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Non-vanishing of n-th derivatives of twisted elliptic L-functions in the critical point

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Résumé. On note $L^{(n)}(s,E)$ la dérivée n-ième de la série L de Hasse-Weil associée à une courbe elliptique modulaire E définie sur $\mathbb{Q}$. On évalue dans cet article le nombre de tordues E_d, $d \leq D$, de la courbe elliptique E telles que $L^{(n)}(1,E_d) \neq 0$.

Abstract. Let E be a modular elliptic curve over $\mathbb{Q}$ and $L^{(n)}(s,E)$ denote the n-th derivative of its Hasse-Weil $L-$series. We estimate the number of twisted elliptic curves $E_d, d \leq D$ such that $L^{(n)}(1,E_d) \neq 0$.

1. Introduction

Let $E = E/\mathbb{Q}$ be a modular elliptic curve over $\mathbb{Q}$ with conductor N defined by the Weierstrass equation $y^2 = f(x)$ and let $\mathbb{D}$ be defined as follows:

$$\mathbb{D} = \{d - \text{square-free} : 0 < d \equiv -\nu^2 \pmod{4N} \text{ for some } \nu \text{ prime to } 4N\}.$$

For any $d \in \mathbb{D}$ we consider the twisted elliptic curve E_d given by the equation $-dy^2 = w(x)$. We denote by $L(s,E)$ and $L(s,E_d)$ the Hasse-Weil L-functions associated to the curves E and E_d, respectively.

The celebrated Birch and Swinnerton-Dyer conjecture (see [B-S]) asserts that the rank of the group of rational points of $E/\mathbb{Q}$ is equal to the vanishing order of the associated Hasse-Weil function $L(s,E)$ at $s = 1$. Kolyvagin [Ko] has proved that

$E/\mathbb{Q}$ has rank equal to zero if:

1) $L(E,1) \neq 0$,

2) There exists $d \in \mathbb{D}$ such that $L(s,E_d)$ has a simple zero at $s = 1$.

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$E/\mathbb{Q}$ has rank equal to one if:

1') $L(s, E)$ has a simple zero at $s = 1$.

2') There exists $d \in \mathbb{D}$ such that $L(1, E_d) \neq 0$.

The condition 2') is true according to Waldspurger's theorem (see [Wa]).

The condition 2) has been proved to hold for infinitely many $d \in \mathbb{D}$ (see [B-F-H], [M-M]).

Iwaniec [Iw] has also proved a quantitative result on this condition. Let

$$N(D) = \#\{d \in \mathbb{D}, d \leq D : \ L'(1, E_d) \neq 0\}.$$

He has obtained the estimate

$$N(D) \gg D^{\frac{2}{3}-\varepsilon}$$

with arbitrary $\varepsilon > 0$.

The above exponent is improved to $1-\varepsilon$ in [P-P]. Here we will generalize this result to the n-th derivative of $L(s, E_d)$ where n is an arbitrary non-negative integer.

In this connection let $w = w(E)$ be the sign in the functional equation (see (1)). We define

$$N_n(D) = \#\{d \in \mathbb{D}, \ d \leq D : \ L^{(n)}(1, E_d) \neq 0\}.$$

We will prove

Theorem 1. *Let ε be an arbitrary positive real number and n be a fixed non-negative integer. Then we have as D tends to infinity*

$$N_n(D) \gg (n + |w - 1|)D^{1-\varepsilon}.$$

where the constant implied in the symbol $\gg$ depends on n and ε.

Our result is based on a recent large sieve type estimates over fundamental discriminants obtained by Heath-Brown [HB] (see Theorem 3 of [P-P]) and the method applied by Iwaniec in [Iw].

2. Outline of the proofs

For Re $s > 1$ the corresponding L-functions are given by

$$L(s, E) = \sum_{k=1}^{\infty} a_k k^{-s}, \qquad L(s, E_d) = \sum_{k=1}^{\infty} \chi_d(k) a_k k^{-s},$$

where $\chi_d(\cdot) = \left(\frac{-d}{\cdot}\right)$ (the Kronecker symbol) is a real character to modulus d prime to $4N$. They have the analytic prolongation to the whole complex plane, where they satisfy the equations

$$\left(\frac{\sqrt{N}}{2\pi}\right)^{s} \Gamma(s) L(s, E) = w\left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(E, 2-s), \tag{1}$$

$$\left(\frac{d\sqrt{N}}{2\pi}\right)^{s} \Gamma(s) L(s, E_d) = w_d\left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s) L(2-s, E_d), \tag{2}$$

where w and $w_d = \pm 1$ are suitable constants depending on the reduction of E at the primes dividing N, which satisfy the equality

$$w_d = w\chi_d(-N). \tag{3}$$

From (1) and (2) it follows that

$$w = (-1)^{\operatorname*{ord}_{s=1} L(s,E)}, \quad w_d = (-1)^{\operatorname*{ord}_{s=1} L(s,Ed)}$$

Theorem 1 is a consequence of the following two theorems.

THEOREM 2. *For any integer $n \geq 0$ we have the asymptotical equality*

$$\sum_{d \leq D}{}^{*} L^{(n)}(1, E_d) F\left(\frac{d}{Y}\right) \sim c_n L(1) \int F(t)\, dt\; D(\ln D)^n, \qquad \text{as } D \to \infty,$$

where F is a smooth function compactly supported in $\mathbb{R}^+$ with positive mean-value and the star (*) above means that the summation is restricted to $d \in \mathbb{D}$. The (nonzero) constant $L(1)$ is described in [Iw], while

$$c_n = \begin{cases} \frac{1}{2}(1-w)c & \text{if } n = 0 \\ w(-2)^{n-1} c & \text{if } n \geq 1 \end{cases} \tag{4}$$

$$c = \frac{3}{\pi^2 N} \prod_{p|4N} \left(1 - \frac{1}{p^2}\right)^{-1} \times$$
$$\times \ \#\{d \pmod{4N} : d \equiv -\nu^2 \pmod{4N}, (\nu, 4N) = 1\}. \tag{5}$$

The proof follows the idea exploited in [Iw] (cf. also [M-M] and [P-S]). We postpone it to sect.3.

Along the same lines as Theorem 3 of [P-P] we obtain

Theorem 3. *Let $\varepsilon > 0$ and n be a fixed non-negative integer. Then we have*

$$\sum_{d \le D}^{*} |L^{(n)}(1, E_d)|^2 \ll_{\varepsilon} D^{1+\varepsilon}$$

Now from Theorem 2 and Theorem 3 we obtain by an application of the Cauchy-Schwarz inequality that

$$(n+|w-1|)D(\ln D)^n \ll \sum_{d\le D}^{*} |L^{(n)}(1, E_d)| \le \Big(\sum_{d \le D}^{*} |L^{(n)}(1, E_d)|^2\Big)^{\frac{1}{2}} N_n(D)^{\frac{1}{2}}$$

hence

$$N_n(D) \gg (n + |w-1|)D^{1-2\varepsilon}$$

and Theorem 1 follows.

3. Proof of Theorem 2

For $n \ge 0$ we introduce the approximate functions

$$A_n(X, \chi_d) = \sum_{m=1}^{\infty} a_m \frac{\chi_d(m)}{m} V_n\Big(\frac{2\pi m}{X}\Big) \tag{6}$$

with

$$V_n(X) = \frac{1}{2\pi i} \int_{(3/4)} \frac{\Gamma(s)}{s^n} X^{-s}\, ds, \tag{7}$$

where we integrate over the line Re $s = \frac{3}{4}$.

By the functional equation (2) we have

$$L(1+s, E_d) = w_d\Big(\frac{d\sqrt{N}}{2\pi}\Big)^{-2s} L(1-s, E_d)\frac{\Gamma(1-s)}{\Gamma(1+s)}.$$

Therefore

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{(-3/4)} L(1+s, E_d)\frac{\Gamma(s)}{s^n}\Big(\frac{X}{2\pi}\Big)^s\, ds \\
&\qquad = \frac{w_d}{2\pi i}\int\limits_{(-3/4)} L(1-s, E_d)\frac{\Gamma(1-s)}{\Gamma(1+s)}\frac{\Gamma(s)}{s^n}\Big(\frac{2\pi X}{d^2 N}\Big)^s\, ds \\
&\qquad = -\frac{w_d}{2\pi i}\int\limits_{(-3/4)} L(1-s, E_d)\frac{\Gamma(-s)}{s^n}\Big(\frac{2\pi X}{d^2 N}\Big)^s\, ds \\
&\qquad = (-1)^{n+1}\frac{w_d}{2\pi i}\int\limits_{(3/4)} L(1+s, E_d)\frac{\Gamma(s)}{s^n}\Big(\frac{2\pi}{d^2 N/X}\Big)^{-s}\, ds \\
&\qquad = (-1)^{n+1} w_d A_n\Big(\frac{d^2 N}{X}, \chi_d\Big).
\end{aligned}$$

Hence defining

$$G_n(s, d, X) = L(1+s, E_d)\frac{\Gamma(s)}{s^n}\Big(\frac{X}{2\pi}\Big)^s$$

we obtain by the Cauchy theorem

$$\begin{aligned}
\operatorname*{res}_{s=0} G_n(s, d, X) &= \frac{1}{2\pi i}\int\limits_{(3/4)} G_n(s, d, X)\, ds - \frac{1}{2\pi i}\int\limits_{(-3/4)} G_n(s, d, X)\, ds \\
&= A_n(X, \chi_d) + (-1)^n w_d A_n\Big(\frac{d^2 N}{X}, \chi_d\Big).
\end{aligned}$$

By the definition of $\mathbb{D}$ we have $w_d = w\chi_d(-N) = -w$ for any $d \in \mathbb{D}$. Hence letting $X = d\sqrt{N}$ we obtain

$$\operatorname*{res}_{s=0} G_n(s, d, d\sqrt{N}) = \begin{cases} 2A_n(d\sqrt{N}, \chi_d) & \text{if } n \equiv \dfrac{w+1}{2} \pmod 2 \\ 0 & \text{if } n \equiv \dfrac{w-1}{2} \pmod 2. \end{cases} \tag{8}$$

On the other hand the residuum of G_n may be expressed in terms of the derivatives $L^{(k)}(s,E_d)$, $k=0,1,2,\dots$, by means of the Laurent expansions

$$\Gamma(s)=s^{-1}+\gamma_0+\gamma_1 s+\dots,$$

$$\Big(\frac{X}{2\pi}\Big)^s=\sum_{k=0}^{\infty}b_k s^k,$$

where $-\gamma_0$ is the Euler constant and $b_k=b_k(X)=\frac{1}{k!}\big(\ln\frac{X}{2\pi}\big)^k$.

Namely we prove

Lemma. *For any $n\ge 0$ we have*

$$\operatorname*{res}_{s=0} G_n(s,d,X)=b_0\frac{L^{(n)}(1,E_d)}{n!} + \sum_{l=0}^{n-1}\Big(b_{l+1}+\sum_{m=0}^{l}\gamma_m b_{l-m}\Big)\frac{L^{(n-1-l)}(1,E_d)}{(n-1-l)!}.$$

Proof. We have

$$\begin{aligned} G_n(s,d,X) &= s^{-n}\Big(\sum_{k=0}^{\infty}\frac{L^{(k)}(1,E_d)}{k!}s^k\Big)\Big(s^{-1}+\sum_{k=0}^{\infty}\gamma_k s^k\Big)\Big(\sum_{k=0}^{\infty}b_k s^k\Big)\\ &= s^{-n}\Big(\sum_{k=0}^{\infty}\frac{L^{(k)}(1,E_d)}{k!}s^k\Big)\Big(\sum_{k=0}^{\infty}b_k s^{k-1}+\sum_{k=0}^{\infty}f_k s^k\Big),\end{aligned}$$

where

$$f_k=\sum_{l=0}^{k}b_l\gamma_{k-l}.$$

The expression in the second bracket is equal to

$$\frac{b_0}{s}+\sum_{k=0}^{\infty}b_{k+1}s^k+\sum_{k=0}^{\infty}f_k s^k=\frac{b_0}{s}+\sum_{k=0}^{\infty}(f_k+b_{k+1})s^k.$$

Hence

$$G_n(s,d,X)=s^{-n}\Big(\frac{b_0}{s}\sum_{k=0}^{\infty}\frac{L^{(k)}(1,E_d)}{k!}s^k+\sum_{k=0}^{\infty}c_k s^k\Big),$$

where

$$c_k = \sum_{l=0}^{k} (f_l + b_{l+1}) \frac{L^{(k-l)}(1, E_d)}{(k-l)!}.$$

Therefore

$$\begin{aligned}\operatorname*{res}_{s=0} G_n(s,d,X) &= b_0 \frac{L^{(n)}(1,E_d)}{n!} + c_{n-1} \\ &= b_0 \frac{L^{(n)}(1,E_d)}{n!} + \sum_{l=0}^{n-1} (b_{l+1} + f_l) \frac{L^{(n-1-l)}(1,E_d)}{(n-1-l)!},\end{aligned}$$

as required.

Since $b_l(X) = \frac{1}{l!}\left(\ln \frac{X}{2\pi}\right)^l$ we obtain asymptotically

$$\operatorname*{res}_{s=0} G_n(s,d,X) \sim \sum_{l=0}^{n} b_l(X) \frac{L^{(n-l)}(1,E_d)}{(n-l)!}, \qquad \text{as } X \to \infty. \tag{9}$$

By (8) and the Lemma we obtain the formula

$$\sum_{l=0}^{n} b_l(d\sqrt{N}) \frac{L^{(n-l)}(1,E_d)}{(n-l)!} \sim \\ \sim \begin{cases} 2A_n(d\sqrt{N}, \chi_d) & \text{if } n \equiv \frac{w+1}{2} \pmod 2 \\ 0 & \text{if } n \equiv \frac{w-1}{2} \pmod 2, \end{cases} \tag{10}$$

as $d \to \infty$. Next we sum both sides of the above equality multiplied by the weighted function $F\left(\frac{d}{D}\right)$ over the numbers $d \in \mathbb{D}$. The contribution of the right-hand side of (10) is evaluated on the basis of the results on square-free sieve obtained in § 7-§ 9 of [Iw]. Precisely we have

$$\sum_{d \in \mathbb{D}}{}^{*} 2A_n(d\sqrt{N}, \chi_d) F\left(\frac{d}{D}\right) \sim cD \int F(t) B_n(tD\sqrt{N})\, dt\,, \tag{11}$$

where

$$B_n(X) = \frac{1}{2\pi i} \int_{(3/4)} L(s+1) \frac{\Gamma(s)}{s^n} \left(\frac{X}{2\pi}\right)^s ds\,,$$

the constant c is defined by (5) and the series $L(s)$ is defined in §9 of [Iw].

In order to find the asymptotical behaviour of the right-hand side of (11) we denote

$$\bar{G}_n(s,X) = L(s+1)\frac{\Gamma(s)}{s^n}\Big(\frac{X}{2\pi}\Big)^s$$

and apply the contour integration to obtain

$$\begin{aligned} B_n(X) &= \frac{1}{2\pi i}\int_{(3/4)} \bar{G}_n(s,X)\,ds \\ &= \operatorname*{res}_{s=0} \bar{G}_n(s,X) + \frac{1}{2\pi i}\int_{(-1/4)} \bar{G}_n(s,X)\,ds \\ &= \operatorname*{res}_{s=0} \bar{G}_n(s,X) + O(X^{-1/4}). \end{aligned}$$

Applying the asymptotical equality (9) with G_n replaced by $\bar{G}_n$ we obtain

$$B_n(X) \sim \sum_{l=0}^{n} b_l(X)\frac{L^{(n-l)}(1)}{(n-l)!} \sim b_n(X)L(1) \sim \frac{1}{n!}(\ln X)^n L(1), \qquad \text{as } X\to\infty.$$

Hence the right-hand side of (11) is asymptotically equal to

$$L(1)\frac{c_0}{n!}D\int F(t)\ln^n(tD\sqrt{N})\,dt \sim L(1)\frac{c_0}{n!}\int F(t)\,dt\,D(\ln D)^n.$$

Therefore in view of (10) we obtain the asymptotic equality

$$\sum_{d\le D}\sum_{l=0}^{n} b_l(d\sqrt{N})\frac{L^{(n-l)}(1,E_d)}{(n-l)!}F\Big(\frac{d}{D}\Big) \sim$$

$$\sim \begin{cases} \dfrac{c}{n!}L(1)\displaystyle\int F(t)\,dt\,D(\ln D)^n & \text{if } n \equiv \dfrac{w+1}{2} \pmod 2 \\ 0 & \text{if } n \equiv \dfrac{w-1}{2} \pmod 2. \end{cases}$$

Hence we obtain immediately that the constant c_0 in Theorem 2 is equal to $\binom{1-w}{2} c$, where c is the constant defined by (5).

Assuming that

$$\sum_{d\in\mathbb{D}} L^{(k)}(1,E_d)F\Big(\frac{d}{D}\Big) \sim c_k L(1)\int F(t)\,dt D(\ln D)^k$$

where c_k are some constants we see that they have to satisfy the equality

$$\sum_{k=0}^{n} \binom{n}{k} c_k = \begin{cases} c & \text{if } n \equiv \dfrac{w+1}{2} \pmod 2 \\ 0 & \text{if } n \equiv \dfrac{w-1}{2} \pmod 2. \end{cases} \tag{12}$$

To complete the proof of Theorem 2 it remains to prove that (12) holds with the constants c_k defined by (4). Indeed we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} c_k &= \left(\frac{1-w}{2}\right) c + \sum_{k=1}^{n} \binom{n}{k} c_k \\ &= c\left(\frac{1-w}{2} - w\Big(\frac{-1}{2}\sum_{k=1}^{n}\binom{n}{k}(-2)^k\Big)\right) \\ &= c\left(\frac{1-w}{2} - \frac{w}{2}\big((-1)^n - 1\big)\right) \\ &= \begin{cases} c & \text{if} \quad n \equiv \dfrac{w+1}{2} \pmod 2 \\ 0 & \text{if} \quad n \equiv \dfrac{w-1}{2} \pmod 2 \end{cases} \end{aligned}$$

as it is claimed. This completes the proof of Theorem 2.

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