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# On the mean values of Dedekind Sums

par Wenpeng ZHANG

Résumé. On étudie dans ce papier le comportement asymtotique de valeurs moyennes de sommes de Dedekind. On donne en particulier une formule asymptotique améliorant un résultat antérieur.

Abstract. In this paper we study the asymptotic behavior of the mean value of Dedekind sums, and give a sharper asymptotic formula.

## 1. Introduction

For a positive integer $k$ and an arbitrary integer $h$, the Dedekind sum $S(h,k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right)$$

where

$$((x))=\begin{cases} x-[x]-\dfrac{1}{2} & \text{if } x \text{ is not an integer;}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various properties of $S(h,k)$ were investigated by many authors. Perhaps the most famous property of Dedekind sums is the reciprocity formula (see [3], [5], [6]):

$$S(h,k)+S(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4} \tag{1}$$

for all $(h,k)=1$, $h>0$, $k>0$. A three term version of (1) was discovered by Rademacher [7]. Walum [8] has shown that for prime $p\geq 3$,

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^4=\frac{\pi^4(p-1)}{p^2}\sum_{h=1}^{p}|S(h,p)|^2 \tag{2}$$

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and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2(p-1)^2(p-2)}{12p^2} \tag{3}$$

Recently, J.B. Conrey et al. [4] studied the mean value distribution of $S(h,k)$, and proved that

$$\sum_{h=1}^{k}{}' |S(h,k)|^{2m} = f_m(k)\left(\frac{k}{12}\right)^{2m} + O\left(\left(k^{\frac{9}{5}} + k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right), \tag{4}$$

where $\sum_{h}'$ denotes the summation over all $h$ such that $(k,h)=1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

In the spirit of [4] and [8], we use the estimates for character sum and the properties of Dirichlet's $L$-functions to give a sharper asymptotic formula for (4), where $m=1$ and $k=p^n$, a power of a prime. That is, we shall prove the following two theorems.

THEOREM 1. *Let $p$ be a prime and $n$ be a positive integer. Then for $k=p^n$, we have the asymptotic formula*

$$\sum_{h=1}^{k}{}' |S(h,k)|^2 = \frac{5}{144}k^2\frac{(p^2-1)^2}{p(p^3-1)} + O\left(k\exp\left(\frac{3\ln k}{\ln\ln k}\right)\right),$$

*where the constant implied by the $O$-symbol does notdepend on any parameter, and* $\exp(y)=e^y$.

THEOREM 2. *Let $n$ be an integer $\geq 1$, $p$ be a prime, and $k=p^n$. Then we have*

$$\sum_{h=1}^{k}{}' \frac{S(h,k)}{h} = \frac{\pi^2}{72}k\left(1-\frac{1}{p^2}\right) + O\left(\sqrt{k}\right).$$

Corollary. *Let $p$ be a prime. Then we have the asymptotic formulas*

1) $$\sum_{h=1}^{p} |S(h,p)|^2 = \frac{5}{144}p^2 + O\left(p\exp\left(\frac{3\ln p}{\ln\ln p}\right)\right).$$

2) $$\sum_{h=1}^{p} \frac{S(h,p)}{h} = \frac{\pi^2}{72}p + O\left(\sqrt{p}\right).$$

## 2. Some Lemmas

To prove the theorems, we need the following lemmas.

Lemma 1. *Let $q$ be an integer $\geq 3$, and let $\chi$ be any Dirichlet character modulo $q$ with $\chi(-1) = -1$. Then we have*

$$L(1,\chi) = \frac{\pi}{2q}\sum_{r=1}^{q}\chi(r)\cot\frac{\pi r}{q}$$

*where $L(s,\chi)$ denotes Dirichlet $L$-function corresponding to $\chi$ modulo $q$.*

*Proof.* Let $N \geq 3$ be an integer. Then we have

$$\sum_{1\leq n\leq Nq}\frac{\chi(n)}{n} = \sum_{a=1}^{q}\sum_{\ell=0}^{N-1}\frac{\chi(\ell q + a)}{\ell q + a}$$

$$(5) \qquad = \frac{1}{q}\sum_{a=1}^{q}\chi(a)\sum_{\ell=0}^{N-1}\frac{1}{\ell + a/q}.$$

Note that since $\chi(-1) = -1$, we also have

$$\sum_{1\leq n\leq Nq}\frac{\chi(n)}{n} = \frac{1}{q}\sum_{a=1}^{q}\chi(q-a)\sum_{\ell=0}^{N-1}\frac{1}{\ell + 1 - a/q}$$

$$(6) \qquad = \frac{-1}{q}\sum_{a=1}^{q}\chi(a)\sum_{\ell=1}^{N}\frac{1}{\ell - a/q}.$$

Thus from (5) and (6) we get

$$2\sum_{1\leq n\leq Nq}\frac{\chi(n)}{n} = \frac{1}{q}\sum_{a=1}^{q}\chi(a)\left(\sum_{\ell=0}^{N-1}\frac{1}{\ell + a/q} - \sum_{\ell=1}^{N}\frac{1}{\ell - a/q}\right)$$

$$= \frac{1}{q}\sum_{a=1}^{q}\chi(a)\left(\frac{1}{a/q} + \sum_{\ell=1}^{N-1}\left(\frac{1}{\ell + a/q} - \frac{1}{\ell - a/q}\right) - \frac{1}{N - a/q}\right)$$

$$= \frac{1}{q}\sum_{a=1}^{q}\chi(a)\left(\frac{1}{a/q} + \sum_{\ell=1}^{N-1}\left(\frac{1}{\ell + a/q} - \frac{1}{\ell - a/q}\right)\right) + O\left(\frac{1}{N-1}\right).$$

Taking $N \to \infty$ in the above, and noting that

$$\lim_{n\to\infty} \sum_{1\le n\le Nq} \frac{\chi(n)}{n} = L(1,\chi)$$

and

$$\frac{a}{q} + \sum_{\ell=1}^{\infty} \left( \frac{1}{\ell + a/q} - \frac{1}{\ell + a/q} \right) = \pi \cot \frac{\pi a}{q},$$

the lemma follows.

LEMMA 2. *Let $k$ be an integer $\ge 3$ and $(h,k)=1$. Then*

$$S(h,k) = \frac{1}{\pi^2 k} \sum_{d\mid k} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) \left| L(1,\chi) \right|^2,$$

*where $\phi(k)$ is the Euler function.*

*Proof.* Note that $\sum_{a=1}^{k} \chi(a) \cot \frac{\pi a}{k} = 0$ if $\chi(-1)=1$. For $(c,k)=1$, from Lemma 1 and the orthogonality of characters modulo $k$ we have the identity

$$\begin{aligned}
\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \overline{\chi}(c) L(1,\chi) &= \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \overline{\chi}(c) \left( \frac{\pi}{2k} \sum_{b=1}^{k} \chi(b) \cot \frac{\pi b}{k} \right) \\
&= \sum_{\chi \bmod k} \overline{\chi}(c) \left( \frac{\pi}{2k} \sum_{b=1}^{k} \chi(b) \cot \frac{\pi b}{k} \right) = \frac{\pi \phi(k)}{2k} \cot \frac{\pi c}{k}.
\end{aligned}$$

Hence

$$\cot \frac{\pi c}{k} = \frac{2k}{\pi \phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \overline{\chi}(c) L(1,\chi) \tag{7}$$

From [2] we know that

$$\left(\left( \frac{uh}{k} \right)\right) = -\frac{1}{2k} \sum_{a=1}^{k-1} \sin \frac{2\pi a h u}{k} \cot \frac{\pi a}{k}. \tag{8}$$

Thus

$$\left(\left(\frac{uh}{k}\right)\right) = -\frac{1}{2k}\sum_{a=1}^{k}\sin\frac{2\pi ahu}{k}\cot\frac{\pi a}{k} \tag{9}$$

$$= -\frac{1}{2k}\sum_{d|k}\sum_{a=1}^{k/d}\left(\frac{2k/d}{\pi\phi(k/d)}\sum_{\substack{\chi \bmod k/d\\ \chi(-1)=-1}}\overline{\chi}(a)L(1,\chi)\right)\frac{e\left(\frac{ahu}{k/d}\right)-e\left(\frac{-ahu}{k/d}\right)}{2i}$$

$$= \frac{i}{2\pi}\sum_{d|k}\frac{1}{d\phi(k/d)}\sum_{\substack{\chi \bmod k/d\\ \chi(-1)=-1}}L(1,\chi)\sum_{a=1}^{k/d}\overline{\chi}(a)\left(e\left(\frac{ahu}{k/d}\right)-e\left(\frac{-ahu}{k/d}\right)\right)$$

$$= \frac{i}{2\pi}\sum_{d|k}\frac{1}{d\phi(k/d)}\sum_{\substack{\chi \bmod k/d\\ \chi(-1)=-1}}\chi(h)L(1,\chi)\sum_{a=1}^{k/d}\overline{\chi}(a)\left(e\left(\frac{au}{k/d}\right)-e\left(\frac{-au}{k/d}\right)\right)$$

$$= \frac{i}{\pi}\sum_{d|k}\frac{1}{d\phi(k/d)}\sum_{\substack{\chi \bmod k/d\\ \chi(-1)=-1}}\chi(h)L(1,\chi)\sum_{a=1}^{k/d}\overline{\chi}(a)e\left(\frac{au}{k/d}\right),$$

where $e(y)=e^{2\pi iy}$, $i^2=-1$, $(h,k)=1$.

Similarly, we also have

$$\left(\left(\frac{u}{k}\right)\right) = \frac{i}{\pi}\sum_{d|k}\frac{1}{d\phi(k/d)}\sum_{\substack{\chi \bmod k/d\\ \chi(-1)=-1}}L(1,\chi)\sum_{a=1}^{k/d}\overline{\chi}(a)e\left(\frac{au}{k/d}\right). \tag{10}$$

Note that from the definitions of $S(h,k)$, and from (9) and (10) we get

$$\begin{aligned}
S(h,k) &= \sum_{c=1}^{k}\left(\left(\frac{c}{k}\right)\right)\left(\left(\frac{ch}{k}\right)\right)\\
&= \frac{-1}{\pi^2}\sum_{u\mid k}\sum_{v\mid k}\frac{1}{u\phi(k/u)}\frac{1}{v\phi(k/v)}\\
&\quad\times \sum_{\substack{\chi_1 \bmod k/u\\ \chi_1(-1)=-1}}\ \sum_{\substack{\chi_2 \bmod k/v\\ \chi_2(-1)=-1}}\chi_1(h)L(1,\chi_1)L(1,\chi_2)\\
&\quad\times \sum_{c=1}^{k}\left(\sum_{a=1}^{k/u}\overline{\chi}_1(a)e\left(\frac{ac}{k/u}\right)\right)\left(\sum_{b=1}^{k/v}\overline{\chi}_2(b)e\left(\frac{bc}{k/v}\right)\right)\\
&= \frac{k}{\pi^2}\sum_{d\mid k}\frac{1}{d^2\phi(k/d)}\sum_{\substack{\chi \bmod k/d\\ \chi(-1)=-1}}\chi(h)\left|L(1,\chi)\right|^2\\
&= \frac{1}{\pi^2 k}\sum_{d\mid k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)\left|L(1,\chi)\right|^2,
\end{aligned}$$

where we have used the identities

$$\begin{aligned}
&\sum_{a=1}^{k/u}\sum_{b=1}^{k/v}\overline{\chi}_1(a)\overline{\chi}_2(b)\sum_{c=1}^{k}e\left(\frac{(au+bv)c}{k}\right)\\
&= \begin{cases} k\displaystyle\sum_{a=1}^{k/u}\overline{\chi}_1(a)\overline{\chi}_2(k/u-a), & \text{if } u=v;\\ 0, & \text{if } u\neq v.\end{cases}\\
&= \begin{cases} -k\phi(k/u), & \text{if } \overline{\chi}_1=\chi_2 \text{ and } u=v;\\ 0, & \text{if } \overline{\chi}_1\neq\chi_2 \text{ or } u\neq v.\end{cases}
\end{aligned}$$

This completes the proof of Lemma 2.

Lemma 3. *Let $q$ be an integer $\geq 3$. Then we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^4 = \frac{5\pi^4}{144}\phi(q)\prod_{p\mid q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\frac{\phi(q)}{q}\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right).$$

*Proof.* For any parameter $N \geq q$, applying Abel's identity we have

$$
\begin{aligned}
L^2(1,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)d(n)}{n} \\
&= \sum_{1\leq n\leq N} \frac{\chi(n)d(n)}{n} + \int_N^{\infty} \frac{A(y,\chi)}{y^2}dy,
\end{aligned} \tag{11}
$$

where $d(n)$ is the divisor function. We have $A(y,\chi) = \sum_{N<n\leq y} \chi(n)d(n)$. Note that the identities

$$
\begin{aligned}
A(y,\chi) = 2\sum_{n\leq\sqrt{y}} \chi(n) \sum_{m\leq y/n} \chi(m) - 2\sum_{n\leq\sqrt{N}} \chi(n) \sum_{m\leq N/n} \chi(m) - \\
- \left(\sum_{n\leq\sqrt{y}} \chi(n)\right)^2 + \left(\sum_{n\leq\sqrt{N}} \chi(n)\right)^2.
\end{aligned}
$$

Applying the Cauchy inequality and estimates for character sums we have

$$
\begin{aligned}
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |A(y,\chi)|^2 &\ll \sqrt{y} \sum_{n\leq\sqrt{y}} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left|\sum_{m\leq y/n} \chi(m)\right|^2 + \\
&+ \sqrt{N} \sum_{n\leq\sqrt{N}} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left|\sum_{m\leq N/n} \chi(m)\right|^2 + \\
&+ y \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left|\sum_{n\leq\sqrt{y}} \chi(n)\right|^2 + N \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left|\sum_{n\leq\sqrt{N}} \chi(n)\right|^2 \\
&\ll y\phi^2(q).
\end{aligned} \tag{12}
$$

Thus from (12) we get

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left| \int_N^\infty \frac{A(y,\chi)}{y^2} dy \right|^2 \\
&\quad = \int_N^\infty \int_N^\infty \frac{1}{y^2 z^2} \left( \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |A(y,\chi)| \cdot |A(z,\chi)| \right) dy dz \\
&\quad \ll \left( \int_N^\infty \frac{1}{y^2} \left( \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |A(y,\chi)|^2 \right)^{\frac{1}{2}} dy \right)^2 \\
&\quad \ll \left( \int_N^\infty \frac{1}{y^{3/2}} \phi(q) dy \right)^2 \ll \frac{\phi^2(q)}{N}. \qquad (13)
\end{aligned}
$$

Note that for $(ab, q) = 1$, from the orthogonality relation for character sums modulo $q$ we have

$$
\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(a)\overline{\chi})b) = \begin{cases} \frac{1}{2}\phi(q), & \text{if } a \equiv b(q); \\ -\frac{1}{2}\phi(q), & \text{if } a \equiv -b(q); \\ 0, & \text{otherwise.} \end{cases}
$$

Hence

$$
\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left| \sum_{1 \le n \le N} \frac{\chi(n) d(n)}{n} \right|^2 = \qquad (14)
$$

$$= \frac{1}{2}\phi(q) \sum_{\substack{1\le a,b\le N\\(ab,q)=1\\a\equiv b(q)}} \frac{d(a)d(b)}{ab} - \frac{1}{2}\phi(q) \sum_{\substack{1\le a,b\le N\\(ab,q)=1\\a\equiv -b(q)}} \frac{d(a)d(b)}{ab}$$

$$= \frac{1}{2}\phi(q) \sum_{\substack{1\le a\le N\\(a,q)=1}} \frac{d^2(a)}{a^2} + O\left(\phi(q)\sum_{b=1}^{N}\sum_{\ell=1}^{[N/q]} \frac{d(b)d(\ell q+b)}{(\ell q+b)b}\right) +$$

$$+ O\left(\phi(q)\sum_{a=1}^{q-1} \frac{d(a)d(q-a)}{a(q-a)}\right) +$$

$$+ O\left(\phi(q) \sum_{1\le a\le N} \sum_{(1+a/q)\le \ell\le N/q} \frac{d(a)d(\ell q-a)}{a(\ell q-a)}\right)$$

$$= \frac{1}{2}\phi(q) \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{d^2(n)}{n^2} + O\left(\frac{\phi(q)}{q}\exp\left(\frac{\ln N}{\ln\ln N}\right)\right)$$

$$= \frac{1}{2}\phi(q)\frac{\zeta^4(2)}{\zeta(4)} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\frac{\phi(q)}{q}\exp\left(\frac{\ln N}{\ln\ln N}\right)\right)$$

$$= \frac{5}{144}\pi^4\phi(q) \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\frac{\phi(q)}{q}\exp\left(\frac{\ln N}{\ln\ln N}\right)\right),$$

where we have used the estimate $d(n) \ll \exp\left(\frac{(1+\epsilon)\ln 2\ln n}{\ln\ln n}\right)$, and $\zeta(s)$ is the Riemann zeta-function.

(15) $$\sum_{\substack{\chi \bmod q\\\chi(-1)=-1}} \left(\sum_{1\le n\le N} \frac{\chi(n)d(n)}{n}\right)\left(\int_N^\infty \frac{A(y,\overline{\chi})}{y^2}dy\right)$$

$$\ll (\ln N)^2 \int_N^\infty \frac{1}{y^2}\left(\sum_{\substack{\chi \bmod q\\\chi(-1)=-1}} |A(y,\overline{\chi})|\right) dy \ll \phi^{\frac{3}{2}}(q)(\ln N)^2 N^{-\frac{1}{2}}.$$

Taking the parameter $N = q^3$, from (11), (13), (14) and (15) we obtain

$$\sum_{\substack{\chi \bmod q\\\chi(-1)=-1}} |L(1,\chi)|^4 = \frac{5}{144}\pi^4\phi(q)\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(\frac{\phi(q)}{q}\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right).$$

This proves lemma 3.

## 3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove theorem 1. For $k = p^n$, applying Lemma 2 and Lemma 3 we have

$$\sum_{h=1}^{k}{}' |S(h,k)|^2 = \frac{1}{\pi^4 k^2} \sum_{h=1}^{k}{}' \left| \sum_{d|k} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h)|L(1,\chi)|^2 \right|^2$$

$$= \frac{1}{\pi^4 k^2} \sum_{u|k} \sum_{v|k} \frac{u^2}{\phi(u)} \frac{v^2}{\phi(v)}$$

$$\times \sum_{\substack{\chi_1 \bmod u \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod v \\ \chi_2(-1)=-1}} |L(1,\chi_1)|^2 |L(1,\chi_2)|^2 \sum_{h=1}^{k} \overline{\chi}_1(h)\chi_2(h)$$

$$= \frac{\phi(k)}{\pi^4 k^2} \left[ 2 \sum_{u|k} \sum_{v|u} \frac{u^2}{\phi(u)} \frac{v^2}{\phi(v)} \sum_{\substack{\chi \bmod v \\ \chi(-1)=-1}} |L(1,\chi)|^4 \right.$$

$$\left. - \sum_{u|k} \frac{u^4}{\phi^2(u)} \sum_{\substack{\chi \bmod u \\ \chi(-1)=-1}} |L(1,\chi)|^4 \right]$$

$$= \frac{\phi(k)}{\pi^4 k^2} \sum_{u|k} \sum_{v|u} \frac{u^2}{\phi(u)} \frac{v^2}{\phi(v)} \left[ \frac{10}{144} \pi^4 \phi(v) \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left( \frac{\phi(v)}{v} \exp\left( \frac{3\ln v}{\ln\ln v} \right) \right) \right]$$

$$- \frac{\phi(k)}{\pi^4 k^2} \sum_{u|k} \frac{u^4}{\phi^2(u)} \left[ \frac{5}{144} \pi^4 \phi(u) \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left( \frac{\phi(u)}{u} \exp\left( \frac{3\ln u}{\ln\ln u} \right) \right) \right]$$

$$= \frac{5}{144} \frac{\phi(k)}{k^2} \frac{(p^2-1)^3}{p^4(p^2+1)} \left[ 2 \sum_{u|k} \sum_{v|u} \frac{u^2 v^2}{\phi(u)} - \sum_{u|k} \frac{u^4}{\phi(u)} \right] + O\left( k \exp\left( \frac{3\ln k}{\ln\ln k} \right) \right)$$

$$= \frac{5}{144} k^2 \frac{(p^2-1)^2}{p(p^3-1)} + O\left( k \exp\left( \frac{3\ln k}{\ln\ln k} \right) \right).$$

This completes the proof of Theorem 1.

*Proof of Theorem 2.* For real $y \geq k = p^n$, let $U(y,\chi) = \sum_{k<n\leq y} \chi(n)$. Note

that

$$\sum_{h=1}^{k}{}' \frac{S(h,k)}{h} = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left(\sum_{h=1}^{k}\frac{\chi(h)}{h}\right)|L(1,\chi)|^2$$

(16)

$$= \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left(L^2(1,\chi)L(1,\overline{\chi}) - |L(1,\chi)|^2\int_k^{\infty}\frac{U(y,\chi)}{y^2}dy\right).$$

Using the method of proving Lemma 3 we get
(17)

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} L^2(1,\chi)L(1,\overline{\chi}) = \frac{\pi^4}{72}\phi(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2 + O\left(\exp\left(\frac{3\ln q}{\ln\ln q}\right)\right).$$

Note that the estimate

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |U(y,\chi)|^2 \ll \phi^2(q)$$

holds. Now, using the Cauchy inequality and Lemma 3 we get

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} |L(1,\chi)|^2\int_k^{\infty}\frac{U(y,\chi)}{y^2}dy \ll \frac{\phi^{\frac{3}{2}}(d)}{k}. \tag{18}$$

From (16), (17) and (18) we obtain

$$\begin{aligned}\sum_{h=1}^{k}{}' \frac{S(h,k)}{h} &= \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)}\left[\frac{\pi^4}{72}\phi(d)\left(1-\frac{1}{p^2}\right)^2 + O\left(\frac{\phi^{\frac{3}{2}}(d)}{d}\right)\right]\\ &= \frac{\pi^2}{72}k\left(1-\frac{1}{p^2}\right) + O\left(\sqrt{\phi(k)}\right).\end{aligned}$$

Remark. *In* [9] *and* [10], *we proved a generalized version of (3). i.e.*

THEOREM A. *Let $q$ be an integer $\geq 3$. Then we have*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\phi^2(q)}{q^2}\left[q\prod_{p|q}\left(1+\frac{1}{p}\right)-3\right],$$

*where the summation is over all odd Dirichlet characters modulo $q$.*

THEOREM B. *Let $q$ be a square-full number ( i.e. $q \geq 4$ and a prime $p|q$ implies $p^2|q$ ). Then we have*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right),$$

*where the summation is over all odd primitive Dirichlet characters modulo $q$.*

Using Lemma 2 we can give a simple proof of Theorem A and Theorem B. In fact, note that for $k \geq 2$, from the definition of $S(h,k)$ we have

$$S(1,k) = \sum_{a=1}^{k-1}\left(\frac{a}{k}-\frac{1}{2}\right)^2 = \frac{(k-1)(k-2)}{12k}$$

From this identity, Lemma 2 with $h=1$ and the Möbius inversion formula we get

$$\begin{aligned}
\frac{q^2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 &= \pi^2\sum_{d|q}\mu(d)\frac{q}{d}S\left(1,\frac{q}{d}\right) \\
&= \pi^2\sum_{d|q}\mu(d)\frac{\left(\frac{q}{d}-1\right)\left(\frac{q}{d}-2\right)}{12} = \frac{\pi^2}{12}\sum_{d|q}\mu(d)\left(\frac{q^2}{d^2}-3\frac{q}{d}+2\right) \\
&= \frac{\pi^2}{12}\left[q^2\sum_{d|q}\frac{\mu(d)}{d^2}-3q\sum_{d|q}\frac{\mu(d)}{d}+2\sum_{d|q}\mu(d)\right] \\
&= \frac{\pi^2}{12}\phi(q)\left[q\prod_{p|q}\left(1+\frac{1}{p}\right)-3\right].
\end{aligned}$$

This proves Theorem A.

If $q$ be a square-full number, then using the Möbius inversion formula and Theorem A we get

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |L(1,\chi)|^2 &= \sum_{d|q}\mu(d)\left(\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}} |L(1,\chi)|^2\right)\\ &= \sum_{d|q}\mu(d)\left\{\frac{\pi^2}{12}\frac{\phi^2(q/d)}{(q/d)^2}\left[\frac{q}{d}\prod_{p|q/d}\left(1+\frac{1}{p}\right)-3\right]\right\}\\ &= \frac{\pi^2}{12}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right).\end{aligned}$$

This completes the proof of Theorem B.

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Wenpeng ZHANG
Department of Mathematics
Northwest University
Xi'an, Shaanxi
P. R. China.