Journal de Théorie des Nombres
de Bordeaux **8** *(1996), 413–428*

On the discrepancy of Markov-normal sequences

par M.B. LEVIN

RÉSUMÉ. On construit une suite normale de Markov dont la discrépance est $O(N^{-1/2}\log^2 N)$, améliorant en cela un résultat donnant l'estimation $O(e^{-c(\log N)^{1/2}})$.

ABSTRACT. We construct a Markov normal sequence with a discrepancy of $O(N^{-1/2}\log^2 N)$. The estimation of the discrepancy was previously known to be $O(e^{-c(\log N)^{1/2}})$.

A number $\alpha \in (0,1)$ is said to be *normal* to the base q, if in a q-ary expansion of α,

$$\alpha = .d_1 d_2 \cdots \quad = \sum_{i=1}^{\infty} d_i/q^i, \qquad d_i \in \{0,1,\cdots,q-1\}$$

each fixed finite block of digits of length k appears with an asymptotic frequency of q^{-k} along the sequence $(d_i)_{i\geq 1}$. Normal numbers were introduced by Borel (1909). Borel proved that almost every number (in the sense of Lebesgue measure) is normal to the base q. But only in 1935 did Champernowne give the explicit construction of such a number, namely

$$\theta = .1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ \ldots\ .$$

obtained by successively concatenating all the natural numbers.

Let $P = (p_{i,j})_{0\leq i,j\leq q-1}$ be an irreducible Markov transition matrix, $(p_i)_{0\leq i\leq q-1}$ the stationary probability vector of P and $\overline{\mu}$ its probability measure.

A number α (sequence $(d_i)_{i\geq 1}$) is said to be *Markov-normal* if in a q-ary expansion of α each fixed finite block of digits $b_0 b_1 \ldots b_k$ appears with an asymptotic frequency of $p_{b_0} p_{b_0 b_1} \ldots p_{b_{k-1} b_k}$.

According to the individual ergodic theorem $\overline{\mu}$-almost all sequences (numbers) are normals.

Markov normal numbers were introduced by Postnikov and Piatecki-Shapiro [1]. They also obtained, by generalizing Champernowne's method, the explicit construction of these numbers. Another Champernowne construction of Markov normal numbers was obtained in Smorodinsky-Weiss

Manuscrit reçu le 7 février 1995.

[2] and in Bertrand-Mathis [3] . In [4] Chentsov gave the construction of Markov normal numbers using *completely uniformly distributed sequences* (for the definition, see [5]) and the standard method of modelling Markov chains. In [6] Shahov proposed using a *normal periodic systems of digits* (for the definition, see [5]) to construct Markov normal numbers. In [7] he obtained the estimate of discrepancy of the sequence $\{\alpha q^n\}_{n=1}^N$ to be $O(e^{-c(\log N)^{1/2}})$. In this article we construct a Markov normal sequence with the discrepancy of sequence $\{\alpha q^n\}_{n=1}^N$ equal to $O(N^{-1/2}\log^2 N)$.

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers, μ - measure on $[0,1)$. The quantity

$$(1) \qquad D(\mu,N) = \sup_{\gamma\in[0,1)} |\frac{1}{N}\#\{n\in[1,N] \mid 0\leq\{x_n\}<\gamma)\} - \mu[0,\gamma)|$$

is called the *discrepancy* of $(x_n)_{n=1}^N$.
The sequence $(\{x_n\})_{n\geq 1}$ is said to be $\mu-$*distributed* in $[0,1)$ if $D(\mu,N)\to 0$.

Let the measure μ be such that

$$(2) \qquad \mu([\gamma_n,\gamma_n+1/q^n)) = p_{c_1}p_{c_1c_2}...p_{c_{n-1}c_n}, \qquad \gamma_n = .c_1...c_n, \quad n=1,2,...,$$

where $c_k\in\{0,1,..q-1\}$, $k=1,2,...$.
It is known that if and only if α is Markov normal number, the sequence $\{\alpha q^n\}_{n=1}^\infty$ is μ-distributed.
The discrepancy $D(\mu,N)$ satisfies $D(\mu,N) = O(N^{-1/2}(\log\log N)^{1/2})$ for almost all α.
The following facts are known from the theory of finite Markov chains [8,9]:

Let a Markov chain have d cyclic class $C_1,...,C_d$. We enumerate the states $e_1,...,e_q$ of the Markov chain in such a way, that if $e_i\in C_m$, $e_j\in C_n$ and $i>j$, then $m\geq n$. Here matrix P has d^2 blocks $(\overline{P}_{i,j})_{0\leq i,j\leq d-1}$, where $\overline{P}_{i,j}=0$ except for $\overline{P}_{1,2},\overline{P}_{2,3},\overline{P}_{d-1,d},\overline{P}_{d,1}$. Matrix P^d has a block-diagonal structure. Let $P_1,...,P_d$ be the block diagonal of matrix P^d. There exists a number k_0 such that all the elements of matrices $P_i^{k_0}$ $(i=1,...,d)$ are greater than zero [9, ch. 4]. Let θ be the minimal element of these matrices, and $p_{ij}^{(k)}$ the ij element of matrix P^k, $k=1,2,....$
It is evident that

$$(3) \qquad \theta = \min_{i,j} p_{ij}^{(dk_0)},$$

where we choose minimum values for i,j so that e_i, e_j are included in the same cyclic class.
Let $f(j)$ be the number of cyclic class states e_j $(e_j\in C_{f(j)}, j=0,...,q-1)$.

According to [9, ch.4] we obtain

$$(4) \qquad |p_{ij}^{(kd+f(j)-f(i))} - dp_j| \leq (1-2\theta)^{-1+k/k_0},$$

$$p_{ij}^{(kd+f(j)-f(i)+l)} = 0, \quad l = 1,2,...,d-1, \quad k = 1,2,\dots .$$

Let

$$(5) \qquad p = \max_{0\leq i,j\leq q-1}(p_i, p_{ij}), \quad A_n = [p^{-n}], \ n = 1,2,\dots .$$

We have, from the irreduciblity of matrix P, that

$$(6) \qquad p < 1 \quad \text{and} \quad A_n \to \infty.$$

We use matrices $P_n = (p_{ij}(n))_{0\leq i,j\leq q-1}$ with the rational elements

$$(7) \qquad p_{ij}(n) = v_{ij}(n)/A_n,$$

and we choose $v_{ij}(n)$ as follows:
Let i be fixed and p_{ij_0} be greater than zero. Then we denote

$$v_{ij}(n) = [A_n p_{ij}], \ \text{if} \quad j \neq j_0, \quad \text{and} \quad v_{ij_0}(n) = A_n - \sum_{j\neq j_0} v_{ij}(n).$$

It is evident that

(8)

$$\sum_{j=0}^{q-1} p_{ij}(n) = 1, \quad |v_{ij}(n) - A_n p_{ij}| \leq q, \quad i,j = 0,...,q-1, \quad n = 1,2,...$$

If k_1 is sufficiently large, then using (3) and (6)-(8), we obtain

$$(9) \qquad \min_{ij} p_{ij}^{(dk_0)}(n) \geq \theta/2, \qquad n > k_1,$$

where we choose minimum values for i,j so that e_i, e_j belong to the same cyclic class.

It is evident that P_n $(n > k_1)$ is an irreducible matrix with a d-cyclic class.
Applying (3),(4) and (9) we obtain

$$(10) \qquad |p_{ij}^{(kd+f(j)-f(i))}(n) - dp_j(n)| \leq (1-\theta)^{-1+k/k_0}, \qquad k = 1,2,...$$

$$p_{ij}^{(kd+f(j)-f(i)+l)}(n) = 0, \quad l = 1,2,...,d-1, \qquad i,j = 0,...,q-1,$$

where $n > k_1$, and $(p_j(n))_{j<q}$ is the stationary probability vector of P_n.

According to [7, 10] there exist integers $v_0(n),...,v_{q-1}(n), L_n > 0$, such that

$$p_j(n) = v_j(n)/L_n \qquad v_0(n) + ... + v_{q-1}(n) = L_n$$

$$(11)\sum_{j=0}^{q-1} v_j(n)p_{ji}(n) = v_i(n), \quad |p_i - v_i(n)/L_n| < B/A_n, \ (i = 0, ... q-1).$$

If k_1 is sufficiently large, then applying (5)-(8) and (11), we obtain

$$(12) \qquad \max_{i,j}(p_i(n), p_{ij}(n))) \leq (p+1)/2 < 1, \qquad n > k_1,$$

$$(13) \qquad \min_{0\leq i\leq q-1} p_i(n) \geq \overline{p} = 1/2 \min_{0\leq i\leq q-1} p_i > 0.$$

Let the measure μ_n on $[0,1)$ be such that

$$(14) \qquad \mu_n([\gamma_r, \gamma_r + 1/q^r)) = p_{c_1}(n)p_{c_1c_2}(n)...p_{c_{r-1}c_r}(n),$$

$$\gamma_r = .c_1...c_r, \quad n, r = 1, 2, ...$$

where $c_r \in \{0, 1, ..q-1\}$, $r = 1, 2, ...$.

LEMMA 1. *Let* $\gamma = .c_1...c_n....,$ *. Then*

$$(15) \qquad \mu[0,\gamma) = \mu_n[0,\gamma_n) + O(np^n),$$

$$(16) \qquad \mu[0,\gamma) = \mu_n[0,\gamma_n + 1/q^n) + O(np^n),$$

where the O-constant depends only on P.
Proof. It follows from (2), (5) and (6) that

$$(17) \ \mu[0,\gamma) = \mu[0,\gamma_n) + \sum_{r\geq n+1}\sum_{b=0}^{c_r-1} p_{c_1}p_{c_1c_2}...p_{c_{r-1}b} = \mu[0,\gamma_n) + O(p^n).$$

We apply (2), (14) and obtain

$$(18) \qquad \mu[0,\gamma_n) = \mu_n[0,\gamma_n) + \sum_{r=1}^{n}\sum_{b=0}^{c_r-1} \sigma_r(b),$$

$$\sigma_r(b) = p_{c_1}p_{c_1c_2}...p_{c_{r-1}b} - p_{c_1}(n)p_{c_1c_2}(n)...p_{c_{r-1}b}(n).$$

If $p_{c_1}p_{c_1c_2}...p_{c_{r-1}b} = 0$, then $p_{c_ic_j} = 0$ and according to (5), (7), (8), (11) we have $p_{c_ic_j}(n) = O(p^n)$ and

$$(19) \qquad \sigma_r(b) = O(p^n).$$

Let $p_{c_1}p_{c_1c_2}...p_{c_{r-1}b} \neq 0$. Then

$$(20) \qquad \sigma_r(b) = p_{c_1}p_{c_1c_2}...p_{c_{r-1}b}\Delta_r,$$

where

$$\Delta_r = 1 - (1 + \frac{a_{c_1}(n) - L_np_{c_1}}{L_np_{c_1}})\prod_{k=1}^{r-1}(1 + \frac{a_{c_kv_k(n)} - A_np_{c_kv_k}}{A_np_{c_kv_k}}),$$

and $v_k = c_{k+1}$ or b.
On the basis of (5), (7), (8) and (11) we deduce that

$$|\Delta_r| \le (1+\frac{B}{p^{'}A_n})(1+\frac{q}{p^{'}A_n})^{r-1} - 1 \le (1+\epsilon p^n)^r - 1, \qquad p^{'} = \min_{i,j \;\; p_{ij}\ne 0} p_{ij},$$

where $|\epsilon| < 2qB/p^{'}$.
It is easy to compute that

$$\Delta_r = O(rp^n), \qquad r \le n.$$

Hence and from (17) - (20) we obtain

$$\mu[0,\gamma) - \mu_n[0,\gamma_n) = O(np^n + np^n \sum_{r=1}^{n} \sum_{b=0}^{c_r-1} p_{c_1}p_{c_1c_2}...p_{c_{r-1}b}) = O(np^n)$$

and formula (15) is proved. Statement (16) is proved analogously. ■

We obtain the Markov normal number $\alpha = .d_1d_2...$ by concatenating blocks $\alpha_n^{'} = (a_1, ..., a_{A_{2n}})$,where $a_i \in \{0,1,...,q-1\}$, $i = 1,2,...$

$$\alpha = .\alpha_1^{'}...\alpha_n^{'}..., \tag{21}$$

We choose the numbers a_i as follows:

Let

(22)

$$\Omega_n = \{\omega_n = (b_0, ..., b_{A_{2n}+n}) \mid b_0 \in \{0,...,L_n-1\},\ b_1, b_2, ... \in \{0,...,A_n-1\}\}$$

$$S_0 = [0, v_0(n)),\ S_j = [v_0(n)+...+v_{j-1}(n), v_0(n)+...+v_j(n)),$$
$$S_{i,0} = [0, v_{i,0}(n)),$$
$$S_{i,j} = [v_{i,0}(n)+...+v_{i,j-1}(n), v_{i,0}(n)+...+v_{i,j}(n))$$
$$(i = 0,...,q-1,\ j = 1,...,q-1).$$

We set $a_0 = i$, if $b_0 \in S_i$, $\;i = 0,...,q-1$. If we choose the numbers $a_0, ..., a_{k-1}$, then we set

$$a_k = i, \text{ if } b_k \in S_{a_{k-1},i}, \quad i = 0,...,q-1. \tag{23}$$

Let

$$\alpha_n = \alpha_n(\omega_n) = .a_1, ..., a_{A_{2n}+n}, \qquad n = 1,2... \tag{24}$$

(25)

$$R_{[\beta,\gamma)}(\mu_n, \alpha, M) \;=\; \#\{n \in [1,M] \mid \beta \le \{\alpha q^n\} < \gamma\} - M\mu_n[\beta,\gamma),$$

$$E_n(\omega_n) = \max_{1\le M\le A_{2n}} \max_{\gamma_n} |R_{[0,\gamma_n)}(\mu_n, \alpha_n(\omega_n), M)|, \tag{26}$$

$$E_n = \min_{\omega_n \in \Omega_n} E_n(\omega_n). \tag{27}$$

We choose ω_n (and consequently $\alpha_n(\omega_n)$) such that

$$E_n(\omega_n) = E_n. \tag{28}$$

LEMMA 2.

$$E_n = O(p^{-n} n^2).$$

Proof. (To follow later.)

Let

$$n_1 = 0, ..., n_{k+1} = n_k + A_{2k}, \qquad k = 1, 2, ... \tag{29}$$

Every natural N can be represented uniquely in the following form with integers k

$$N = n_k + M_1\ , \ \ 0 \le M_1 < A_{2k}, \quad k = 1, 2, ... \tag{30}$$

Let

$$T_\gamma(\alpha, Q, M) = \#\{n \in (Q, Q+M] \ | \ \{\alpha q^n\} < \gamma\},$$

$$R_\gamma(\mu, \alpha, Q, M) = T_\gamma(\alpha, Q, M) - M\mu[0, \gamma). \tag{31}$$

For $Q = 0$ we use the symbols $T_\gamma(\alpha, M)$ and $R_\gamma(\mu, \alpha, M)$.

THEOREM 1. *Let the number α be defined by (21), (23), (24) and (27). Then α is Markov-normal and the following estimate is true:*

$$D(\mu, N) \ = \ O(N^{-1/2} \log^2 N), \tag{32}$$

where the O-constant depends only on P.
Proof. Using (29), (30) and (31), we obtain

$$R_\gamma(\mu, \alpha, N) = \sum_{r=1}^{k-1} R_\gamma(\mu, \alpha, n_r, A_{2r}) + R_\gamma(\mu, \alpha, n_k, M_1). \tag{33}$$

According to (21), (24) and (25) we have

$$R_\gamma(\mu, \alpha, n_r, M) = R_\gamma(\mu, \alpha_r, M), \qquad M < A_{2r} - 2r. \tag{34}$$

It follows from (31) that

$$R_\gamma(\mu, \alpha_r, M) = T_\gamma(\alpha_r, M) - M\mu[0, \gamma),$$

and

$$T_{\gamma_r}(\alpha_r, M) \le T_\gamma(\alpha_r, M) \le T_{\gamma_r + 1/q^r}(\alpha_r, M).$$

It is evident that

$$|R_\gamma(\mu,\alpha_r,M)| \le |T_{\gamma_r}(\alpha_r,M) - M\mu[0,\gamma)| + |T_{\gamma_r+1/q^r}(\alpha_r,M) - M\mu[0,\gamma)|.$$

We apply (31) and obtain

$$|R_\gamma(\mu,\alpha_r,M)| \le |R_{\gamma_r}(\mu,\alpha_r,M)| + |R_{\gamma_r+1/q^r}(\mu,\alpha_r,M)| + M(|\mu[0,\gamma) - \mu[0,\gamma_r)|$$

$$+|\mu[0,\gamma) - \mu[0,\gamma_r+1/q^r)|).$$

On the basis of (26)-(28), Lemma 1 and Lemma 2 we deduce that

$$R_\gamma(\mu,\alpha_r,M) = O(p^{-r}r^2).$$

According to (34) we have for $M < A_{2r} - r$

$$R_\gamma(\mu,\alpha,n_r,M) = O(p^{-r}r^2). \tag{35}$$

It follows from (31) that

$$R_\gamma(\mu,\alpha_r,M) = R_\gamma(\mu,\alpha_r,M-2r) + O(r),$$

It is evident from this that statement (35) is valid both for $M < A_{2r} - 2r$ as well as for $M \in [A_{2r} - 2r, A_{2r}]$.
Substituting (35) into (33) and bearing in mind (30) we deduce

$$R_\gamma(\mu,\alpha,N) = \sum_{r=1}^{k-1} O(p^{-r}r^2) + O(p^{-k}k^2) = O(p^{-k}k^2).$$

Using (29), (30) and (5) we obtain

$$R_\gamma(\mu,\alpha,N) = O(N^{1/2}\log^2 N).$$

Hence and from (1), (31) the statement of the theorem follows. ∎

We denote

$$\delta(a) = \begin{cases} 1, & \text{if } a = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

It is easy to see that

$$\delta(a) = \frac{1}{N}\sum_{m=1}^{N} e^{2\pi i \frac{ma}{N}}, \quad 0 \le a \le N-1. \tag{37}$$

Lemma 3. *Let* $1 \le M \le A_{2n}$ *and*

$$G_M = \sum_{x=1}^{M} g_x.$$

Then

$$|G_M| \leq \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1} \Big|\sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{mx}{A_{2n}}}\Big|. \tag{38}$$

Proof. According to (36) we have

$$G_M = \sum_{y=1}^{M} \sum_{x=1}^{A_{2n}} g_x \delta(x-y).$$

Using (37), we obtain

$$|G_M| = \Big|\sum_{m=0}^{A_{2n}-1} \frac{1}{A_{2n}} \sum_{y=1}^{M} \sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{m(x-y)}{A_{2n}}}\Big| \leq \tag{39}$$

$$\leq \sum_{m=0}^{A_{2n}-1} \frac{1}{A_{2n}} \Big|\sum_{y=1}^{M} e^{2\pi i \frac{-my}{A_{2n}}}\Big| \Big|\sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{mx}{A_{2n}}}\Big|.$$

Let $0 < N_2 - N_1 < A_{2n}$. It is known [5, p. 1] that

$$\frac{1}{A_{2n}} \Big|\sum_{y=N_1}^{N_2} e^{2\pi i \frac{-my}{A_{2n}}}\Big| \leq \min\Big(1, \frac{1}{A_{2n}|\sin \frac{\pi m}{A_{2n}}|}\Big) \leq \frac{1}{m+1}. \tag{40}$$

From (39) and (40) we give the assertion of the lemma. ■

LEMMA 4. *Let* $0 \leq u_1 \leq u_2 < A_{2n}$, $m \geq 0$ $i, j = 0, ..., q-1$, $n > k_1$. *Then*

$$S = \sum_{x=u_1}^{u_2} e^{2\pi i \frac{mx}{A_{2n}}} (p_{ij}^{(x)}(n)/p_j(n) - 1) = O(1),$$

where the constant in symbol O *depends only on* P.

Proof. Let $N_1 = [u_1/d]$, $N_2 = [u_2/d]$. We change the variable $x = dy + z$ and obtain according to (13)

$$S = \epsilon \frac{2d}{\overline{p}} + \sum_{y=N_1}^{N_2} \sum_{z=1}^{d} e^{2\pi i \frac{m(yd+z)}{A_{2n}}} (p_{ij}^{(dy+z)}(n)/p_j(n) - 1), \qquad where\ \ |\epsilon| < 1$$

Let

$$\sigma_y = \sum_{z=1}^{d} e^{2\pi i \frac{mz}{A_{2n}}} (p_{ij}^{(dy+z)}(n)/p_j(n) - 1).$$

It follows that

$$S = \epsilon \frac{2d}{\overline{p}} + \sum_{y=N_1}^{N_2} e^{2\pi i \frac{myd}{A_{2n}}} \sigma_y. \tag{41}$$

Applying (13), (10), we obtain

$$\sigma_y = de^{2\pi i \frac{mz_1}{A_{2n}}} + \epsilon_1 \frac{d}{p_j(n)}(1-\theta)^{-1+y/k_0} - \sum_{z=1}^{d} e^{2\pi i \frac{mz}{A_{2n}}},$$

where $|\epsilon_1| < 1$, $z_1 = f(j) - f(i)$.
Substituting this formula into (41), we obtain according to (13), that

$$S = S_1 S_2 + \epsilon_1 \sum_{y=N_1}^{N_2} \frac{d}{\overline{p}}(1-\theta)^{-1+y/k_0}, \qquad |\epsilon_1| \le 1, \tag{42}$$

where

$$S_1 = \sum_{y=N_1}^{N_2} e^{2\pi i \frac{myd}{A_{2n}}} \quad S_2 = \sum_{z=1}^{d}\left(e^{2\pi i \frac{mz_1}{A_{2n}}} - e^{2\pi i \frac{mz}{A_{2n}}}\right). \tag{43}$$

It is known that

$$|e^{2\pi i \frac{m(z_1-z)}{A_{2n}}} - 1| = 2|\sin \pi m(z_1 - z)/A_{2n})| \le 2\pi md/A_{2n}. \tag{44}$$

Using (40) we get

$$S_1 \le A_{2n}/(md+1).$$

Hence and from (42-44) the assertion of the lemma follows. ∎

We consider further that a_i, $i = 1, 2, \ldots$ is the sign of the number $\alpha_n(\omega_n)$.
It follows from (25), that

$$R_{[0,\gamma_n)}(\mu_n, \alpha_n, M) = \sum_{r=1}^{n} \sum_{b=0}^{c_r - 1} R_{[\gamma_{r-1}+b/q^r, \gamma_{r-1}+(b+1)/q^r)}(\mu_n, \alpha_n, M), \tag{45}$$

and
$R_{[\gamma_{r-1}+b/q^r, \gamma_{r-1}+(b+1)/q^r)}(\mu_n, \alpha_n, M) =$

$$\sum_{x=1}^{M} \delta(a_{x+1} - c_1) \ldots \delta(a_{x+r} - b) - M\mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r).$$

Hence and from (45) we get

$$R_{[0,\gamma_n)}(\mu_n, \alpha_n, M) = \tag{46}$$

$$= \sum_{r=1}^{n} \sum_{b=0}^{c_r-1} \sum_{x=1}^{M} (\delta(a_{x+1} - c_1) \ldots \delta(a_{x+r} - b) - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r)).$$

LEMMA 5. *Let $n > k_1$,*

$$B(r,c) = \tag{47}$$

$$= \sum_{x,y=1}^{A_{2n}} e^{2\pi i \frac{m(x-y)}{A_{2n}}} (\mu_n^2[\gamma_r, \gamma_r + \frac{1}{q^r}) + \sigma_1(x,y) - \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r})(\sigma_2(x) + \sigma_2(y))),$$

where

(48)

$$\sigma_1(x,y) = \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \delta(a_{x+1} - c_1)...\delta(a_{x+r} - c_r)\delta(a_{y+1} - c_1)...\delta(a_{y+r} - c_r),$$

$$\sigma_2(x) = \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \delta(a_{x+1} - c_1)...\delta(a_{x+r} - c_r). \tag{49}$$

Then

$$E_n \le \sum_{m=0}^{A_{2n}-1} \frac{(nq)^{1/2}}{m+1} (\sum_{r=1}^{n} \sum_{c_1=0}^{q-1} ... \sum_{c_r=0}^{q-1} B(r,c))^{1/2}. \tag{50}$$

Proof. It follows from (46) and Lemma 3 that

$$|R_{[0,\gamma_n)}(\mu_n, \alpha_n, M)| \le \sum_{r=1}^{n} \sum_{b=0}^{c_r-1} \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1} \Big| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}}$$

$$(\delta(a_{x+1} - c_1)...\delta(a_{x+r} - b) - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r)) \Big|.$$

Changing the order of summation and applying the Cauchy inequality

$$|\frac{1}{N} \sum_{n=1}^{N} g_n| \le (\frac{1}{N} \sum_{n=1}^{N} |g_n|^2)^{1/2}, \tag{51}$$

we obtain that

$$|R_{[0,\gamma_n)}(\mu_n, \alpha_n, M)| \le$$

$$\le \sum_{m=0}^{A_{2n}-1} \frac{(qn)^{1/2}}{m+1} \Big(\sum_{r=1}^{n} \sum_{b=0}^{c_r-1} | \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1} - c_1)...\delta(a_{x+r} - b) -$$

$$- \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r))|^2 \Big)^{1/2}.$$

We change the variable b to c_r and assume, on the right-hand side, the summation on c_i, $i = 1, ..., r-1$.

It is evident that

$$|R_{[0,\gamma_n)}(\mu_n, \alpha_n, M)| \le \sum_{m=0}^{A_{2n}-1} \frac{(qn)^{1/2}}{m+1} \Big(\sum_{r=1}^{n} \sum_{c_1=0}^{q-1} ... \sum_{c_r=0}^{q-1} \tag{52}$$

$$\Big| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1}-c_1)...\delta(a_{x+r}-c_r) - \mu_n[\gamma_r, \gamma_r + 1/q^r)) \Big|^2 \Big)^{1/2}.$$

We denote by $S(\omega_n)$ the right-hand side of formula (52).
It is evident that $S(\omega_n)$ does not depend on M and γ_n.
Applying (26), we obtain

$$E_n(\omega_n) \leq S(\omega_n)$$

and

$$E_n \leq \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} E_n(\omega_n) \leq \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} S(\omega_n).$$

Changing the order of summation and using (51), we obtain

$$E_n \leq \sum_{m=0}^{A_{2n}-1} \frac{(qn)^{1/2}}{m+1} \Big(\sum_{r=1}^{n} \sum_{c_1=0}^{q-1} ... \sum_{c_r=0}^{q-1} \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n}$$

$$\Big| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1}-c_1)...\delta(a_{x+r}-c_r) - \mu_n[\gamma_r, \gamma_r + 1/q^r)) \Big|^2 \Big)^{1/2}.$$

Hence and from (47)-(49) we deduce formula (50). ∎

Lemma 6. *Let $n > k_1$. Then*

$$\sigma_2(x) = \mu_n[\gamma_r, \gamma_r + 1/q^r).$$

Proof. Applying (49) and (22), we get

$$\sigma_2(x) = \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} ... \sum_{b_{x+r}=0}^{A_n-1} \delta(a_{x+1}-c_1)...\delta(a_{x+r}-c_r).$$

According (23), we obtain

(53) $\qquad a_{x+i} = c_i \quad$ if and only if $\quad b_{x+i} \in S_{c_{i-1}c_i} \;\; i = 2, 3, \dots .$

It follows that

$$\sigma_2(x) = \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} ... \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1}-c_1) \sum_{b_{x+2} \in S_{c_1 c_2}} ... \sum_{b_{x+r} \in S_{c_{r-1} c_r}} 1 =$$

$$= \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} ... \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1}-c_1) v_{c_1 c_2}(n) ... v_{c_{r-1} c_r}(n).$$

Using (7) we get

(54) $$\sigma_(x) = \sigma p_{c_1 c_2}(n) ... p_{c_{r-1} c_r}(n),$$

where

(55)

$$\sigma = \frac{1}{L_n A_n^{x+1}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{x+1}=0}^{A_n-1} \delta B_2 = \sum_{1\leq x,y\leq A_{2n},\ y-x>r} \sigma(x,y), \quad B_3 = \sum_{1\leq x,y\leq A_{2n},\ x-y>r} \sigma(x,y).$$

According to Lemma 8, (12) and (14) we obtain

$$|B_1| \leq \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}) \sum_{1\leq x,y\leq A_{2n},\ |y-x|\leq r} (\frac{1+p}{2})^{|y-x|-1} =$$

$$= O(A_{2n}\mu_n[\gamma_r, \gamma_r + \frac{1}{q^r})). \tag{56}$$

It follows from Lemma 7 that

$$B_2 = \mu_n^2[\gamma_r, \gamma_r + 1/q^r) \sum_{x=1}^{A_{2n}} \sum_{y=x+r}^{A_{2n}} e^{2\pi i \frac{m(x-y)}{A_{2n}}} (p_{c_r c_1}^{(y-x-r)}(n)/p_{c_1}(n) - 1).$$

Changing the variable y to $y_1 = y - x - r$ and applying Lemma 3, we obtain

$$B_2 = O(A_{2n}\mu_n^2[\gamma_r, \gamma_r + 1/q^r)).$$

Similarly estimate is valid for B_3.
Hence and from (61)-(62) we obtain the assertion of the lemma. ■

Proof of Lemma 2. Substituting (60) into (50) and bearing in mind (5), we deduce

$$E_n = O(\sum_{m=0}^{A_{2n}-1} \frac{(nq)^{1/2}}{m+1} (\sum_{r=1}^{n} \sum_{c_1=0}^{q-1} \dots \sum_{c_r=0}^{q-1} A_{2n}\mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}))^{1/2}) =$$

$$O(\sqrt{A}_{2n} n \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1}) = O(p^{-n} n^2).$$

Lemma 2 is proved. ■

Remark. By a similar method and the method in [12] a Markov normal vector for the multidimensional case can be be constructed. By the method in [12] one can reduce the logarithmic multiplier in (32) to $O(\log N^{3/2})$. To reduce the logarithmic multiplier forther see [15].

Problem. According to [12-14] the Borel and Bernoulli normal numbers exist with discrepancy $O(N^{-2/3+\epsilon})$. It would be interesting to know whether Markov normal numbers exist with discrepancy $O(N^{-c})$ where $c > 1/2$.

Acknowledgements. I am very grateful to Professor David Gilat and to Professor Meir Smorodinsky for their hospitality and support.

References

[1] A.G. Postnikov and I.I. Piatetski-Shapiro, *A Markov sequence of symbols and a normal continued fraction*, Izv. Akad. Nauk SSSR, Ser. Mat., 1957, v. 21, 501–514.

[2] M. Smorodinsky and B. Weiss, *Normal sequences for Markov shifts and intrinsically ergodic subshifts*, Israel Journal of Mathematics, 1987, v. 59, 225–233.

[3] A. Bertrand-Mathis, *Points generiques de Champernowne sur certains systems codes; application aux θ-shifts*, Ergod. Th. & Dynam. Sys., 1988, v. 8, 35–51.

[4] N.N. Chentsov, *Pseudorandom numbers for modeling Markov chains*, U.S.S.R. Comput. Maths. Math. Phis., 1967, vol. 7, no 3, 218–233.

[5] N.M. Korobov, *Exponential sums and their applications*, Dordrecht, 1992, 209 pages.

[6] U.N. Sahov, *Imitation of simplist Markov processes*, Izv. Akad. Nauk SSSR, Ser. Mathem., 1959, v. 23, 815–822.

[7] U.N. Sahov, *The construction of sequence of signs that is normal in the sen se of Markov*, Moskovskii Gosudarstvennyi Pedagogiceskii institute im. V.I. Lenina, Ucenye Zapiski, 1971, v. 375, 143–155.

[8] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, New York, 1965.

[9] J.L. Kemeny and J.L. Snell, *Finite Markov chains*, New York, 1960, 210 pages.

[10] I.S. Berezin and N.P.Zhidkov, Computing methods, vol. 2, Pergamon Press, Oxford, 1965, 267, 268.

[11] N.M. Korobov, *Distribution of fractional parts of exponential function*, Vestnic Moskov. Univ., Ser. 1 Mat. Meh., 1966, v. 21, no. 4, 42–46.

[12] M.B. Levin, *The distribution of fractional parts of the exponential function*, Soviet. Math. (Iz. Vuz.), 1977, v. 21, no. 11, 41–47.

[13] U.N. Sahov, *Some bounds in the construction of Bernoulli-normal sequences of signs*, Math. Notes, 1971, v. 10, 724–730.

[14] M.B. Levin, *On normal sequence for Markov and Bernoulli shifts*, 49–53, Proccedings of the Israel Mathematical Union Conference ,1994, Beer Sheva, 97–100.

[15] W. Philipp, *Limit theorems for lacunary series and uniform distribution mod 1*, Acta Arithm., 1975, v. 26, 241–251.

M.B. Levin
University of Tel-aviv
e-mail: mlevin@math.tau.ac.il