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On the Piatetski-Shapiro-Vinogradov Theorem

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Résumé. Dans cet article, nous considérons la formule asymptotique pour le nombre de représentations d'un entier impair N sous la forme $p_1 + p_2 + p_3 = N$, où les p_i sont des nombres premiers du type $p_i = [n^{1/\gamma_i}]$; nous utilisons la méthode de van der Corput en dimension deux et nous étendons le domaine de validité de la formule asymptotique en affaiblissant les hypothèses sur les γ_i. Dans le cas le plus intéressant $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, notre résultat entraîne que tout entier impair assez grand s'écrit comme la somme de trois nombres premiers de Piatetski–Shapiro du type γ pour $50/53 < \gamma < 1$.

Abstract. In this paper we consider the asymptotic formula for the number of the solutions of the equation $p_1 + p_2 + p_3 = N$ where N is an odd integer and the unknowns p_i are prime numbers of the form $p_i = [n^{1/\gamma_i}]$. We use the two-dimensional van der Corput's method to prove it under less restrictive conditions than before. In the most interesting case $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ our theorem implies that every sufficiently large odd integer N may be written as the sum of three Piatetski–Shapiro primes of type γ for $50/53 < \gamma < 1$.

1. Introduction.

In 1937 I. M. Vinogradov [15] solved the Goldbach ternary problem. He proved that for a sufficiently large odd integer N,

$$\sum_{p_1+p_2+p_3=N} 1 = \frac{1}{2}(1+o(1))\mathfrak{S}(N)\frac{N^2}{\log^3 N}\ , \tag{1}$$

where $\mathfrak{S}(N)$ is the singular series

$$\mathfrak{S}(N) = \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right)\ . \tag{2}$$

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In 1986 E. Wirsing [16] considered the question of the existence of thin sets of primes S such that every sufficiently large odd integer is the sum of three elements of S (the set of prime numbers S is called to be thin if $\sum\limits_{p\le x, p\in S} 1 = o(\pi(x))$). Wirsing proved that there exists such a set of primes S with the property that

$$\sum_{p\le x, p\in S} 1 \ll (x\log x)^{1/3}\ .$$

The set of the Piatetski-Shapiro primes of type $\gamma < 1$

$$P_\gamma = \{p \mid p = [n^{1/\gamma}] \quad \text{for some } n\in\mathbb{N}\}$$

is a well-known thin set of prime numbers. The counting function $\pi_\gamma(x)$ of P_γ was studied by a number of authors [1, 4–6, 8–14]. The best results are given by [14] and [10] where it is proved that

$$\pi_\gamma(x) \sim \frac{x^\gamma}{\log x}$$

for $5302/6121 < \gamma < 1$, and

$$\pi_\gamma(x) \gg \frac{x^\gamma}{\log x}$$

for $38/45 < \gamma < 1$.

In 1992 A. Balog and J. P. Friedlander [2] considered the ternary Goldbach problem with variables restricted to Piatetski-Shapiro primes. They proved that if γ_1, γ_2, γ_3 are fixed real numbers such that $\gamma_i \le 1$ and γ_i is close to 1, and N is a sufficiently large odd integer, then the asymptotic formula (3) below is valid.

There are two interesting special cases of this theorem. If $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ in (3) we obtain an asymptotic formula for the number of representations of a large odd integer as the sum of three Piatetski-Shapiro primes of type γ. If $\gamma_1 = \gamma_2 = 1$ in (3), we obtain an asymptotic formula for the number of representations of a large odd integer as the sum of two primes and a Piatetski-Shapiro prime.

The theorem of Balog and Friedlander implies that in the first case the asymptotic formula is valid for $20/21 < \gamma \le 1$, and in the second— for $8/9 < \gamma \le 1$. J. Rivat [14] extended the range $20/21 < \gamma \le 1$ to

$188/199 < \gamma \leq 1$, and C.-H. Jia [7] used a sieve method to show that there exists a positive constant $\rho_0 = \rho_0(\gamma)$ such that

$$T_1(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_i \in P_\gamma}} 1 \geq \rho_0 \frac{\mathfrak{S}(N) N^{3\gamma-1}}{\log^3 N}$$

for $15/16 < \gamma \leq 1$.

In this paper we use better estimates of an exponential sum to prove:

Theorem 1. *Let γ_1, γ_2, γ_3 be fixed real numbers such that $0 < \gamma_i \leq 1$ and*

$$\begin{gathered} 73(1-\gamma_3) < 9 \\ 73(1-\gamma_2) + 43(1-\gamma_3) < 9 \\ 73(1-\gamma_1) + 43(1-\gamma_2) + 43(1-\gamma_3) < 9 \,. \end{gathered}$$

Denote by $T(N)$ the number of the representations of the integer N as the sum of three primes p_1, p_2, p_3 such that $p_i \in P_{\gamma_i}$. Then the asymptotic formula

$$T(N) = (1+o(1)) \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\Gamma(\gamma_3+1)}{\Gamma(\gamma_1+\gamma_2+\gamma_3)} \cdot \frac{\mathfrak{S}(N) N^{\gamma_1+\gamma_2+\gamma_3-1}}{\log^3 N} \tag{3}$$

holds. Here $\mathfrak{S}(N)$ is defined by (2).

In the special cases mentioned above this theorem gives:

Corollary 1. *For any fixed $50/53 < \gamma \leq 1$ every sufficiently large odd integer may be written as the sum of three Piatetski-Shapiro prime numbers of type γ.*

Corollary 2. *For any fixed $64/73 < \gamma \leq 1$ every sufficiently large odd integer may be written as the sum of two primes and a Piatetski-Shapiro prime number of type γ.*

In both cases we may obtain an asymptotic formula for the number of solutions. Thus Theorem 1 improves the known results of this type contained in [2] and [14]. Note also that $64/73 = 0.8767\ldots$ is not much greater than $5302/6121 = 0.8661\ldots$.

2. Notation.

In this paper p, $p_1, \ldots$ are primes; ε is an arbitrary small positive number, not necessary the same in the different appearances. The constants c_1, $c_2, \ldots$ in Section 4 depend at most on γ. We use $[x]$, $\{x\}$ and $\|x\|$ to denote the integral part of x, the fractional part of x and the distance from x to the nearest integer correspondingly. $\Lambda(n)$ is von Mangoldt's function; $e(x) = \exp(2\pi i x)$; $\psi(x) = x - [x] - \frac{1}{2}$.

$f(x) \ll g(x)$ means that $f(x) = \mathcal{O}(g(x))$;

$f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$;

$f(x_1, \ldots, x_n)\tilde{\Delta} g(x_1, \ldots, x_n)$ means that

$$\frac{\partial^{j_1+\cdots+j_n}}{\partial x_1^{j_1} \cdots x_n^{j_n}} f(x_1, \ldots, x_n) = \frac{\partial^{j_1+\cdots+j_n}}{\partial x_1^{j_1} \cdots x_n^{j_n}} g(x_1, \ldots, x_n)(1 + \mathcal{O}(\Delta))$$

for all n-tuples $(j_1, \ldots, j_n)$ for which it makes sence.

$x \sim X$ means that x runs through a *subinterval* of $(X, 2X]$, which endpoints are not necessary the same in the different equations and may depend on the outer summation variables; thus we may write, for example,

$$\sum_{\substack{m\sim M, n\sim N \\ mn\sim x}} \sum a(m,n) = \sum_{m\sim M, n\sim N} \sum a(m,n) \quad .$$

We also define

$$F_i = F_i(\alpha) = \sum_{p\le N} e(\alpha p)([-p^{\gamma_i}] - [-(p+1)^{\gamma_i}]) \, ,$$

$$G_i = G_i(\alpha) = \sum_{p\le N} \gamma_i p^{\gamma_i - 1} e(\alpha p) \, ,$$

$$R(N) = \sum_{p_1+p_2+p_3=N} \gamma_1\gamma_2\gamma_3 \; p_1^{\gamma_1-1} p_2^{\gamma_2-1} p_3^{\gamma_3-1} \, .$$

3. Preliminaries.

The idea of the proof of Theorem 1 is to reduce it to the following weighted version of the Vinogradov estimate (1):

$$R(N) = (1+o(1)) \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\Gamma(\gamma_3+1)}{\Gamma(\gamma_1+\gamma_2+\gamma_3)} \cdot \frac{\mathfrak{S}(N) N^{\gamma_1+\gamma_2+\gamma_3-1}}{\log^3 N} \, . \quad (4)$$

The reduction depends on the asymptotic formula for an exponential sum (Theorem 2 below) and indeed was done by Balog and Friedlander in Section 2 of [2]. They showed that

$$T(N) = R(N) + \mathcal{O}(N^{\gamma_1+\gamma_2+\gamma_3-1-\varepsilon}) , \tag{5}$$

provided that for $1 \le i \le 3$ the estimate

$$\sup_{\alpha\in(0,1)} |F_i - G_i| \ll N^{\gamma_i-\delta_i-\varepsilon}$$

holds with $\delta_1 = \frac{1}{2}(1-\gamma_2) + \frac{1}{2}(1-\gamma_3)$, $\delta_2 = \frac{1}{2}(1-\gamma_3)$, and $\delta_3 = 0$, correspondingly. Thus Theorem 1 follows from (4), (5) and the following improved version of Theorem 4 of [2].

Theorem 2. *Assume that $0 < \gamma < 1$, $0 \le \delta \le 1-\gamma$ and*

$$73(1-\gamma) + 86\delta < 9 .$$

Then, uniformly in $\alpha \in (0,1)$, we have

$$\sum_{p\le N} e(\alpha p)([-p^\gamma] - [-(p+1)^\gamma]) = \sum_{p\le N} \gamma\, p^{\gamma-1} e(\alpha p) + \mathcal{O}(N^{\gamma-\delta-\varepsilon}) , \tag{6}$$

where the implied constant depends at most on γ, δ and ε.

Throughout the rest of this section we reduce the proof of Theorem 2 to the estimation of a double exponential sum. Denoting the sum in the left-hand side of (6) by $F(\alpha)$, and that in the right-hand side by $G(\alpha)$ we get

$$F(\alpha) = G(\alpha) + \sum_{p\le N} e(\alpha p)\big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) + \mathcal{O}(\log N) .$$

It is easy to derive (6) from this equality and the estimate

$$\sum_{p\le x} e(\alpha p)(\log p)\big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) \ll x^{\gamma-\delta-\varepsilon} , \tag{7}$$

provided that the last is proved for each $1 \le x \le N$. Since

$$\begin{aligned}
&\sum_{p\le x} e(\alpha p)(\log p)\big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) \\
&= \sum_{n\le x} \Lambda(n) e(\alpha n)\big(\psi(-(n+1)^\gamma) - \psi(-n^\gamma)\big) + \mathcal{O}(x^{1/2}) ,
\end{aligned}$$

using the simplest splitting up argument, we obtain that to prove (7) it is sufficient to obtain

$$\sum_{n\sim x}\Lambda(n)e(\alpha n)\big(\psi(-(n+1)^\gamma)-\psi(-n^\gamma)\big)\ll x^{\gamma-\delta-\varepsilon} \tag{8}$$

for any $1\le x\le N$.

From here on x is a fixed sufficiently large number subjected to $x\le N$. We recall the well-known expansions

$$\psi(t)=-\sum_{0<|h|\le H_0}\frac{e(ht)}{2\pi ih}+\mathcal{O}\left(\min\left(1,\frac{1}{H_0\|t\|}\right)\right)\ ,$$

$$\min\left(1,\frac{1}{H_0\|t\|}\right)=\sum_{k=-\infty}^{\infty}b_k e(kt)\ ,$$

where

$$|b_k|\ll\min\left(\frac{\log H_0}{H_0},\frac{1}{|k|},\frac{H_0}{|k|^2}\right)\ .$$

We put $H_0=x^{1-\gamma+\delta+\varepsilon}$ and apply the first expansion for the left-hand side of (8). Similarly to [4, p.246] and [2, p.51] we treat the error terms via the second and the estimate of van der Corput [3, Theorem 2.2]. The obtained estimate is admissible if $2(1-\gamma)+3\delta<1$, so it remains to prove that for each $H\le H_0$

$$\sum_{h\sim H}\frac{1}{h}\left|\sum_{n\sim x}\Lambda(n)e(\alpha n)\big(e(hn^\gamma)-e(h(n+1)^\gamma)\big)\right|\ll x^{\gamma-\delta-\varepsilon}\ .$$

Working similarly to [4, p.247] we find that for $H\le x^{1-\gamma}$ to establish the last inequality it is sufficient to prove

$$\sum_{h\sim H}\left|\sum_{n\sim x}\Lambda(n)e(\alpha n+hn^\gamma)\right|\ll x^{1-\delta-\varepsilon}\ .$$

Otherwise we treat the sums involving $e(hn^\gamma)$ and $e(h(n+1)^\gamma)$ separately. Thus in all the cases, it suffices to prove the following

PROPOSITION. *Assume that* $0<\gamma<1$, $0\le\delta\le 1-\gamma$ *and*

$$73(1-\gamma)+86\delta<9\ .$$

Assume further that $H \le x^{1-\gamma+\delta+\varepsilon}$ *and* u *is either* 0, *or* 1. *Then*

$$\min\left(1, \frac{x^{1-\gamma}}{H}\right) \sum_{h\sim H} \left| \sum_{n\sim x} \Lambda(n) e(\alpha n + h(n+u)^\gamma) \right| \ll x^{1-\delta-\varepsilon} . \tag{9}$$

We prove this proposition in Section 5, and therefore complete the proofs of both Theorem 1 and Theorem 2. The proof depends on the estimation of some triple exponential sums, which we estimate in the next section.

4. Exponential sums estimates.

In this section we consider sums of the form

$$\Phi(H) \sum_{h\sim H} \left| \sum_{\substack{m\sim M, n\sim N \\ mn\sim x}} \sum a(m)\, b(n)\, e(\alpha mn + h(mn+u)^\gamma) \right|$$

where

$$\Phi(H) = \min\left(1, \frac{x^{1-\gamma}}{H}\right) .$$

If the coefficients $a(m)$ and $b(n)$ satisfy the conditions

$$|a(m)| \le 1 \quad , \quad b(n) = 1 \text{ or } b(n) = \log n \tag{10}$$

we denote the sum by S_I, and if they satisfy the conditions

$$|a(m)| \le 1 \quad , \quad |b(n)| \le 1 \tag{11}$$

we denote it by S_{II}.

For S_{II} we use the estimate obtained in [2, Proposition 2]:

Lemma 1. *Let* N *satisfy the conditions*

$$x^{1-\gamma+2\delta+\varepsilon} \le N \le x^{5\gamma-4-6\delta-\varepsilon} .$$

Then

$$S_{II} \ll x^{1-\delta-\varepsilon} .$$

For S_I we give a new estimate contained in the following lemma which we prove using van der Corput's method as it is described in [3].

LEMMA 2. *Let γ, δ be subjected to*

$$16(1-\gamma)+19\delta<2\,, \tag{12}$$

and N satisfy the condition

$$N \ge \max(x^{36(1-\gamma)+42\delta-4+\varepsilon},\, x^{2(1-\gamma)+4\delta+\varepsilon})\,. \tag{13}$$

Then

$$S_I \ll x^{1-\delta-\varepsilon}\,.$$

Proof of Lemma 2. Since for γ, δ, N such that

$$6(1-\gamma)+(19/3)\delta<1 \quad \text{and} \quad N \ge x^{4(1-\gamma)+5\delta}\,,$$

the lemma is proved in [2, Proposition 3], it is sufficient to consider the case

$$N \le x^{4(1-\gamma)+5\delta}\,. \tag{14}$$

In the rest of the proof we suppose that the inequality (14) holds. We also note that the condition $mn \sim x$ implies $MN \asymp x$.

We begin with the remark that since

$$\begin{aligned} \alpha mn + h(mn+u)^\gamma &= \alpha mn + h(mn)^\gamma + \gamma uh(mn)^{\gamma-1} + \mathcal{O}(HX^{\gamma-2}) \\ &= f_1(m,n) + \mathcal{O}(Hx^{\gamma-2})\,, \end{aligned}$$

one has

$$S_I \ll \Phi(H) \sum_{h\sim H} \left| \sum_{m\sim M} \sum_{n\sim N} a(m)\, e(f_1(m,n)) \right| + x^{1-\delta-\varepsilon}$$

provided that $1-\gamma+3\delta<1$. Now we apply the Cauchy–Schwarz inequality and Weyl–van der Corput lemma [3, Lemma 2.5] to the sum over n and get

$$S_I \ll \Phi(H) \left(\frac{Hx}{Q^{\frac12}} + \sum_{h\sim H} \sqrt{\frac{x}{Q} \sum_{q\le Q} \sum_{m\sim M} \sum_{n\sim N} e(f_2(m,n))} \right) + x^{1-\delta-\varepsilon}, \tag{15}$$

where $f_2(m,n) = f_1(m,n+q) - f_1(m,n)$ and $Q \le N$ is a parameter at our disposal. We choose

$$Q = [x^{2(1-\gamma)+2\delta+\varepsilon}] + 1\,, \tag{16}$$

which makes the first term in (15) admissible. Applying partial summation to the sums over n and m successively we find that if $N \geq x^{2(1-\gamma)+3\delta+\varepsilon}$ (which is not restrictive in view of (13)), then

$$\sum_{m\sim M,n\sim N}\sum e(f_2(m,n)) \ll \left|\sum_{m\sim M,n\sim N}\sum e(\alpha qm + hm^\gamma((n+q)^\gamma - n^\gamma))\right| . \quad (17)$$

We write $k = [\alpha q]$ and $\theta = \{\alpha q\}$ (note that k and θ do not depend on m and n, and $0 \leq \theta < 1$) and we derive from (17) that

$$\sum_{m\sim M,n\sim N}\sum e(f_2(m,n)) \ll \left|\sum_{m\sim M,n\sim N}\sum e(f_3(m,n))\right|$$

where $f_3(m,n) = \theta m + hm^\gamma((n+q)^\gamma - n^\gamma)$. Hence

$$S_I \ll \Phi(H)\sum_{h\sim H}\left(\frac{x}{Q}\sum_{q\leq Q}\left|\sum_{m\sim M}\sum_{n\sim N} e(f_3(m,n))\right|\right)^{1/2} + x^{1-\delta-\varepsilon} . \quad (18)$$

Now we apply the Poisson summation formula [3, Lemma 3.6] over m. We remove the arising smooth weights via partial summation and get

$$\begin{aligned}&\sum_{m\sim M}\sum_{n\sim N} e(f_3(m,n))\\ \ll &MF^{-1/2}\left|\sum_{n\sim N}\sum_{m} e(f_4(m,n))\right| + xF^{-1/2} + N\log x\end{aligned}$$

where

$$F = qx^\gamma HN^{-1}$$

and

$$f_4(m,n) = c_1 h^{1/(1-\gamma)}((n+q)^\gamma - n^\gamma)^{1/(1-\gamma)}(m-\theta)^{\gamma/(\gamma-1)}$$

and m runs through an interval $[M_1, M_1']$ which endpoints are monomials of n such that $M_1, M_1' \asymp FM^{-1}$. We substitute this estimate in (18) and use (12), (14) to get

$$S_I \ll \Phi(H)\sum_{h\sim H}\sqrt{\frac{xM}{Q}\sum_{q\leq Q}F^{-\frac{1}{2}}\left|\sum_{n\sim N}\sum_{m} e(f_4(m,n))\right|} + x^{1-\delta-\varepsilon}. \quad (19)$$

Note that if $FM^{-1} \ll x^{\varepsilon}$, i.e. the summation over m in the last sum is too short, then the trivial estimate of the sums over m and n in (19) is sufficient to obtain the desired result. Thus we consider further the case $FM^{-1} \gg x^{\varepsilon}$. We will estimate the sum over m, n in two different ways.

Our first tool is van der Corput's estimate as it is stated in [3, Theorem 2.9]. We change the order of summation and apply it with $q = 1$ to the sum over n (this is equivalent to the use of the exponent pair $(\frac{1}{6}, \frac{2}{3})$). Summing up the obtained estimate we find that

$$S_I \ll x^{1-\delta-\varepsilon}$$

for

$$x^{20(1-\gamma)+24\delta-2+\varepsilon} \leq N \leq x^{4(1-\gamma)+5\delta} .$$

Thus, we may suppose further that

$$N \leq x^{20(1-\gamma)+24\delta-2+\varepsilon} . \tag{20}$$

Note that when (12) holds this bound is always smaller than (14).

We can also estimate the sum in the left-hand side of (19) differently. First we apply the Weyl–van der Corput inequality over n, introducing in this way a new parameter $Q_1 \leq N$. We get

$$\begin{aligned} &\left|\sum_m \sum_{n\sim N} e(f_4(m,n))\right| \\ &\ll \frac{NF}{MQ_1^{1/2}} + \left(\frac{NF}{MQ_1} \sum_{q_1 \leq Q_1} \left|\sum_m \sum_{n \sim N} e(f_5(m,n))\right|\right)^{1/2} \end{aligned} \tag{21}$$

where $f_5(m,n) = f_4(m,n+q_1) - f_4(m,n)\tilde{\Delta}_1 c_2 q_1 (qh)^{1/(1-\gamma)}(m-\theta)^{\gamma/(1-\gamma)} n^{-2} \asymp q_1 F N^{-1}$, $\Delta_1 = (q+q_1)/N$. We choose

$$Q_1 = [x^{6(1-\gamma)+7\delta-1+\varepsilon} N] + 1 \tag{22}$$

which makes the contribution of the first term sufficiently small.

Then we use the Poisson summation formula over n. Let $[N_1(m), N_2(m)]$ be the interval through which runs n in (21) and let $[N_3(m), N_4(m)]$ be the interval through which runs $\frac{\partial}{\partial y} f_5(m,y)$ when n runs through $[N_1(m), N_2(m)]$ (then $N_3(m), N_4(m) \asymp q_1 F N^{-2}$). Denoting by y_n the unique solution of the equation

$$\frac{\partial}{\partial y} f_5(m, y_n) = n \qquad (n \in [N_3(m), N_4(m)])$$

we obtain from [3, Lemma 3.6] and partial summation that

$$\begin{aligned}
&\left|\sum_m \sum_{N_1(m)<n\le N_2(m)} e(f_5(m,n))\right| \\
&\ll \left|\sum_m \sum_{N_3(m)\le n\le N_4(m)} \left|\frac{\partial^2}{\partial y^2} f_5(m,y_n)\right|^{-1/2} \cdot e(f_6(m,n))\right| \\
&\qquad + \frac{F}{M}\left(\frac{N^{3/2}}{(q_1F)^{1/2}} + \log x\right) \\
&\ll \frac{N^{3/2}}{(q_1F)^{1/2}} \left|\sum_m \sum_n e(f_6(m,n))\right| + \frac{F}{M}\left(\frac{N^{3/2}}{(q_1F)^{1/2}} + \log x\right) \qquad (23)
\end{aligned}$$

where

$$f_6(m,n) = f_5(m,y_n) - ny_n \; .$$

If $q_1FN^{-2} \ll x^\varepsilon$ then the first term in the right-hand side of (23) may be omitted at the cost of a x^ε factor and the lemma follows from (12), (13), (16), (19)–(23). We consider further the case $q_1FN^{-2} \gg x^\varepsilon$. Then the argument of the proof of Lemma 3.9 of [3] shows that

$$f_6(m,n)\tilde{\Delta}_1 c_3 q_1^{1/3}(hq)^{1/3(1-\gamma)}(m-\theta)^{\gamma/3(1-\gamma)} n^{2/3} \asymp q_1F/N \; . \qquad (24)$$

Let us denote the sum over m,n in (23) by T. We apply to it Weyl-van der Corput inequality over m and obtain

$$T \ll \frac{q_1F^2}{xNQ_2^{1/2}} + \left(\frac{q_1F^2}{xNQ_2} \sum_{q_2\le Q_2} \left|\sum_{(m,n)} e(f_7(m,n))\right|\right)^{1/2} \qquad (25)$$

where $Q_2 \le FM^{-1}$ is a parameter, (m,n) runs through a subdomain of the domain of summation in (23) and

$$f_7(m,n) = f_6(m+q_2,n) - f_6(m,n) \; . \qquad (26)$$

We choose

$$Q_2 = [x^{20(1-\gamma)+24\delta-2+\varepsilon} N^{-1}] + 1 \; , \qquad (27)$$

and make the contribution of the first term in the right-hand side of (25) admissible. From (24) and (26) we derive that

$$f_7(m,n)\; \tilde{\Delta}_2 \; c_4 q_2 q_1^{1/3}(hq)^{1/3(1-\gamma)}(m-\theta)^{\gamma/3(1-\gamma)-1} n^{2/3} \asymp q_1 q_2 \frac{M}{N}$$

where $\Delta_2 = \Delta_1 + q_2 M F^{-1}$. Now we estimate the sum over m, n via [3, Lemma 6.11] with $(x, y) = (m, n)$ and we get

$$x^{-\varepsilon} \sum_{(m,n)} e(f_7(m,n)) \ll q_1 q_2 \frac{M}{N} + \frac{q_1^{1/2} F^2}{q_2^{1/2} x^{3/2}} .$$

Combining the last inequality with (12), (13), (16), (19)–(23), (25) and (27), we complete the proof of the lemma.

5. Proof of the Proposition.

The inner sum in the left-hand side of (13) is an exponential sum over primes. It is well-known that the sum

$$\sum_{n \sim x} \Lambda(n)\, F(n)$$

may be decomposed into double sums of two types—Type I and Type II sums. Both Type I and Type II sums are sums of the form

$$\sum_{\substack{m \sim M, n \sim N \\ mn \sim x}} \sum a(m)\, b(n)\, F(mn) .$$

We call the sum Type I if the coefficients $a(m)$ and $b(n)$ satisfy the conditions (10), and Type II if they satisfy the conditions (11).

We make the decomposition using an identity due Heath-Brown [4, Lemma 3]

LEMMA 3. *Let $3 < U < V < Z < x$ and suppose that $Z - 1/2 \in \mathbb{N}$, $x \geq 64 Z^2 U$, $Z \geq 4U^2$, $V^3 \geq 32x$. Assume further that $F(n)$ is a complex valued function such that $|F(n)| \leq 1$. Then the sum*

$$\sum_{n \sim x} \Lambda(n)\, F(n)$$

may be decomposed into $\mathcal{O}(\log^{10} x)$ sums, each either of Type I *with $N > Z$, or of Type* II *with $U < N < V$.*

We apply Lemma 3 with $F(n) = e(\alpha n + h(n+u)^\gamma)$, $U = 2^{-10} x^{1-\gamma+2\delta+\varepsilon}$, $V = 4x^{1/3}$ and

$$Z = \max([x^{36(1-\gamma)+42\delta-4+\varepsilon}], [5x^{1/3}], [x^{2(1-\gamma)+4\delta+\varepsilon}]) + 1/2 .$$

Then the Proposition follows from Lemmas 1 and 2.

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