Journal de Théorie des Nombres
de Bordeaux **9** *(1997), 395–403*

Classical and Overconvergent Modular Forms of Higher Level

par Robert F. COLEMAN

Résumé. Nous définissons la notion de forme modulaire surconvergente pour $\Gamma_1(Np^n)$ où p est un nombre premier, N et n sont des entiers et N est premier à p. Nous démontrons que toute forme primitive surconvergente pour $\Gamma_1(Np^n)$, de poids k et dont la valeur propre de U_p associée est de valuation strictement inférieure à $k-1$ est une forme modulaire au sens classique.

Abstract. We define the notion overconvergent modular forms on $\Gamma_1(Np^n)$ where p is a prime, N and n are positive integers and N is prime to p. We show that an overconvergent eigenform on $\Gamma_1(Np^n)$ of weight k whose U_p-eigenvalue has valuation strictly less than $k-1$ is classical.

In this note we define a notion of overconvergent modular form of level $\Gamma_1(Np^n)$ where N is a positive integer, p is a prime, $(N,p)=1$ and $n \geq 1$ and generalize the main result of [2]. That is, we show that overconvergent forms of level $\Gamma_1(Np^n)$, weight k and slope strictly less that $k-1$ are classical.

This allows one to recover most of the finite dimensional space of classical forms in the immense (of uncountable dimension) space of overconvergent forms. Using the extra freedom one has with overconvergent modular forms one can use this to prove important results about classical forms (see below).

The only essentially new wrinkle in this paper is the proof of the equality of the dimensions:

$$\dim S_k^{prim}(p^n) = \dim S_k^0(p^n)/\theta^{k-1}M_{2-k}(p^n) \tag{0.1}$$

which is a generalization of [2, Prop. 8.4]. The terms on the right are defined in terms of overconvergent forms of level $\Gamma_1(Np^n)$ in §1 and the term on the left is defined in terms of modular forms on $X_1(Np^n)$ in §2 (and in [6]). The following observation of Ogus is crucial for our proof.

Manuscrit reçu le 9 avril 1996

OBSERVATION (Ogus). *Suppose $\mathcal{S}$ is a locally free rank m coherent sheaf on a curve X over a field K and Ω is an invertible sheaf. Suppose $\mathcal{S}^{\cdot}$: $\mathcal{S}\xrightarrow{\nabla}\mathcal{S}\otimes\Omega$ is a complex. Then $\chi(\mathcal{S}^{\cdot}) =: \mathbf{H}^0(X,\mathcal{S}^{\cdot})-\mathbf{H}^1(X,\mathcal{S}^{\cdot})+H^2(X,\mathcal{S}^{\cdot}) = -m\deg\Omega$.*

Proof. This follows from Riemann-Roch and the magic of Euler characteristics. Namely, using the "Hodge to de Rham" spectral sequence we deduce

$$\chi(\mathcal{S}^{\cdot}) = \chi(\mathcal{S}) - \chi(\mathcal{S}\otimes\Omega).$$

Now using using the fact that the alternating sum of the Euler characteristics of the terms in an exact sequence of locally free sheaves is zero and Riemann-Roch we see $\chi(\mathcal{S}\otimes\Omega) = \chi(\mathcal{S}) + m\deg\Omega$. ∎

(Of course we have connections in mind.)

A consequence of this are the results in [10].

THEOREM. *Suppose $\alpha\in\mathbf{Q}$ and χ is a character on $(\mathbf{Z}/Np^n\mathbf{Z})^*$. Then the dimension of the space of modular forms on $X_1(Np^n)$ of weight $k>\alpha+1$, nebentypus χ and slope α is locally constant as a function of k where k is considered as an element of $(\mathbf{Z}/(p-1)\mathbf{Z})^*\times\mathbf{Z}_p^*$.*

1. Overconvergent forms of level Np^n.

In this section we will suppress the level N from the notation when convenient.

Recall from [2], we have a commutative diagram

$$\begin{array}{ccc} E_2(p) & \xrightarrow{\Phi} & E_1(p) \\ \downarrow & & \downarrow \\ W_2(p) & \xrightarrow{\phi} & W_1(p) \end{array} \tag{1.1}$$

in level Np. Here $W_i(p)$ is the connected component of the rigid subspace of $X_1(Np)$ $\{v(E_{p-1}(x)) < p^{2-i}/(p+1)\}$ containing the cusp ∞ and $E_i(p)$ is the pullback of the universal elliptic curve over $X_1(Np)$ to $W_i(p)$. Also, Φ and ϕ are the Tate-Deligne morphisms. We let $W_i(p^n)$, for $n>1$, denote the inverse image in $X_1(Np^n)$ of $\phi^{1-n}W_i(p)$ with respect to the morphism from $X_1(Np^n)\to X_1(Np)$ which takes the point corresponding to $(E,\alpha\colon\mu_{p^n}\to E)$ to the point corresponding to (E',α') where $E' = E/\alpha(\mu_{p^{n-1}})$ and α' is the map induced by α from $\mu_p\cong\mu_{p^n}/\mu_{p^{n-1}}$ to E'. Better, $W_i(p^n)$ consists of points corresponding to pairs $(E,\alpha\colon\mu_{p^n}\to E)$ such that the point corresponding to $(E,\alpha|_{\mu_p})$ is in $\phi^{-(n-1)}W_i$ and

$$\phi^{n-1}\circ\alpha(\mu_{p^n}) = \mathcal{K}(\Phi^{n-1}E).$$

Also let $Z(p^n)$ be the inverse image of $Z(p)$, where $Z(p)$ is the ordinary locus in $W_2(p)$. (and also $W_1(p)$). Then $Z(p^n)$ is the minimal underlying affinoid of $W_i(p^n)$ containing the cusps in this wide open for $i = 1$ or 2 (the requirement about the cusps is only necessary when the genus of the Igusa curve $I_n(N)$ (see below) is 0 and the number of supersingular points on $\overline{X}_1(N)$ is less than 3 (see [1]).

We have a natural lifting of ϕ to a map from $W_2(p^n)$ to $W_1(p^n)$ which takes the point corresponding to a pair $(E, \alpha\colon \mu_{p^n} \to E)$ to the point corresponding to $(\Phi E, \alpha')$ where $\alpha'(\zeta) = \Phi(Q)$ if $Q \in Ker(\Phi^n)$ and $pQ = \alpha(\zeta)$. Then we have a commutative diagram analogous to (1.1).

Let $f_n\colon E_1(Np^n) \to X_1(Np^n)$ be the universal generalized elliptic curve over $X_1(Np^n)$ (with the singular points above the cusps removed) and let $\omega = f_{n*}\Omega^1_{E_1(Np^n)/X_1(Np^n)}$. For $k \in \mathbf{Z}$ we call sections of ω^k on $Z(p^n)$ **convergent forms** of level $\Gamma_1(Np^n)$, sections on any strict neighborhood of $Z(p^n)$ **overconvergent forms** of level $\Gamma_1(Np^n)$. We note that any overconvergent form of finite slope extends to an section of ω^k on $W_1(p^n)$. Therefore, we set $M_k(p^n) = M_k(\Gamma_1(Np^n)) =: \omega^k(W_1(p^n))$. The space of modular forms of weight k on $X_1(Np^n)$, which we denote $M_{k,cl}(p^n)$, naturally injects into $M_k(p^n)$ and we say that elements in the image are **classical**. We identify $M_{k,cl}(p^n)$ with the space of classical modular forms. Then we can define an operator U on $M_k(p^n)$ in the same way as in [2,§2]. On $M_{k,cl}(p^n)$ this acts as the Hecke operator U_p and so it follows that the slope of any classical form of weight $k+2$ is at most $k+1$ if it is finite.

THEOREM 1.1. *Every p-adic overconvergent form of weight $k+2$ and slope strictly less that $k+1$ in $M_{k+2}(p^n)$ is classical.*

The proof of this theorem follows the same lines as the proof of Theorem 8.1 of [2]. The definitions and results of Sections 2-4 of [2] carry over to this situation without difficulty. In particular, there is a linear map θ from $M_{-k}(p^n)$ to $M_{k+1}(p^n)$ which on q-expansion s is $(qd/dq)^{k+1}$ and which satisfies

$$U \circ \theta^{k+1} = p^{k+1}\theta^{k+1} \circ U.$$

We will use the notation $\mathcal{S}ym^k$ to denote the k^{th} symmetric power of the relative de Rham cohomology complex of $E_1(Np^n)$ (minus its singular points) over $X_1(Np)$. Let $]C(p^n)[$ denote the union of the cuspidal residue classes,
$]SS(p^n)[= W_1(p^n) - Z(p^n)$ and let $\mathbf{H}^1_{par}(W, Sym^k)^*$ denote the kernel of the map

$$\mathbf{H}^1(W, Sym^k) \to \mathbf{H}^1(((]C(p^n)[\cup]SS(p^n)[) \cap W, Sym^k).$$

We may deduce in the same way as in [2] that there exists an endomorphism Ver of $\mathbf{H}^1(W_1(p^n), Sym^k)$, the quotient $M_{k+2}(p^n)/\theta^{k+1}M_k(p^n)$ is naturally isomorphic to this space and the following diagram commutes:

$$\begin{array}{ccc} M_{k+2}(p^n)/\theta^{k+1}M_{-k}(p^n) & \xrightarrow{U} & M_{k+2}(p^n)/\theta^{k+1}M_{-k}(p^n) \\ \downarrow & & \downarrow \\ \mathbf{H}^1(W_1(p^n), Sym^k) & \xrightarrow{Ver} & \mathbf{H}^1(W_1(p^n), Sym^k). \end{array}$$

For $k \geq 0$, we set $S^0_{k+2}(p^n)$ equal to the subspace of $M_k(p^n)$ consisting of cusp forms with trivial residues (in the sense of [11]) on $]SS(p^n)[$. This is also equal to the inverse image with respect to the above map of $H^1_{par}(W_1(p^n), Sym^k)^*$ in $M_{k+2}(p^n)$.

2. Dimensions: Modular forms and Cohomology

Let X denote the curve $X_1(Np^n)$ over $\mathbf{C}_p$. We let $M_k^{prim}(p^n)$ denote the subspace of primitive forms in $M_{k,cl}(p^n)$. Recall, this is the space spanned by the images of the weight k forms of level $\Gamma_1(Np^r)$ of primitive nebentypus at p of $0 \leq r \leq n$ and $S_k^{prim}(p^n)$ is the subspace of primitive cusp forms. (We point out that two copies of the level N forms are contained in $M_k^{prim}(p^n)$ when $n \geq 1$ via the two degeneracy maps.) It follows from the work of Ogg and Li that the subspace of $M_k^{prim}(p^n)$ on which $(\mathbf{Z}/p^n\mathbf{Z})^*$ acts non-trivially is the same as the subspace of $\cap_s U_p^s M_{k,cl}(p^n)$ on which $(\mathbf{Z}/p^n\mathbf{Z})^*$ acts non-trivially and $\cap_s U_p^s M_k(p^n) = M_k^{prim}(p^n) + M_k(N;p)$.

We set $\Gamma(N;p^r) = \Gamma_1(Np^r, Np^{r-1}) = \Gamma_1(Np^{r-1}) \cap \Gamma_0(p^r)$,
$X(N;p^r) = X_1(Np^r, Np^{r-1})$ etc., in the notation of Mazur-Wiles [MW].

For a representable moduli problem $\mathcal{P}$ over $E11/R$ we let $I_n(\mathcal{P})$ denote the Igusa problem of level p^n, $n \geq 0$, over $\mathcal{P}$ (so that $I_0(\mathcal{P}) = \mathcal{P}$). If $\mathcal{P}$ is a representable moduli problem over $E11/k$ we let $X(\mathcal{P})$ denote the completion of representation space $Y(\mathcal{P})$, $C(\mathcal{P})$ the cuspidal divisor on $X(\mathcal{P})$ and $\Omega(\mathcal{P}) = \Omega^1_{X(\mathcal{P})}(\log C(\mathcal{P}))$. We also set

$$\Omega_1(M) = \Omega(\Gamma_1(M)) \qquad \text{and} \qquad \Omega(N;p^r) = \Omega(\Gamma(N;p^r))$$

etc. We will also use Sym^k to denote the k-th symmetric product of the first relative de Rham cohomology complex of the universal elliptic curve over $X(\mathcal{P})$ and ∇ will denote the relevant connection. The group $\mathbf{H}^1(X, Sym^k)^{prim}$ is defined similarly to $M_k^{prim}(p^n)$.

We let $\mathcal{X}$ be stable model of (X, C) over R (we only need to worry about C when X has genus 0 or 1). Then let $I_n(N)$ be the "good" Igusa component of $\overline{\mathcal{X}}$ containing ∞. Let $I =: I(N)$ be the rigid space in X

whose closed points reduce to points in $I_n(N)$ (this is the generic fiber of the tube of $I_n(N)$ in $\mathcal{X}$) and $I' = I'(N)$ be the rigid space corresponding to the other good Igusa component (which equals $w_\zeta I$ for any primitive p^n-th root unity ζ). We also let $U = U_1(Np^n)$ denote the reduction inverse of the union of all the components of the stable model of X besides the good ones. Let SS denote the supersingular divisor on $I_n(N)$ or its degree when no confusion will arise. Then the spaces $I \cap U$ and $I' \cap U$ are each the union of SS annuli.

For a rigid subspace W of X we let $\mathbf{H}^1(W, Sym^k)^*$ denote the kernel of the restriction map from $\mathbf{H}^1(W, Sym^k)$ to $\mathbf{H}^1(W \cap U, Sym^k)$.

Theorem 2.1. *The natural map from* $\mathbf{H}^1(X, Sym^k)^{prim}$ *to*

$$\mathbf{H}^1(I, Sym^k)^* \oplus \mathbf{H}^1(I', Sym^k)^*$$

is an isomorphism.

Note: The map from $\mathbf{H}^1(X, Sym^k)^{prim}$ to $\mathbf{H}^1(I, Sym^k)$ factors through $\mathbf{H}^1(I, Sym^k)^*$ because $(\mathbf{Z}/p^n\mathbf{Z})^*$ acts trivially on the cohomology of the complex Sym^k when restricted to the supersingular annuli.

Proof. First, using [5, Thm. 10.13.12], we see that when $n \geq 2$

$$\begin{aligned}\deg \Omega_1(Np^n) - \deg \Omega(Np^{n-1}; p^n) &\\ &= 2(|SL_2(\mathbf{Z}/p^n\mathbf{Z})|/p^n - |SL_2(\mathbf{Z}/p^n\mathbf{Z})|/p^{n+1}) \deg \Omega_1(N)\\ &= 2\big(p^n\phi(p^n) - p^{n-1}\phi(p^{n-1})\big) \deg \Omega_1(N)\end{aligned}$$

and similarly

$$\deg \Omega_1(Np) - \deg \Omega(N; p) = 2(p\phi(p) - 2) \deg \Omega_1(N).$$

Hence, as $\mathbf{H}^1(X, Sym^k)^{prim}$ modulo $\mathbf{H}^1(X_1(Np^{n-1}), Sym^k)^{prim}$ is isomorphic to

$$\mathbf{H}^1(X_1(Np^n), Sym^k)/\mathbf{H}^1(X(N; p^n), Sym^k)$$

when $n \geq 1$ (all the maps involved here are injections) we see using the observation of Ogus and [5, Cor. 12.9.4], that the dimension of $\mathbf{H}^1(X, Sym^k)^{prim}$ is

$$\begin{aligned}2(k+1)(p^n\phi(p^n) - p^{n-1}\phi(p^{n-1})) \deg \Omega_1(N) &\\ &= 2(k+1)\big(\deg \Omega(I_n(N))) - \deg \Omega(I_n(N))\big).\end{aligned}$$

Now we have to compute $\dim \mathbf{H}^1(I, Sym^k)^*$. We do this by adding disks to I to make a complete smooth lifting of the Igusa curve $I_n(N)$, $\hat{I}$, and extending the connection Sym^k to a connection $\hat{\nabla}$ on $\hat{I}$ which only has log poles at the cusps C in I (see [2, §8]). Then the observation of Ogus mentioned in the introduction implies that $\chi(\hat{\nabla}) = \deg \Omega^1_{\hat{I}_n}(\log C) = \deg \Omega(I_n(N))$. Since $H^0(\hat{\nabla}) = 0$ if $k > 2$ this gives us what we want. Similarly, we can work on the reduction inverse image $I(N, p^n)$ of the good component containing ∞ on the stable reduction of the curve $X(N, p^n)$ define $\mathbf{H}^1(I(N, p^n), Sym^k)^*$ and prove its dimension is $\deg \Omega(I_{n-1}(N))$. We also let $U(N, p^n)$ denote the reduction inverse of the bad components of the stable model of $X(N, p^n)$.

To prove the isomorphism in the theorem we use Meyer-Vietoris and induction on n. It is true when $n = 1$ by [2]. We have exact sequences

$$\mathbf{H}^0(V \cap (J \cup J'), Sym^k) \to \mathbf{H}^1(Y, Sym^k) \to \\ \mathbf{H}^1(J \cup J', Sym^k) \oplus \mathbf{H}^1(V, Sym^k) \to \mathbf{H}^1(V \cap (J \cup J'), Sym^k)\,.$$

Where Y is either $X_1(Np^n)$ or $X(N, p^n)$, V is either $U_1(Np^n)$ or $U(N; p^n)$, J is either $I(N)$ or $I(N, p^n)$ etc. accordingly. When $k > 0$ the first maps are injections and the last maps are surjections. In any case we deduce that

$$\mathbf{H}^1(X_1(Np^n), Sym^k)/\mathbf{H}^1(X(N; p^n), Sym^k)$$

(it is easy to see that the map from the second of these spaces to the first is an injection, and we identify it with its image) is isomorphic to the direct sum of

$$\mathbf{H}^1(I, Sym^k)/\mathbf{H}^1(I(N; p^n), Sym^k),$$
$$\mathbf{H}^1(I', Sym^k)/\mathbf{H}^1(I'(N; p^n), Sym^k)$$

and

$$\mathbf{H}^1(U_1(Np^n), Sym^k)/\mathbf{H}^1(U(N; p^n), Sym^k).$$

But, using the triviality of the action of $(\mathbf{Z}/p^n\mathbf{Z})^*$ on the cohomology over the supersingular annuli, either of the first two spaces is isomorphic under the natural map to

$$\mathbf{H}^1(I, Sym^k)^*/\mathbf{H}^1(I(N; p^n), Sym^k)^*$$

and it follows from the above that this space has dimension

$$(k+1)\big(p^n\phi(p^n) - p^{n-1}\phi(p^{n-1})\big) \deg \Omega_1(N).$$

Hence the map to the sum of the first two spaces above is an isomorphism, which proves the theorem, and it follows that the dimension of the third is zero. ∎

This last assertion implies:

COROLLARY 2.2.

$$\mathbf{H}^1(X, Sym^k)^{prim} = \mathbf{H}^1(X, Sym^k)^*.$$

In particular, every primitive modular form of weight $k+2$ when considered as a section of $Sym^k \otimes \Omega_1(Np^n)$ and restricted to $U_1(Np^n)$ is equal to ∇G for some section G of Sym^k on $U_1(Np^n)$.

For a subspace W of X we set $\mathbf{H}^1_{par}(W, Sym^k)$ equal to

$$Ker\big(\mathbf{H}^1(W, Sym^k) \to \mathbf{H}^1(W \cap]C(p^n)[, Sym^k)\big).$$

Then

$$\mathbf{H}^1_{par}(W, Sym^k)^* = \mathbf{H}^1_{par}(W, Sym^k) \cap \mathbf{H}^1(W, Sym^k)^*.$$

COROLLARY 2.3. *We have $\dim S^{prim}_{k+2}(p^n) = \dim \mathbf{H}^1_{par}(I, Sym^k)^*$ which equals*

$$(k+1)\deg \Omega(I_n(N)) - c(I_n(N)) + \delta$$

where $\delta = 2$ if $k = 0$ and $\delta = 0$ otherwise.

Proof. By Riemann-Roch and [5, Thm. 10.13.12]

$$2\dim H^0\big(X_1(Np^r), \omega_1(Np^r)^{k+2} \otimes \mathcal{I}_{C_1(Np^r)}\big) = (k+1)\deg \Omega_1(Np^r) - c_1(Np^r)$$

when $k > 0$. It follows that, in this case,

$$\dim S^{prim}_{k+2}(p^n) = (k+1)p^n\phi(p^n)\deg \Omega_1(N) - \phi(p^n)c_1(N)$$

which implies the result using Ogus' observation and [5, Cor. 12.9.4]. ∎

This and the isomorphism between $M_{k+2}(p^n)/\theta^{k+1}M_k(p^n)$ and $\mathbf{H}^1(W_1(p^n), Sym^k)$ enables us to prove (0.1).

REMARK. *It follows from the results in [8,§2 & §3] that the stable reduction of $X(N;p^n)$ is the same as that of $X_1(Np^n)$ except that the two good components in the latter are each replaced by a copy of $I_{n-1}(N)$. On the other hand we point out that the stable reduction of $X_1(Np^n)$ is not known (but see [3]).*

3. Conclusion

As in [2, §5] we can put a pairings $(\ ,\ )_I$ and $(\ ,\ )_{I'}$ on $\mathbf{H}^1_{par}(I, Sym^k)^*$ and on $\mathbf{H}^1_{par}(I', Sym^k)^*$. We can conclude in the same way as in [2, §5] that

$$(Ver(a), Ver(b))_I = p(Ver(a), Ver(b))_I.$$

Moreover, if $(\ ,\ )$ is the natural pairing on $H^1_{par}(X, Sym^k)$, $(\ ,\ )$ restricts to a perfect pairing on $H^1_{par}(X, Sym^k)^{prim}$ and if a and b are elements of $H^1_{par}(X, Sym^k)$,

$$(a,b) = (res_I(a), res_I(b))_I + (res_{I'}(a), res_{I'}(b))_{I'}.$$

This and Theorem 2.1 implies that $(\ ,\ )_I$ is non-degenerate which allows us to generalize [2, Prop. 8.3].

We can conclude that the map from the slope α subspace of S^{prim}_{k+2} to the slope α subspace of $\mathbf{H}^1_{par}(W_1(p^n), Sym^k)^*$ is an isomorphism if $\alpha < k+1$ exactly as in [2, §8]. The proof of Theorem 1.1 may be completed by arguments similar to those used in [2].

REFERENCES

[1] Coleman R., *Reciprocity Laws on Curves*, Compositio **72** (1989), 205–235.

[2] Coleman R., *Classical and Overconvergent modular forms*, Invent. Math. **124** (1996), 215–241.

[3] Edixhoven B., *Stable models of modular curves and applications*, Thesis, University of Utrecht (unpublished).

[4] Katz N., *P-adic properties of modular schemes and modular forms Modular Functions of one Variable III*, Springer Lecture Notes **350** (197), 69–190.

[5] Katz N. and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Math. Stud. **108**, Princeton University Press, 1985.

[6] Mazur B. and A. Wiles, *"Class fields and abelian extensions of* $\mathbf{Q}$*"*, Invent. Math. **76** (1984), 179–330.

[7] Li W., *"Newforms and functional equations,"*, Math. Ann. **212** (1975), 285–315.

[8] Mazur B. and A. Wiles, *"On p-adic analytic families of Galois representations"*, Compositio Math. **59** (1986), 231–264.

[9] Ogg A., *"On the eigenvalues of Hecke operators"*, Math. Ann. **179** (1969), 101–108.

[10] Coleman R., *p-adic Banach spaces and families of modular forms*, Invent. math? **127** (1992), 917–979.

[11] Coleman R., *p-adic Shimura Isomorphism and p-adic Periods of modular forms*, Contemp. Math. **165** (1997), 21–51.

Robert F. Coleman
Department of Mathematics
Berkeley, California 94720-3840 USA
email: coleman@math.berkeley.edu