^{-1}, R_i = I_{ii}, e_i = |R_i^\times|$.

To our cusp forms $f_1, f_2$ there correspond by the work of Eichler [E] two functions $\varphi_1, \varphi_2$ in the space

$$\mathcal{A}(D_{\mathbb{A}}^\times, R_{\mathbb{A}}^\times) = \{\varphi : D_{\mathbb{A}}^\times \to \mathbb{C} \mid \varphi(\gamma x u) = \varphi(x) \text{ for } \gamma \in D^\times, x \in D_{\mathbb{A}}^\times, u \in R_{\mathbb{A}}^\times\}.$$

With

$$\vartheta_{ij}^{(2)}(Z) := \vartheta^{(2)}(I_{ij}, Z) = \sum_{\underline{x}=(x_1,x_2)\in I_{ij}^2} \exp(2\pi i \text{ tr } (Q(\underline{x}) \cdot Z))$$

$(Q(\underline{x})_{\nu\mu} = \frac{1}{2}\operatorname{tr}(x_\nu \overline{x}_\mu), Z \in \mathbb{H}_n)$ let the Yoshida-lifting $Y^{(2)}(\varphi_1, \varphi_2)$ of $(\varphi_1, \varphi_2)$ be defined by

$$Y^{(2)}(\varphi_1, \varphi_2) := \sum_{i,j=1}^{h} \frac{\varphi_1(y_i)\varphi_2(y_j)}{e_i e_j} \vartheta_{ij}^{(2)}.$$

Then from [B-SP 1] we know that $F = Y^{(2)}(\varphi_1, \varphi_2)$ is a nonzero Siegel cusp form of degree 2 for the group $\Gamma_0^{(2)}(N)$. Moreover, if the eigenvalue of $f_1, f_2$ under the Fricke involution $w_N$ is $+1$, the Koecher-Maaß series $D_{KM}(F, s)$ is identically zero [B-SP 3, Remark 2.1] (This remark mentions only the $A(F, \Delta)$ for fundamental discriminants $\Delta$. The proof for arbitrary discriminants proceeds as the proof of Corollary 2.2 in [B-SP 3], using Proposition 3 of [B-SP 2] and the Remark on page 373 of [B-SP 2]). In fact, more is true:

Proposition 2. *Let $F$ and $N$ be as above. Then all Bessel functionals of special type associated to the corresponding function $F_o$ on $H(\mathbb{A})$ are zero.*

*Proof.* As in [B-SP 1, Lemma 8.1] we have to consider $F|\gamma_i$ for $\gamma_1 = 1, \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$. The Koecher-Maaß series for $F$ is zero from the argument given above, and $F|\gamma_3(NZ)$ is proportional to $F(Z)$ by [B-SP 1, Lemma 9.1]. For $F|\gamma_2$ we recall from [B-SP 1, Lemma 8.2] that

$$\vartheta_{ij}^{(2)}|\gamma_2(Z) = c \sum_{\substack{x_1 \in I_{ij}^{\#} \\ x_2 \in I_{ij}}} \exp(2\pi i \operatorname{tr}(Q((x_1, x_2))Z)).$$

with some constant $c$ (where $I_{ij}^{\#}$ is the dual lattice of $I_{ij}$). Each pair $x_1, x_2$ as in the summation generates a sublattice $K$ of discriminant in $N^{-1}\mathbb{Z}$ of $I_{ij}^{\#}$. The completion at the prime $N$ of the latter lattice is the set of vectors of integral length in $D \otimes \mathbb{Q}_N$. It is therefore clear that any binary sublattice of discriminant in $N^{-1}\mathbb{Z}$ of $I_{ij}^{\#}$ has a basis with one basis vector in $I_{ij}$. But this implies that any binary sublattice of discriminant $\Delta \in N^{-1}\mathbb{Z}$ of $I_{ij}^{\#}$ is counted by the coefficient at $\Delta$ of the Koecher-Maaß series of $\vartheta_{ij}^{(2)}|\gamma_2$, and it is easily seen that it is counted $2(SL_2(\mathbb{Z}) : \Gamma_i') = 2N(N+1)$ times. Since by definition it is counted twice by the coefficient at $\Delta$ of the Koecher-Maaß series of the theta series of degree 2 of $I_{ij}^{\#}$ and since we already know that the Koecher-Maaß series of

$$\sum_{i,j=1}^{h} \frac{\varphi_1(y_i)\varphi_2(y_j)}{e_i e_j} \vartheta^{(2)}(I_{ij}^{\#})$$

(which is proportional to $F|\gamma_3$) is zero, it follows that $D_{KM}(F|\gamma_2, s)$ is zero as well, which by Proposition 1 implies the assertion.

## References

[B-F-SP] S. Böcherer, M. Furusawa and R. Schulze-Pillot, *On Whittaker coefficients of some metaplectic forms*, Duke Math. Journal (to appear).

[B-SP 1] S. Böcherer and R. Schulze-Pillot, *Siegel modular forms and theta series attached to quaternion algebras*, Nagoya Math. J. **121** (1991), 35-96.

[B-SP 2] S. Böcherer and R. Schulze-Pillot, *On a theorem of Waldspurger and on Eisenstein series of Klingen type*, Math. Annalen **288** (1990), 361-388.

[B-SP 3] S. Böcherer and R. Schulze-Pillot, *The Dirichlet series of Koecher and Maaß and modular forms of weight* 3/2, Math. Zeitschr. **209** (1992), 273-287.

[E] M. Eichler, *The basis problem for modular forms and the trace of the Hecke operators*, Modular functions of one variable 1, Lecture Notes in Math. 320, Springer Verlag, Berlin, Heidelberg, New York, 1973, pp. 75-151.

[Fu] M. Furusawa, *On L-functions for* $GSp(4) \times GL(2)$ *and their special values*, J. f. d. reine u. angew. Math. **438** (1993), 187-218.

[N-PS] M. E. Novodvorski and I. I. Piatetski-Shapiro, *Generalized Bessel models for a symplectic group of rank* 2, Math. USSR Sbornik **19** (1973), 243-255.

[PS-S] I. I. Piatetski-Shapiro and D. Soudry, *On a correspondence of automorphic forms on orthogonal groups of order five*, J. Math. pures et appl. **66** (1987), 407-436.

[Si] C. L. Siegel, *Symplectic geometry*, Gesammelte Abhandlungen, Bd. II, Springer Verlag, Berlin, Heidelberg, 1966, pp. 274-359.

[Su] T. Sugano, *On holomorphic cusp forms on quaternion unitary groups of degree* 2, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **31** (1984), 521-568.

Rainer Schulze-Pillot
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
D-50931 Köln
e-mail : schupi@mi.uni-koeln.de