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Sums of squares in $\mathbb{Z}[\sqrt{k}]$

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Résumé. Nous étudions une généralisation du fameux problème du cercle aux anneaux d'entiers quadratiques réels : nous nous intéressons à $\mathcal{C}(N,M)=\sum_{n\le N}\sum_{m\le M} r(n+m\sqrt{k})$, le nombre de représentations de $n+m\sqrt{k}$ comme somme de deux carrés dans $\mathbb{Z}[\sqrt{k}]$ (où $k>1$ et sans facteur carré). En utilisant la théorie spectrale dans $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$, nous obtenons une formule asymptotique avec terme erreur pour $\mathcal{C}(N,M)$, démontrant que certaines techniques d'estimations de fonctions L automorphes fournissent précisément des majorations de ce terme erreur.

Abstract. We study a generalization of the classical circle problem to real quadratic rings. Namely we study $\mathcal{C}(N,M)=\sum_{n\le N}\sum_{m\le M} r(n+m\sqrt{k})$ where $r(n+m\sqrt{k})$ is the number of representations of $n+m\sqrt{k}$ as a sum of two squares in $\mathbb{Z}[\sqrt{k}]$ (with $k>1$ and squarefree). Using spectral theory in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$, we get an asymptotic formula with error term for $\mathcal{C}(N,M)$, showing that some techniques on the estimation of automorphic $L-$functions can be applied to get upper bounds of the error term.

1. Introduction.

The classical circle problem asks about the asymptotic behaviour of $\sum_{n\le N} r(n)$ where $r(n)$ is the number of representations as a sum of two squares. The problem comes back to Gauss [Ga] and it was conjectured by Hardy [Ha] that for every $\alpha>1/4$

$$\sum_{n\le N} r(n)=\pi N+O(N^{\alpha}). \tag{1.1}$$

Several methods of exponential sums have been applied along this century to extend the range of α in which (1.1) holds true. The best known result is due to M.N. Huxley [Hu] who has proved (1.1) for every $\alpha>23/73$.

A natural generalization of the circle problem to the ring $\mathbb{Z}[\sqrt{k}]$ with k squarefree, $k>1$, leads to consider

$$\mathcal{C}(N)=\sum_{n,m\le N} r(n+m\sqrt{k})$$

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where $r(n+m\sqrt{k})$ is the number of representations of $n+m\sqrt{k}$ as a sum of two squares in $\mathbb{Z}[\sqrt{k}]$. Curiously, with this formulation circle problem in $\mathbb{Z}[\sqrt{k}]$ reduces to an ellipsoid problem that can be completely solved. Other formulations due to W. Schall [Sc] and U. Rausch [Ra] lead to difficult and interesting problems about the distribution of the conjugates of an algebraic number. In this paper we study

$$\mathcal{C}(N,M)=\sum_{n\le N}\sum_{m\le M} r(n+m\sqrt{k})$$

i.e. we substitute the square $1\le n,m\le N$ in the definition of $\mathcal{C}(N)$ by the rectangle $1\le n\le N$, $1\le m\le M$.

To get an optimal asymptotic formula for thin enough rectangles (M small) appears as a difficult problem involving harmonic analysis in the hyperbolic plane. In fact, in the limiting case $M=2$, the problem can be interpreted as a hyperbolic circle problem which has strong resemblances with the classical circle problem (see Ch. 12 of [Iw2], [Ph-Ru] and [Ch1]), and in some ranges the estimation of the error term in the asymptotic formula for $\mathcal{C}(N,M)$ is related to the estimation of some automorphic $L-$functions.

In [Ch1] it was stated as an example that the asymptotic of

$$\sum_{n\le N} r(n+m\sqrt{k})$$

can be decided using the spectral theory of automorphic forms (the needed results are summarized in section **2** below). In this paper we study the uniformity in m to get an asymptotic formula with error term for $\mathcal{C}(N,M)$. Using well known results in lattice point theory (when M is large) and average results for Hecke eigenvalues (when M is small), we obtain in section **3** bounds for the error term. Finally, in section **4** (that we consider the main part of this paper) we show that there is a relation between the estimation of automorphic L-functions and bounding the error term in our problem. Combining a method due to W. Luo (see [Lu]) with other results, we obtain several bounds improving in some ranges the previously obtained.

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2. Notation and review on some known results.

The main purpose of this section is to expand $\mathcal{C}(N,M)$ in terms of non-holomorphic cusp forms (see Lemma 2.3 below). The process was, essentially, described in [Ch1] but to ease references we prefer to give here a specific statement of the results that we need with a very brief comment about the underlying ideas.

We shall follow the notation of [Iw2] and we also refer the reader to that monography for any basic result on spectral theory used in this paper. Nevertheless we find convenient to refresh briefly at least the most basic concepts.

We shall denote by $u_j(z)$ Maass cusp forms for $\mathrm{PSL}_2(\mathbb{Z})$; i.e. automorphic functions vanishing at infinity which are normalized eigenfunctions of Laplace-Beltrami operator. The eigenvalues will be denoted by $1/4 + t_j^2$, taking (by notational convenience in Lemma 2.3) t_j with a different sign for $u_j(z)$ and $\overline{u_j(z)}$.

We shall assume that the $u_j(z)$ are Hecke eigenforms, i.e. they satisfy $T_m u_j(z) = \lambda_j(m) u_j(z)$ where T_m is the Hecke operator

$$T_m f(z) = \frac{1}{\sqrt{m}} \sum_{ad=m} \sum_{b=1}^{d} f\left(\frac{az+b}{d}\right).$$

On $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ there is only one Eisenstein series, $E_\infty(z,s)$, and $E_\infty(\cdot, 1/2 + it)$ is an eigenfunction of T_m whose eigenvalue is $\eta_t(m) = \sum_{ab=m}(a/b)^{it}$.

The hyperbolic circle problem will play an important role in the subsequent arguments. As the classical lattice point problem consists of counting the images of a point under integral translations, the hyperbolic circle problem deals with

$$\#\big\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) \;:\; \rho(\gamma z, w) \le R\big\}$$

where ρ is the Poincaré distance given by

$$\rho(z,w) = \operatorname{arc\,cosh}\left(1 + 2u(z,w)\right) \quad \text{with } u(z,w) = \frac{|z-w|^2}{4\,\mathrm{Im}\, z\,\mathrm{Im}\, w}.$$

By simplicity it is convenient to write $\operatorname{arc\,cosh}(X/2)$ instead of R, defining for $X \ge 2$ and $z, w \in \mathbb{H}$

$$H(X; z, w) = \#\big\{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) \;:\; 4u(\gamma z, w) + 2 \le X\big\}.$$

Let us define

$$c(N,m) = \sum_{n \le N} r(n + m\sqrt{k}).$$

We shall relate $c(N,m)$ to the hyperbolic circle problem thanks to the following elementary result whose proof reduces to expand $(a + d\sqrt{k})^2 + (c - b\sqrt{k})^2$

LEMMA 2.1. *If m is odd, $r(n+m\sqrt{k})=0$ and if m is even, $r(n+m\sqrt{k})$ is the number of integral solutions of a,b,c,d of*

$$\begin{cases} a^2+kb^2+c^2+kd^2=n \\ ad-bc=m/2. \end{cases}$$

A calculation proves

$$\sqrt{k}\big(4u(\gamma i/\sqrt{k}, i)+2\big)=a^2+kb^2+c^2+kd^2$$

where a,b,c,d are the entries of the matrix $\gamma\in \mathrm{SL}_2(\mathbb{Z})$ (see Lemma 3.8 of [Ch1] for a generalization of this fact). On the other hand, the Hecke operator T_m can be used, roughly speaking, to pass from matrices of determinant one to matrices of determinant m, namely

$$\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \;:\; ad-bc=m\right\}=\bigcup_{ad=m}\bigcup_{1\le b\le d}\mathrm{SL}_2(\mathbb{Z})\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}.$$

With this idea in mind, one can prove (see [Iw2] or [Ch1])

LEMMA 2.2. *For $N,m\ge 1$*

$$c(N,2m)=2\sqrt{m}\,T_m\big|_{z=i/\sqrt{k}}H(N/m\sqrt{k};z,i).$$

$H(X;z,w)$ is automorphic as a function of z or w, so it can be analysed with the spectral theory of $L^2(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H})$. The lack of regularity of H requires some smoothing before applying the pretrace formula. These steps were worked out in Lemma 2.3 of [Ch1] (see also Proposition 2.3 of [Ch2]). With the arguments used there, it is deduced

LEMMA 2.3. *Given $0<\Delta<1$, $z,w\in\mathbb{H}$ and $X>3$, it holds*

$$T_mH(X;z,w)=\frac{3\sigma(m)}{\sqrt{m}}X+\mathcal{E}_\Delta(X;z,w)+O\big(\frac{\sigma(m)}{\sqrt{m}}X\Delta\big)$$

where σ denotes the sum of positive divisor and

$$\mathcal{E}_\Delta(X;z,w)=\sum_j\lambda_j(m)g(t_j)u_j(z)\overline{u_j(w)}+ \\ +\frac{1}{4\pi}\int_{-\infty}^{\infty}\eta_t(m)g(t)E_\infty(z,1/2+it)\overline{E_\infty(w,1/2+it)}dt$$

with

$$g(t) = \frac{\sinh \Delta \sinh S}{\sinh^2(\Delta/2)} P^{-1}_{-1/2+it}(\cosh\ \Delta) P^{-1}_{-1/2+it}(\cosh\ S)$$

where $P^{-1}_{-1/2+it}$ *is an associated Legendre function (see* [Gr-Ry]) *and* $S = \operatorname{arc\,cosh}(X/2) + \Delta_0$ *for some* Δ_0, $-\Delta \le \Delta_0 \le \Delta$.

Remarks: i) Actually, the condition $X > 3$ can be replaced by $X > c$ where c is any constant greater than 2, but the $O-$constant in the error term is unbounded when $c \to 2$. The same situation occurs with the factor 3 in the statement of Lemma 2.4 and in the results based on this lemma.

ii) Note that it holds (use 8.723.1 of [Gr-Ry] and Lemma 2.4 (c) of [Ch1])

$$g(t) \ll |t|^{-3/2} X^{1/2} \min\big(1, (\Delta|t|)^{-3/2}\big)$$

for $t \in \mathbb{R}$ far away form zero, and $g(0) \ll X^{1/2} \log X$ (see 8.713.2 of [Gr-Ry]).

The Eisenstein series $E_\infty(i/\sqrt{k}, 1/2 + it)$ and $E_\infty(i, 1/2 + it)$ are essentially Epstein zeta-functions, namely

$$E_\infty(i,s) = \frac{1}{\zeta(2s)} \sum \frac{r(n)}{n^s} \qquad \text{and} \qquad E_\infty(i/\sqrt{k}, s) = \frac{k^{s/2}}{\zeta(2s)} \sum \frac{r_k(n)}{n^s}$$

where $r_k(n) = \#\big\{(a,b) \in \mathbb{Z}^2 \ : \ a^2 + kb^2 = n\big\}$. Using standard arguments involving an approximate functional equation of the Epstein zeta-functions (compare with [Go] or with Theorem 4.2 of [Iv], they can be approximated in $s = 1/2 + it$ by a Dirichlet polynomial of length less than $|t|^{1+\epsilon}$ and coefficients $a_n \ll n^\epsilon$, then

$$\int_T^{2T} |E_\infty(i/\sqrt{k}, 1/2 + it)|^2 dt, \ \int_T^{2T} |E_\infty(i, 1/2 + it)|^2 dt \ll T^{1+\epsilon}.$$

Hence the contribution of the continuous spectrum in $\mathcal{E}_\Delta(X; i/\sqrt{k}, i)$ is $O(m^\epsilon X^{1/2+\epsilon})$, and it follows at once from Lemma 2.2 and Lemma 2.3

Lemma 2.4. *Given* $0 < \Delta < 1$ *and* $N > 3M\sqrt{k}/2$, *for every* $\epsilon > 0$

$$\mathcal{C}(N,M) = \frac{\pi^2 s(M)}{2\sqrt{k}} NM + O\big(N^\epsilon \sup_T E_\Delta(N,T)\big) + O\big(NM^{1+\epsilon}\Delta + N^{1/2+\epsilon} M\big)$$

where

$$s(M)=\frac{12}{\pi^2 M}\sum_{m\le M/2}\frac{\sigma(m)}{m}$$

and

$$E_\Delta(N,T)=\Big|\sum_{T<|t_j|\le 2T} u_j(i/\sqrt{k})\overline{u_j(i)}\sum_{m\le M/2}\lambda_j(m)g(t_j)\sqrt{m}\Big|$$

with g *as in* Lemma 2.3 *taking* $X=N/m\sqrt{k}$.

Remarks: i) Note that $s(M)\to 1$ when $M\to\infty$, indeed $s(M)=1+O\big(M^{-1+\epsilon}\big)$.

ii) It is important to note that although Δ_0 in Lemma 2.3 depends on X and Δ, once N and Δ are fixed the same Δ_0 can be chosen for every $g(t_j)$ appearing in the summation over m in $E_\Delta(N,T)$. The reason of this fact is that Δ_0 is obtained by the mean value theorem using the monotonicity of $f(t)=H(t+N/m\sqrt{k};i/\sqrt{k},i)$ and the function $F(t)=\sum_{m\le M/2}f(t)$ is also monotonic.

iii) In the applications, Δ^{-1} will be bounded by a power of N and consequently calculating the supremum of $E_\Delta(N,T)$ it is enough to consider T less than a certain power of N because otherwise $E_\Delta(N,T)$ is absorbed by the other error terms. We shall implicitly use this fact in the proofs.

3. The asymptotics of $\mathcal{C}(N,M)$.

As we mentioned in the introduction, when $N=M$ it is possible to get a sharp estimate of the error term as a consequence of a similar result for rational ellipsoids. This is the content of the following result

PROPOSITION 3.1. *If* $N\le M\sqrt{k}$ *then*

$$\mathcal{C}(N,M)=\frac{\pi^2}{4k}N^2+O(N^\alpha)$$

holds for $\alpha>1$ *and does not hold for any* $\alpha\le 1$.

Remark: Note that this implies that

$$\mathcal{C}(N)=\frac{\pi^2}{4k}N^2+O(N^{1+\epsilon})$$

is the best possible up to sharpenning in the N^ϵ factor.

Proof. By Lemma 2.1, $\mathcal{C}(N, M)$ is one half of the integral solutions of

$$\begin{cases} a^2 + kb^2 + c^2 + kd^2 \leq N \\ 0 \neq |ad - bc| \leq M/2. \end{cases} \tag{3.1}$$

but the inequalities $a^2+kb^2 \geq 2|ab|\sqrt{k}$, $c^2+kd^2 \geq 2|cd|\sqrt{k}$ and our hypothesis $N \leq M\sqrt{k}$ implies that the second inequality in (3.1) is superfluous. Moreover the values of a, b, c, d with $ad - bc = 0$ contribute $O(N^{1+\epsilon})$, so, up to a term of that order, $\mathcal{C}(N, M)$ is one half of the number of lattice points in the ellipsoid

$$a^2 + kb^2 + c^2 + kd^2 \leq N$$

and when $\alpha \neq 1$ the proposition follows from the classical results of Landau [La] (see also §21 in [Fr]).

To deal with the case $\alpha = 1$, it is enough to prove

$$\limsup_{N\to\infty} \frac{1}{N} \#\{a, b, c, d \ : \ a^2 + kb^2 + c^2 + kd^2 = N, \ ad - bc \neq 0\} = \infty.$$

By the work of Kloosterman [Kl], for infinitely many values of N, with suitably chosen prime factors, the equation $a^2 + kb^2 + c^2 + kd^2 = N$ has more than $cN\prod_{p|N}(1 + 1/p) \gg N \log\log N$ solutions. If $ad - bc = 0$ then $a = d_1 a'$, $c = d_1 c'$, $b = d_2 a'$, $d = d_2 c'$ with $\gcd(a', c') = 1$ and

$$N = a^2 + kb^2 + c^2 + kd^2 = (d_1^2 + kd_2^2)(a'^2 + c'^2),$$

hence all of the solutions excepting at most $O(N^\epsilon)$ satisfy $ad - bc \neq 0$. $\square$

Now we shall state an asymptotic formula for $\mathcal{C}(N, M)$ when M is small in comparison with N. Estimating the error term we shall employ sharp average bounds for $|\lambda_j(m)|$ and $|u_j(z)|$; but, as we shall see in the next section, in some ranges it is possible to proceed in a better way studying the cancellation induced by the oscillation of $\lambda_j(m)g(t_j)$ when m varies.

Although the bound

$$c(N, 2m) = \frac{6\sigma(m)}{m} N + O\big(N^{2/3+\epsilon}\big)$$

(uniformly for $m < N/3\sqrt{k}$) is only known conjecturally under Ramanujan-Petersson conjecture $|\lambda_j(m)| = O\big(m^\epsilon\big)$, it is possible to prove the following result (the same situation occurs, in a different context, in Theorem 4.3 of [Ch 2] studying linear forms on $\sum_n r(n)r(n + m)$)

Theorem 3.2. *If $N \geq 3M\sqrt{k}/2$, then for any $\epsilon > 0$*

$$\mathcal{C}(N,M) = \frac{\pi^2 s(M)}{2\sqrt{k}} NM + O\big(MN^{2/3+\epsilon}\big)$$

Proof. By Lemma 2.4 it is enough to prove for some choice of Δ and every $T > 1$

$$E_\Delta(N,T) + NM^{1+\epsilon}\Delta \ll N^{2/3+\epsilon}M. \tag{3.2}$$

As we remarked in Lemma 2.3

$$g(t) \ll N^{1/2}m^{-1/2}|t|^{-3/2}\min(1,(\Delta|t|)^{-3/2}).$$

On the other hand Theorem 8.3 of [Iw2] assures

$$\sum_{m\leq M/2} |\lambda_j(m)|^2 \ll T^\epsilon M.$$

Hence, by Cauchy's inequality, for $T < |t_j| \leq 2T$

$$\sum_{m\leq M/2} g(t_j)\lambda_j(m)\sqrt{m} \ll T^{-3/2+\epsilon}N^{1/2}M\min\big(1,(\Delta T)^{-3/2}\big). \tag{3.3}$$

Using Proposition 7.2 of [Iw2] we deduce

$$\sum_{|t_j|<T} |u_j(i/\sqrt{k})||u_j(i)| \ll T^2,$$

which combined with (3.3) implies

$$E_\Delta(N,T) \ll (NT)^{1/2+\epsilon}M\min\big(1,(\Delta T)^{-3/2}\big).$$

Finally, choosing $\Delta = N^{-1/3}$, (3.2) is proved. □

4. $\mathcal{C}(N,M)$ and automorphic L-functions.

The growth of the Riemann zeta-function in the critical strip is closely related to the study of the sums $\sum_{m\leq M} m^{-it}$. For instance, an upper bound of order $|t|^\epsilon M^{1/2+\epsilon}$ for $1 < M \ll |t|^{1/2}$ implies Lindelöf hypothesis. In the same way, the sums

$$\sum_{m\leq M} \lambda_j(m)m^{-it}$$

are related to analytic properties of the automorphic (Hecke) L−function

$$H(s) = \sum_{m=1}^{\infty} \lambda_j(m) m^{-s}.$$

The estimation of these L−functions (beyond convexity) is more difficult than the one of the Riemann zeta-function because there is not an analogue of van der Corput's method on exponential sums after twisting with $\lambda_j(m)$. H. Iwaniec and other authors have developed an amplification technique that allows to surpass convexity arguments getting individual bounds from average results (see [Iw1]).

After Lemma 2.2, Lemma 2.3 and the asymptotic expansion of $P^{-1}_{-1/2+it}$, summation over m leads in some ranges to partial sums of $H(1/2+it_j)$, let us name them as

$$\mathcal{H}_j(M) = \sum_{m\le M} \lambda_j(m) m^{-it_j}.$$

In this way it is found a relation between the error term in the asymptotic formula for $\mathcal{C}(N,M)$ and the estimation of these automorphic L−functions.

Lindelöf hypothesis in s and in spectral aspect is in this context

$$H(s) \ll |s|^{\epsilon} |t_j|^{\epsilon} \qquad \text{for } \operatorname{Re} s = 1/2,$$

which suggests

Conjecture 4.1. *For $1 \le M \le |t_j|$ and any $\epsilon > 0$*

$$\mathcal{H}_j(M) \ll M^{1/2+\epsilon} |t_j|^{\epsilon}.$$

The previous conjecture, as Lindelöf conjecture, is out of reach by current methods, but there are some average results supporting it that will be useful in our arguments, namely, from the work of W. Luo [Lu] it is deduced (apply Theorem 1 after expanding $\mathcal{H}_j^q(M)$ using the multiplicativity of $\lambda_j(m)$)

(4.1)
$$\sum_{|t_j|\le T} |\mathcal{H}_j(M)|^{2q} \ll \big(M^q T^2 + M^{3q/2} T^{3/2} + M^{9q/4}\big)(MT)^{\epsilon} \qquad \text{for } q \in \mathbb{Z}^+.$$

Note that this gives Conjecture 4.1 in average whenever $M^q T^2$ is the leading term of the right hand side.

In the spectral analysis of $\mathcal{C}(N,M)$ the partial sums $\mathcal{H}_j(M)$ appear multiplied by the factor $u_j(i/\sqrt{k})\overline{u_j(i)}$, then the average result (4.1) can be only used if we take control of the cusp forms evaluated in these special values. The inequality $|u_j(z)| \ll |t_j|^{\epsilon}$ is conjectured (see (0.8) in [Iw-Sa], in fact it can be understood as a stronger form of Lindelöf hypothesis) but for our purposes it will be enough the following weaker bound

CONJECTURE 4.2. *For $T>1$ and every $\epsilon>0$*

$$\sum_{T<|t_j|\le 2T}|u_j(z)|^4\ll T^{2+\epsilon}.$$

Remark: Recently N. Pitt has proved (personal communication) a Hardy-Voronoi type summation formula implying the previous conjecture for infinitely many special values of z but not including $z=i/\sqrt{k}$ nor $z=i$.

On the other hand, the best known $L^\infty-$bound for $|u_j(z)|$ is

$$|u_j(z)|\ll|t_j|^{5/12+\epsilon}$$

due to H. Iwaniec and P. Sarnak (see [Iw-Sa]), which implies (by Proposition 7.2 of [Iw2])

$$\sum_{T<|t_j|\le 2T}|u_j(z)|^4\ll T^{17/6+\epsilon},\qquad \sum_{T<|t_j|\le 2T}|u_j(z)|^{8/3}\ll T^{41/18+\epsilon}. \tag{4.2}$$

In order to relate $E_\Delta(N,T)$ and $\mathcal{H}_j(M)$, it will be convenient to use the following result

LEMMA 4.3. *Let g as in* Lemma 2.4 *and $T<|t_j|\le 2T$, then*

$$\sum_{m\le M/2}g(t_j)\lambda_j(m)\sqrt{m}=Re\sum_{m\le M/2}G(m,t_j)\lambda_j(m)m^{1/2-it_j}$$
$$+O\big(T^{-1/2+\epsilon}N^{-3/2}M^3\min\big(1,(\Delta T)^{-3/2}\big)\big)$$

where G is decreasing in m and

$$G(m,t_j)\ll T^{-3/2}N^{1/2}m^{-1/2}\min\big(1,(\Delta T)^{-3/2}\big).$$

Proof. By 8.723.1 of [Gr-Ry]

$$P^{-1}_{-1/2+it}(\cosh S)=\frac{2}{\sqrt{2\pi\sinh S}}\,\mathrm{Re}\,\frac{\Gamma(it)}{\Gamma(it+3/2)}\big(e^{itS}+O(T^{-1}N^{-2}M^2)\big). \tag{4.3}$$

It is plain that (recall that $S=\operatorname{arc\,cosh}(X/2)+\Delta_0$ and $X=N/m\sqrt{k}$)

$$e^{itS}=e^{it\Delta_0}(N/\sqrt{k})^{it}m^{-it}\big(1+O(TN^{-2}M^2)\big),$$

which substituted in (4.3) and in the definition of $g(t_j)$, gives

$$\sum_{m\le M/2} g(t_j)\lambda_j(m)\sqrt{m} = \operatorname{Re}\sum_{m\le M/2} G(m,t_j)\lambda_j(m)m^{1/2-it_j} + O(E)$$

with $G(m,t_j)$ verifying the required properties and

$$E = \sum_{m\le M/2} T^{-3/2}N^{1/2}m^{-1/2}\min\big(1,(\Delta T)^{-3/2}\big)\cdot TM^2N^{-2}\cdot|\lambda_j(m)|\cdot M^{1/2}.$$

Finally, Theorem 8.3 of [Iw2] proves the desired bound for E. $\square$

Although the different level of difficulty and depth of Conjecture 4.1 and Conjecture 4.2 (the former is an out of reach Lindelöf hypothesis and there are partial results toward a proof of the latter) both of them imply almost the same result in our problem.

THEOREM 4.4. *For $N > 3M\sqrt{k}/2$ and any $\epsilon > 0$ we have under* Conjecture 4.1

$$\mathcal{C}(N,M) = \frac{\pi^2 s(M)}{2\sqrt{k}}NM + O\big((N^{2/3}M^{2/3} + N^{1/2}M + N^{-1}M^{7/2})N^{\epsilon}\big).$$

and under Conjecture 4.2

$$\mathcal{C}(N,M) = \frac{\pi^2 s(M)}{2\sqrt{k}}NM + O\big((N^{2/3}M^{2/3} + N^{1/2}M^{17/16} + N^{-1}M^{7/2})N^{\epsilon}\big).$$

Proof. By Lemma 2.4 it is enough to prove for some $0 < \Delta < 1$

$$E_\Delta(N,T) + NM\Delta \ll (N^{2/3}M^{2/3} + N^{-1}M^{7/2})N^{\epsilon} \tag{4.4}$$

and

$$E_\Delta(N,T) + NM\Delta \ll (N^{2/3}M^{2/3} + N^{1/2}M^{17/16} + N^{-1}M^{7/2})N^{\epsilon}. \tag{4.5}$$

Assume firstly Conjecture 4.1, then Lemma 4.3 (after partial summation) implies

$$\sum_{m\le M/2} g(t_j)\lambda_j(m)\sqrt{m} \ll$$

$$\big(T^{-3/2}N^{1/2}M^{1/2} + T^{-1/2}N^{-3/2}M^3\big)T^{\epsilon}\min\big(1,(\Delta T)^{-3/2}\big),$$

and by Proposition 7.2 of [Iw2]

$$E_\Delta(N,T) \ll \big(T^{1/2}N^{1/2}M^{1/2} + T^{3/2}N^{-3/2}M^3\big)T^\epsilon \min\big(1,(\Delta T)^{-3/2}\big).$$

Choosing $\Delta = N^{-1/3}M^{-1/3}$, (4.4) is proved.

Now, let us assume Conjecture 4.2, then by Cauchy's inequality

$$|E_\Delta(N,T)|^2 \ll$$

$$\big(T^{-1}N \sum_{T<|t_j|\le 2T} |\mathcal{H}_j(M')|^2 + T^3N^{-3}M^6\big)(TN)^\epsilon \min\big(1,(\Delta T)^{-3}\big)$$

for some $M' \le M/2$.

By (4.1) with $q=1$

$$|E_\Delta(N,T)|^2 \ll$$

$$\ll \big(TNM + T^{1/2}NM^{3/2} + T^{-1}NM^{9/4} + T^3N^{-3}M^6\big)(TN)^\epsilon \min\big(1,(\Delta T)^{-3}\big).$$

Taking $\Delta = N^{-1/3}M^{-1/3}$ we get for $T > M^{1/8}$

$$|E_\Delta(N,T)|^2 \ll \big(N^{4/3}M^{4/3} + N^{7/6}M^{5/3} + NM^{17/8} + N^{-2}M^7\big)N^\epsilon,$$

and the second term in the parenthesis is absorbed by the other obtaining (4.5). On the other hand, if $T \le M^{1/8}$, we have as in the proof of Theorem 3.2

$$E_\Delta(N,T) \ll (NT)^{1/2+\epsilon}M \min\big(1,(\Delta T)^{-3/2}\big) \ll N^{1/2}M^{17/16}$$

which gives a better bound. $\square$

THEOREM 4.5. *For $N > 3M\sqrt{k}/2$ and any $\epsilon > 0$ we have unconditionally*

$$\mathcal{C}(N,M) = \frac{\pi^2 s(M)}{2\sqrt{k}}NM + O(N^\epsilon L)$$

where $L = \min\big(L_1, L_2\big)$ with

$$L_1 = N^{31/46}M^{85/92} + N^{1/2}M^{9/8} + N^{-51/46}M^{78/23}$$

and

$$L_2 = N^{29/41}M^{29/41} + N^{55/82}M^{151/164} + N^{23/41}M^{389/328}.$$

Remark: When M is very small in comparison with N this is not the best possible result obtainable from (4.1); in fact, taking q large enough it is possible to deduce $L \ll (NM)^\theta$ for $1 \le M < N^{f(\theta)}$ with θ as close as we wish to $2/3$ (and $f(\theta) \to 0$ as $\theta \to 2/3$), but we do not consider worthy to complicate the statement of the previous theorem covering all the cases because in these short ranges Theorem 3.2 gives similar bounds. For instance, when $M = N^{0.1}$, Theorem 4.5 could be slighty improved to $L \ll N^{0.759\ldots}$ using (4.1) with $q = 7$, but Theorem 3.2 gives in this range $L \ll N^{0.766\ldots}$.

Proof. By Lemma 4.3 and Proposition 7.2 of [Iw2]
(4.6)
$$E_\Delta(N,T) \ll \sum_{T<|t_j|\le 2T} |u_j(i/\sqrt{k})||u_j(i)|\Big|\sum_{m\le M/2} G(m,t_j)\lambda_j(m)m^{1/2-it_j}\Big| + E$$

with
$$E = T^{-3/2+\epsilon}N^{-3/2}M^3 \min\big(1,(\Delta T)^{-3}\big).$$

Using the first bound of (4.2), Cauchy's inequality and partial summation give

$$|E_\Delta(N,T)|^2 \ll \Big(T^{-1/6}N \sum_{T<|t_j|\le 2T} |\mathcal{H}_j(M')|^2 + T^3N^{-3}M^6\Big)\times \\ \times (TN)^\epsilon \min\big(1,(\Delta T)^{-3}\big)$$

for some $M' \le M/2$, and so it is bounded by (4.1) with $q = 1$ getting

$$|E_\Delta(N,T)|^2 \ll \big(T^{11/6}NM + T^{4/3}NM^{3/2} + T^{-1/6}NM^{9/4} + T^3N^{-3}M^6\big)\times \\ \times (TN)^\epsilon \min\big(1,(\Delta T)^{-3}\big).$$

Choosing $\Delta = N^{-6/23}M^{-6/23}$ we have

$$|E_\Delta(N,T)|^2 \ll \\ \ll \big(N^{34/23}M^{34/23} + N^{31/23}M^{85/46} + NM^{9/4} + N^{-51/23}M^{156/23}\big)N^\epsilon.$$

Therefore by Lemma 2.4, it can be taken

$$(4.7)\quad L \ll N^{17/23}M^{17/23} + N^{31/46}M^{85/92} + N^{1/2}M^{9/8} + N^{-51/46}M^{78/23}.$$

On the other hand, applying Hölder inequality in (4.6) with exponents $p = 4/3$, $p' = 4$, and using (4.2) to bound $\sum |u_j(i/\sqrt{k})u_j(i)|^{4/3}$, we get

$$|E_\Delta(N,T)|^4 \ll \Big(T^{5/6}N^2 \sum_{T<|t_j|\le 2T} |\mathcal{H}_j(M'')|^4 + T^6N^{-2}M^8\Big)\times \\ \times (TN)^\epsilon \min\big(1,(\Delta T)^{-6}\big)$$

for some $M'' \le M/2$.

Using (4.1) with $q = 2$

$$|E_\Delta(N,T)|^4 \ll \big(T^{17/6}N^2M^2 + T^{7/3}N^2M^3 + T^{5/6}N^2M^{9/2} + T^6N^{-6}M^{12}\big)\times \\ \times (TN)^\epsilon \min\big(1,(\Delta T)^{-6}\big)$$

and choosing $\Delta = N^{-12/41}M^{-12/41}$ we get

$$|E_\Delta(N,T)|^4 \ll \big(N^{116/41}M^{116/41}+N^{110/41}M^{151/41}+N^{92/41}M^{389/82}+N^{-174/41}M^{564/41}\big)N^\epsilon,$$

which implies by Lemma 2.4

$$L \ll N^{29/41}M^{29/41} + N^{55/82}M^{151/164} + N^{23/41}M^{389/328} + N^{-87/82}M^{141/41}. \tag{4.8}$$

Some calculations prove that when the first term in (4.7) is the main term, the bound (4.8) is better. The same situation occurs with the last term in (4.8), hence both of them can be omitted in the final result. □

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