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Irrationality of quick convergent series

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Résumé. On démontre une généralisation d'un résultat dû à Badea concernant l'irrationnalité de certaines séries à convergence rapide.

Abstract. We generalize a previous result due to Badea relating to the irrationality of some quick convergent infinite series.

There are many papers concerning the irrationality of infinite series. Erdös [4] proved that if the sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers converges quickly to infinity, then the series $\sum_{n=1}^{\infty} 1/a_n$ is an irrational number. The author [7] defined the irrational sequences and proved criterion for them. Another result is due to Erdös and Strauss [5]. They proved that if $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive integers with $\limsup_{n\to\infty} a_1 \cdots a_n/a_{n+1} < \infty$ and $\limsup_{n\to\infty} a_n^2/a_{n+1} \leq 1$, then the number $\sum_{n=1}^{\infty} 1/a_n$ is rational if and only if $a_{n+1} = a_n^2 - a_n + 1$ holds for every $n > n_0$. Sándor [8] proved that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\limsup_{n\to\infty} a_n/(a_1 \cdots a_{n-1} b_n) = \infty$ and $\liminf_{n\to\infty} a_n b_{n-1}/(a_{n-1} b_n) > 1$, then the number $\sum_{n=1}^{\infty} b_n/a_n$ is irrational.

Finally Badea [1] proved that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $b_{n+1} > (b_n^2 - b_n)a_{n+1}/a_n + 1$, then the sum $\sum_{n=1}^{\infty} a_n/b_n$ is an irrational number. Later he generalized his result ([2]).

In this paper we will generalize Badea's result in another way and prove the following theorem.

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THEOREM. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers. If there is a natural number m such that the following three inequalities hold for every $n > n_o$*

$$b_n > m+1 \tag{1}$$

$$b_n \sum_{k=1}^{m} (-1)^k \binom{m}{k} \left(\prod_{j=n-m}^{n-k-1} b_j \right) \sum_{j=m-k}^{m-1} a_{n-m+j}/b_{n-m+j} > \tag{2}$$

$$> -a_n \sum_{k=0}^{m} (-1)^k \binom{m}{k} \prod_{j=n-m}^{n-1-k} b_j + \sum_{i=1}^{m} \sum_{k=0}^{m} (-1)^{i+k+1} \operatorname{sgn}(k+1-i) \binom{m}{i} \times$$

$$\times \binom{m}{k} \left(\prod_{s=n-m}^{n-i} b_s / \prod_{s=n-k}^{n-1} b_s \right) \sum_{j=\min(m-i,m-k-1)+1}^{j=\max(m-i,m-k-1)} a_{n-m+j}/b_{n-m+j}$$

and

(3)

$$b_n \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} \left(\prod_{j=n-m-1}^{n-k-1} b_j \right) \sum_{j=m+1-k}^{m} a_{n-m+j-1}/b_{n-m+j-1} <$$

$$< -a_n \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \prod_{j=n-m-1}^{n-1-k} b_j + \sum_{i=1}^{m+1} \sum_{k=0}^{m+1} (-1)^{i+k+1} \operatorname{sgn}(k+1-i) \times$$

$$\times \binom{m+1}{i} \binom{m+1}{k} \left(\prod_{s=n-m-1}^{n-i} b_s / \prod_{s=n-k}^{n-1} b_s \right) \times$$

$$\times \sum_{j=min(m+1-i,m-k)+1}^{\max(m+1-i,m-k)} a_{n-m+j-1}/b_{n-m+j-1}$$

then the number $A = \sum_{n=1}^{\infty} a_n/b_n$ is irrational.

Proof : For the sake of simplicity we will suppose that (1) – (3) hold for every n. (If not, we define $a'_n = a_{n+m+n_0}$, $b'_n = b_{n+m+n_0}$ for every $n =$

$1, 2, \cdots$ and these two sequences $\{a'_n\}_{n=1}^{\infty}$ and $\{b'_n\}_{n=1}^{\infty}$ satisfy then our above requirements.)

Let us denote

$$B_n = B_{n,0} = \prod_{i=1}^{n} b_i \tag{4}$$

$$A_n = A_{n,0} = B_n \sum_{i=1}^{n} a_i/b_i \tag{5}$$

$$B_{n,i} = B_{n,i-1} - B_{n-1,i-1} \qquad i = 1, \cdots, m+1$$

$$A_{n,i} = A_{n,i-1} - A_{n-1,i-1} \qquad i = 1, \cdots, m+1$$

One can prove by induction that

$$B_{n,i} = \sum_{j=0}^{i} \binom{i}{j} B_{n-j} (-1)^j \tag{6}$$

and

$$A_{n,i} = \sum_{j=0}^{i} \binom{i}{j} A_{n-j} (-1)^j \tag{7}$$

hold for $i = 0, 1, \cdots, m+1$. (1) and (4) yield

$$\binom{i}{j} B_{n-j} - \binom{i}{j+1} B_{n-j-1} = \binom{i}{j} B_{n-j-1} (b_{n-j} - \frac{i-j}{j+1}) > 0 \tag{8}$$

and

$$\binom{i}{j} A_{n-j} - \binom{i}{j+1} A_{n-j-1} = \binom{i}{j} (A_{n-j} - \frac{i-j}{j+1} A_{n-j-1}) = \tag{9}$$

$$= \binom{i}{j} B_{n-j-1} \left(a_{n-j} + (b_{n-j} - \frac{i-j}{j+1} \sum_{k=1}^{n-j-1} a_k/b_k \right) > 0$$

for every natural number n. Then (4)–(9) imply that $B_{n,i} > 0$ and $A_{n,i} > 0$ for every positive integer n and $i = 0, 1, \cdots, m-1$.

First we will prove that

$$A_{n,m}/B_{n,m} < A_{n+1,m}/B_{n+1,m} < \cdots \tag{10}$$

and secondly

$$A_{n,m+1}/B_{n,m+1} > A_{n+1,m+1}/B_{n+1,m+1} \tag{11}$$

(11) implies that there is a number $c \geq 0$ such that

$$c = \lim_{n\to\infty} A_{n,m+1}/B_{n,m+1}.$$

Using the famous theorem of Stolz (see e.g. [6]), we obtain

$$A = \lim_{n\to\infty} A_n/B_n = \cdots = \lim_{n\to\infty} A_{n,m+1}/B_{n,m+1} = c. \tag{12}$$

On the other hand (10), (11), (12) and Brun's Theorem (see e.g. [3]) imply the irrationality of the number A.

Now we will prove (10) and (11). Using (4) and (5) we have

$$\frac{A_{n,m}}{B_{n,m}} - \frac{A_{n-1,m}}{B_{n-1,m}} = \frac{\sum\limits_{i=0}^{m} \binom{m}{i} A_{n-i}(-1)^i}{\sum\limits_{i=0}^{m} \binom{m}{i} B_{n-i}(-1)^i} - \frac{\sum\limits_{i=0}^{m} \binom{m}{i} A_{n-1-i}(-1)^i}{\sum\limits_{i=0}^{m} \binom{m}{i} B_{n-1-i}(-1)^i} = \tag{13}$$

$$= \frac{\sum\limits_{i=0}^{m}(-1)^i\binom{m}{i} B_{n-i}(A_{n-m}/B_{n-m} + \sum\limits_{j=1}^{m-i} a_{n-m+j}/b_{n-m+j})}{\sum\limits_{i=0}^{m} \binom{m}{i} B_{n-i}(-1)^i} -$$

$$-\frac{\sum\limits_{i=0}^{m}(-1)^i\binom{m}{i} B_{n-i-1}(A_{n-m-1}/B_{n-m-1} + \sum\limits_{j=1}^{m-i} a_{n-m+j-1}/b_{n-m+j-1})}{\sum\limits_{i=0}^{m}(-1)^i\binom{m}{i} B_{n-i-1}} =$$

$$= \frac{\sum_{i=0}^{m}(-1)^i\binom{m}{i}B_{n-i}\sum_{j=0}^{m-i}a_{n-m+j}/b_{n-m+j}}{\sum_{i=0}^{m}(-1)^i\binom{m}{i}B_{n-i}} -$$

$$-\frac{\sum_{i=0}^{m}(-1)^i\binom{m}{i}B_{n-i-1}\sum_{j=1}^{m-i}a_{n-m+j-1}/b_{n-m+j-1}}{\sum_{i=0}^{m}(-1)^i\binom{m}{i}B_{n-i-1}} =$$

$$= \sum_{i=0}^{m}\sum_{k=0}^{m}(-1)^{i+k}\binom{m}{i}\binom{m}{k}B_{n-i}B_{n-k-1}\left(\sum_{j=0}^{m-i}a_{n-m+j}/b_{n-m+j}-\right.$$

$$\left.-\sum_{s=1}^{m-k}a_{n-m-1+s}/b_{n-m-1+s}\right)/(B_{n,m}B_{n-1,m}) =$$

$$= \left(B_n\sum_{k=0}^{m}(-1)^k\tbinom{m}{k}B_{n-1-k}\sum_{j=m-k}^{m}a_{n-m+j}/b_{n-m+j}-\right.$$

$$-\sum_{i=1}^{m}\sum_{k=0}^{m}(-1)^{i+k+1}\binom{m}{i}\binom{m}{k}B_{n-i}B_{n-k-1}\Big(\sum_{j=0}^{m-i}a_{n-m+j}/b_{n-m+j}-$$

$$\left.-\sum_{s=1}^{m-k}a_{n-m+s-1}/b_{n-m+s-1}\Big)\right)/(B_{n,m}B_{n-1,m}) =$$

$$= \left(B_n\sum_{k=1}^{m}(-1)^k\binom{m}{k}B_{n-1-k}\sum_{j=m-k}^{m-1}a_{n-m+j}/b_{n-m+j}+\right.$$

$$+B_{n-1}a_n\sum_{k=0}^{m}(-1)^k\binom{m}{k}B_{n-k-1}-\sum_{i=1}^{m}\sum_{k=0}^{m}(-1)^{i+k+1}\binom{m}{i}\binom{m}{k}B_{n-i}B_{n-k-1}$$

$$\left.\times\sum_{j=\min(m-i,m-k-1)+1}^{\max(m-i,m-k-1)}\operatorname{sgn}(k+1-i)a_{n-m+j}/b_{n-m+j}\right)/(B_{n,m}B_{n-1,m}).$$

(13) and (2) yield (10). Similarly (13) (if we substitute $m+1$ instead of m) and (3) yield (11).

Remark: If we put $m = 0$ in the main theorem, then we receive $b_n > 1$, $0 > -a_n$ and $-b_n a_{n-1}/b_{n-1} < -a_n(b_{n-1} - 1) - a_{n-1}/b_{n-1}$. ¿Thus $b_n > (b_{n-1}^2 - b_{n-1})a_n/a_{n-1} + 1$ and this is the famous theorem due to Badea (see e.g. [1]).

Consequence 1: Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers. If

$$b_n > 2 \tag{14}$$

$$b_n < (b_{n-1}^2 - b_{n-1})a_n/a_{n-1} + 1 \tag{15}$$

$$\begin{aligned} b_n(-b_{n-1}a_{n-2} + 2b_{n-2}^2 a_{n-1} - b_{n-2}a_{n-1}) > \\ > a_n b_{n-1} b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3a_{n-1}b_{n-2}^2 - 2b_{n-1}a_{n-2} \\ - 2b_{n-2}a_{n-1} + a_{n-2} \end{aligned} \tag{16}$$

hold for every $n > n_0$, then the number $A = \sum\limits_{n=1}^{\infty} a_n/b_n$ is irrational.

Proof: Let us put $m = 1$ in the main theorem. Then (14) is (1), (15) is (2) and (16) is (3).

Consequence 2: Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $b_1 > 2$ and

$$kb_{n-1}^2 - (3k-1)b_{n-1} < b_n < kb_{n-1}^2 - kb_{n-1} \tag{17}$$

hold for every $n > n_0$ where k is a positive integer. Then the number $A = \sum\limits_{n=1}^{\infty} k^n/b_n$ is irrational.

Proof: Let us put $a_n = k^n$ in consequence 1. Then (17) immediately implies (15) and

$$b_n > kb_{n-1}^2 - (3k-1)b_{n-1} = kb_{n-1}(b_{n-1} - 3) + b_{n-1}.$$

This and $b_1 > 2$ imply that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing. Thus (15) is fulfilled too. Condition (16) can be rewritten in the following way

$$b_n(-b_{n-1} + 2kb_{n-2}^2 - kb_{n-2}) > \tag{18}$$
$$> k^2 b_{n-1}b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3kb_{n-2}^2 - 2b_{n-1} - 2kb_{n-2} + 1.$$

Let us define the sequence $\{s_n\}_{n=1}^{\infty}$ of nonnegative integers such that

$$s_n = kb_{n-1}^2 - kb_{n-1} - b_n. \tag{19}$$

(17) implies that

$$0 < s_n < (2k-1)b_{n-1}. \tag{20}$$

Substituting (19) for (18) we obtain the equivalent inequality (21) with (18) :

$$(kb_{n-1}^2 - kb_{n-1} - s_n)(kb_{n-2}^2 + s_{n-1}) > \tag{21}$$
$$> k^2 b_{n-1}b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3kb_{n-2}^2 - 2b_{n-1} - 2kb_{n-2} + 1.$$

Carrying out the equivalent calculations step by step, we receive

$$kb_{n-1}^2 s_{n-1} - s_n kb_{n-2}^2 - s_n s_{n-1} - kb_{n-1}s_{n-1} + k^2 b_{n-1}b_{n-2}^2 - k^2 b_{n-1}b_{n-2}$$
$$- 3kb_{n-2}^2 + 2b_{n-1} + 2kb_{n-2} - 1 > 0.$$

Using (20) and the fact that $\{b_n\}_{n=1}^{\infty}(b_1 > 2)$ is an increasing sequence, it is enough to prove that

$$kb_{n-1}^2 - (k-1)kb_{n-1}b_{n-2}^2 - Kb_{n-1}b_{n-2} > 0, \tag{22}$$

where K is a suitable constant. (22) is equivalent with

$$kb_{n-1}(b_{n-1} - kb_{n-2}^2) + kb_{n-1}b_{n-2}^2 - Kb_{n-1}b_{n-2} > 0.$$

(17) implies that

$$-(3k-1)b_{n-2} < b_{n-1} - kb_{n-2}^2. \tag{23}$$

Because of (23), it is enough to prove that

$$kb_{n-1}b_{n-2}^2 - K_1 b_{n-1}b_{n-2} > 0, \tag{24}$$

where K_1 is a suitable constant too. But (24) is true for every $n > n_0$. Thus (18) is right and the number A is irrational.

Examples: The numbers $\sum\limits_{n=1}^{\infty} 2^n/b_n$ and $\sum\limits_{n=1}^{\infty} 3^n/a_{n'}$ where $a_1 > 2$, $b_1 > 2$, $b_n = 2b_{n-1}^2 - 2b_{n-1} - 1$ and $a_n = 3a_{n-1}^2 - 3a_{n-1} - 4$ are irrational.

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