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A new lower bound for the football pool problem for 7 matches

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Résumé. Notons $K_3(7,1)$ le cardinal minimal d'un code ternaire de longueur 7 et de rayon de recouvrement un. Dans un précédent article, nous avons amélioré la minoration $K_3(7,1) \geq 147$ en montrant que $K_3(7,1) \geq 150$. Dans cette note, nous prouvons que $K_3(7,1) \geq 153$.

Abstract. Let $K_3(7,1)$ denote the minimum cardinality of a ternary code of length 7 and covering radius one. In a previous paper, we improved on the lower bound $K_3(7,1) \geq 147$ by showing that $K_3(7,1) \geq 150$. In this note, we prove that $K_3(7,1) \geq 153$.

1. Introduction

Let $\mathbb{F}_3$ be the finite field with three elements and n be some positive integer. Define the Hamming distance between two elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $\mathbb{F}_3^n$ by $\mathrm{d}(x,y) = |\{i \in \{1,\ldots,n\} \ : \ x_i \neq y_i\}|$. For $x \in \mathbb{F}_3^n$ and $r \in \mathbb{N}$, the sphere of center x and radius r is denoted $S_r(x)$ and is defined by $S_r(x) = \{y \in \mathbb{F}_3^n \ : \ \mathrm{d}(x,y) = r\}$.

A covering code with covering radius one is a subset C of $\mathbb{F}_3^n$ such that the following condition holds:

$$\forall x \in \mathbb{F}_3^n\,,\ \exists y \in C \ : \ \mathrm{d}(x,y) \leq 1\,. \tag{1}$$

Let A denote the characteristic function of C. For $i \in \mathbb{Z}$ let, as usual, the function A_i be defined by $A_i(x) = |C \cap S_i(x)|$. More generally, for any function F defined on $\mathbb{F}_3^n$, we define the function F_i by

$$F_i(x) = \sum_{y \in S_i(x)} F(y)\,,$$

as in [4]. Then the covering condition (1) becomes

$$\forall x \in \mathbb{F}_3^n\,,\ (A_0 + A_1)(x) \geq 1\,.$$

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The problem of determining $K_3(n,1)$, the minimal cardinality of C, has been widely studied [2] and is known as the "football pool problem". Summing the covering conditions over $\mathbb{F}_3^n$ leads to the sphere covering bound $(2n+1)K_3(n,1) \geq 3^n$. For instance this gives $K_3(7,1) \geq 146$. Chen and Honkala [1] found the lower bound 147 and in [3] we gave the lower bound 150. The purpose of this paper is to prove the following Theorem.

THEOREM 1. $K_3(7,1) \geq 153$.

2. Proof of Theorem 1

Let us consider the case $n = 7$. Let C be a covering code with covering radius one and suppose that $|C| \leq 152$. Up to adding some points to C, we may assume that $|C| = 152$. Let xy denotes the concatenation of x and y. For $x \in \mathbb{F}_3^3$, let us put

$$N(x) = \sum_{y \in \mathbb{F}_3^4} A(xy) = |\{y \in \mathbb{F}_3^4 \ : \ xy \in C\}|\,,$$

so that $\sum_{i=0}^{3} N_i(x) = |C| = 152$. We proved in [3] that the following condition holds:

$$\forall x \in \mathbb{F}_3^3\,,\ 9N_0(x) + N_1(x) \geq 81\,. \tag{2}$$

By summing this inequality over $S_2(x)$ and $S_3(x)$, we obtain the additional conditions

$$\forall x \in \mathbb{F}_3^3\,,\ 4N_1(x) + 11N_2(x) + 3N_3(x) \geq 972\,, \tag{3}$$

$$\forall x \in \mathbb{F}_3^3\,,\ 2N_2(x) + 12N_3(x) \geq 648\,. \tag{4}$$

The linear combination $43(2)+5(3)+4(4)$ now gives the inequality $63|C| + 324N(x) \geq 10935$, which implies the lower bound $N(x) \geq 5$.

Let us put $D(x) = N(x) - 5$, so that $\sum_{i=0}^{3} D_i(x) = 152 - 5 \times 27 = 17$. The condition (2) becomes

$$\forall x \in \mathbb{F}_3^3\,,\ 9D(x) + D_1(x) \geq 6\,,$$

which is equivalent to the new condition

$$\forall x \in \mathbb{F}_3^3\,,\ 6D(x) + D_1(x) \geq 6\,. \tag{5}$$

We shall need the following lemma.

Lemma 2. *Let u be an element of $\mathbb{F}_3^3$ such that $\mathrm{d}(0,u) = 3$. Then the following property holds*

$$\forall x \in \mathbb{F}_3^3\,,\ D(x-u) + D(x) + D(x+u) \geq 1\,. \tag{6}$$

Proof. Up to isometry, we may assume that $u = 111$. Let us define the function ϕ on $\mathbb{F}_3^3$ by $\phi(x) = D(x-u) + D(x) + D(x+u)$. Let us add the condition (5) for $x-u$, x and $x+u$. We obtain $6\phi(x) + \phi_1(x) \geq 18$. Since we know that

$$\phi_1(x) \leq \sum_{x \in \mathbb{F}_3^n} D(x) = 17\,,$$

we have $6\phi(x) \geq 1$ and the lemma follows. $\square$

Since $\sum_{i=0}^{3} D_i(x) = 17 < 2 \times 9$, there exists some $a \in \mathbb{F}_3^2$ such that $D(a0)+D(a1)+D(a2) \leq 1$, say $a = 00$. If $D(000) = D(001) = D(002) = 0$, it follows from condition (5) that $3 \times 6 \leq \sum_{i=0}^{3} D_i(x) = 17$, which is impossible. Therefore, we may assume that, up to translation, $D(000) = D(001) = 0$ and $D(002) = 1$. The condition (5) and Lemma 2 then give us the following inequalities:

$$D(100) + D(200) + D(010) + D(020) \geq 5\,, \tag{7}$$

$$D(101) + D(201) + D(011) + D(021) \geq 5\,, \tag{8}$$

$$D(111) + D(222) \geq 1\,, \tag{9}$$

By using (5) for $x = 222$, (7-9) and the evaluation $D(002) = 1$, we get

$$\begin{aligned} 18 &\leq \sum_{x \in S_1(000) \cup S_1(001) \cup S_1(222)} D(x) + D(111) + 7D(222) \\ &\leq \sum_{x \in \mathbb{F}_3^3} D(x) + 6D(222) = 17 + 6D(222)\,, \end{aligned}$$

which shows that $D(222) \geq 1$. By symmetry we also have $D(112)$, $D(122), D(212) \geq 1$, which implies the inequality

$$D(222) + D(112) + D(122) + D(212) \geq 4\,. \tag{10}$$

We take the following linear combination of inequalities: the inequalities (5) with $x \in \{110, 111, 120, 121, 210, 211, 220, 221\}$, plus seven times the inequalities (7-8) and (10) plus nine times the equation $D(002) = 1$. In this way, we obtain

$$155 \leq 9 \left(\sum_{x \in \mathbb{F}_3^3 \setminus \{012,022,102,202\}} D(x) \right) \leq 9 \times 17 = 153\,,$$

which is impossible. This completes the proof of Theorem 1.

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