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On the number of subgroups of finite abelian groups

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Résumé. Soit

$$T(x) = K_1 x \log^2 x + K_2 x \log x + K_3 x + \Delta(x),$$

où $T(x)$ désigne le nombre de sous groupes des groupes abéliens dont l'ordre n'excède pas x et dont le rang n'excède pas 2, et $\Delta(x)$ est le terme d'erreur. On démontre que

$$\int_1^X \Delta^2(x)\,\mathrm{d}x \ll X^2 \log^{31/3} X, \int_1^X \Delta^2(x)\,\mathrm{d}x = \Omega(X^2 \log^4 X).$$

Abstract. Let

$$T(x) = K_1 x \log^2 x + K_2 x \log x + K_3 x + \Delta(x),$$

where $T(x)$ denotes the number of subgroups of all Abelian groups whose order does not exceed x and whose rank does not exceed 2, and $\Delta(x)$ is the error term. It is proved that

$$\int_1^X \Delta^2(x)\,\mathrm{d}x \ll X^2 \log^{31/3} X, \quad \int_1^X \Delta^2(x)\,\mathrm{d}x = \Omega(X^2 \log^4 X).$$

1. Introduction

Let

$$t_2(n) = \sum_{|\mathcal{G}|=n, r(\mathcal{G})\le 2} \tau(\mathcal{G}), H(s) = \sum_{n=1}^{\infty} t_2(n) n^{-s} \quad (\Re\mathrm{e}\ > 1),$$

where $\tau(\mathcal{G})$ denotes the number of subgroups of a finite Abelian group $\mathcal{G}, r(\mathcal{G})$ is the rank of $\mathcal{G}$, and $|\mathcal{G}|$ is the order of $\mathcal{G}$. The group $\mathcal{G}$ has rank r if

$$\mathcal{G} \ \cong\ \mathbb{Z}/n_1\mathbb{Z} \otimes \cdots \otimes \mathbb{Z}/n_r\mathbb{Z},$$

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where $n_j \mid n_{j+1}$ for $j = 1, \cdots, r-1$. We set

$$T(x) \;=\; \sum_{n\le x} t_2(n) \;=\; \sum_{|\mathcal{G}|\le x, r(\mathcal{G})\le 2} \tau(\mathcal{G})$$

so that one has

$$T(x) = K_1 x \log^2 x + K_2 x \log x + K_3 x + \Delta(x), \tag{1.1}$$

where K_j are effective constants and $\Delta(x)$ is to be considered as the error term in the asymptotic formula for $T(x)$. One has the Dirichlet series representation (this is due to G. Bhowmik [1]; the generating Dirichlet series for Abelian groups of rank ≥ 3 are more complicated)

$$H(s) = \zeta^2(s)\zeta^2(2s)\zeta(2s-1)\prod_p (1 + p^{-2s} - 2p^{-3s}) \qquad (\Re\mathrm{e} \, > 1/2). \tag{1.2}$$

Using (1.2) and the estimate in the four-dimensional asymmetric divisor problem of H.-Q. Liu [6], G. Bhowmik and H. Menzer [2] obtained the bound

$$\Delta(x) \;\ll\; x^{c+\varepsilon} \tag{1.3}$$

with $c = 31/43 = 0.72093\ldots$. Recently H. Menzer [6] used two new estimates in the three-dimensional asymmetric divisor problem to prove (1.3) with the better value $c = 9/14 = 0.64285\ldots$, and this is further improved in the forthcoming paper by G. Bhowmik and J. Wu [3] to $\Delta(x) \ll x^{5/8}\log^4 x$. Note that we can write (1.2) as

$$\begin{aligned} &H(s) = \zeta^2(s)\zeta^3(2s)\zeta(2s-1)U(s), \\ &U(s) = \prod_p (1 - 2p^{-3s} - p^{-4s} + 2p^{-5s}), \end{aligned} \tag{1.4}$$

where the Dirichlet series for $U(s)$ is absolutely convergent for $\Re\mathrm{e}\, > 1/3$. This prompts one to think that in (1.1) there should be a new main term corresponding to the pole of order 3 of $H(s)$ at $s = 1/2$, namely that we should have

$$\begin{aligned} &\Delta(x) = x^{1/2}(C_1\log^2 x + C_2 \log x + C_3) + E(x), \\ &E(x) = o(x^{1/2}\log^2 x) \;(x\to\infty), \end{aligned} \tag{1.5}$$

where $C_1 \neq 0$ (one cannot hope for $E(x) = o(x^{1/2})$ since in [3] it was shown that $E(x) = \Omega(x^{1/2}(\log\log x)^6)$ holds). Even if the relation (1.5) is perhaps too optimistic, it is very likely that $\Delta(x) \ll x^{1/2+\varepsilon}$ holds, and that $\Delta(x)$ cannot be of order lower than $x^{1/2}\log^2 x$. In fact H. Menzer [5] conjectured that

$$\Delta(x) \;=\; \Omega(x^{1/2}\log^2 x). \tag{1.6}$$

This was proved by Bhowmik and Wu [3], which is a corollary of their bound

$$\int_0^X E(x)\,\mathrm{d}x \;\ll\; X^{11/8+\varepsilon}.$$

Since heuristically in (1.5) the terms $x^{1/2}(C_1\log^2 x + C_2\log x + C_3)$ are the residue of $H(s)x^s/s$ at $s = 1/2$ it is not difficult to see that the constant C_1 is negative, so actually in (1.6) Ω is Ω_-, i.e. the Ω–result of Bhowmik and Wu is

$$\liminf_{x\to\infty}\frac{\Delta(x)}{x^{1/2}\log^2 x} \;<\; 0.$$

The object of this note is to investigate $\Delta(x)$ in mean square, and we shall prove two fairly precise results contained in

THEOREM 1. *We have*

$$\int_1^X \Delta^2(x)\,\mathrm{d}x \;\ll\; X^2(\log X)^{31/3}. \tag{1.7}$$

THEOREM 2. *We have*

$$\int_1^X \Delta^2(x)\,\mathrm{d}x \;=\; \Omega(X^2\log^4 X). \tag{1.8}$$

Remark 1. The omega-result (1.8) implies another proof of Menzer's conjecture (1.6).

Remark 2. It is plausible to conjecture that, for $X \to \infty$ and suitable $C > 0$, one has

$$\int_1^X \Delta^2(x)\,\mathrm{d}x \;\sim\; CX^2\log^4 X,$$

although this seems to be out of reach at present.

Remark 3. It will be clear from the proof of Theorem 2 that the method is capable of generalization to the case where the error term in question corresponds to the Dirichlet series generated by suitable factors of the form $\zeta(as+b)$ (a, b integers).

2. Proof of the upper bound estimate

To prove Theorem 1 we start from the relation

$$\int_0^\infty \Delta^2(x)x^{-2c-1}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{H(c+it)}{c+it}\right|^2 \mathrm{d}t, \tag{2.1}$$

where $c > 0$ is a suitable constant. The formula (2.1) follows from the properties of Mellin transforms, similarly as in the case of the classical divisor problem (see (13.23) on p. 357 of [4]). If the integral on the left-hand side of (2.1) converges, so does the integral on the right-hand side and conversely. We shall need the following facts about $\zeta(s)$ (see [4] for proofs):

$$\zeta(\sigma+it) \ll \left(t^{C(1-\sigma)^{3/2}}+1\right)\log^{2/3} t \qquad (1/2 \le \sigma \le 2,\, t \ge t_0 > 0, C > 0),$$

$$\begin{aligned} &\zeta(s) = \chi(s)\zeta(1-s),\\ &t^{1/2-\sigma} \ll |\chi(s)| \ll t^{1/2-\sigma} \quad (s = \sigma+it,\, t \ge t_0 > 0),\\ &\int_1^T |\zeta(\sigma+it)|^4\,\mathrm{d}t \ll T\log^4 T \qquad (1/2 \le \sigma \le 1). \end{aligned} \tag{2.2}$$

The last bound follows e.g. from Th. 4.4 and Th. 5.2 of [4]. Now we take $c = 1/2 + 1/\log X$, $X \ge X_0 > 0$. Then from (1.4) and (2.1) we obtain first

$$\begin{aligned} &\int_X^{2X} \Delta^2(x)\,\mathrm{d}x\\ &\ll X^2\int_{-\infty}^\infty |\zeta^2(c+it)\zeta(2c-1+2it)\zeta^3(2c+2it)|^2|c+it|^{-2}\,\mathrm{d}t. \end{aligned} \tag{2.3}$$

Let

$$U := \exp\left(10\log X\log\log X\right). \tag{2.4}$$

By symmetry we have

$$\int_{-\infty}^{\infty} \leq 2\left(\int_0^1 + \int_1^U + \int_U^{\infty}\right) = 2I_1 + 2I_2 + 2I_3,$$

say. Since $\zeta(s) \ll 1/|s-1|$ near $s=1$, we have $I_1 \ll \log^6 X$. Using (2.2) it follows that ($C_1 > 0$ is a constant)

$$\begin{aligned} I_2 &\ll \int_1^U t^{-1-4/\log X} |\zeta(\frac{1}{2} + \frac{1}{\log X} + it)|^4 \times \\ &\times |\zeta(1 - \frac{2}{\log X} + 2it)|^2 |\zeta(1 + \frac{2}{\log X} + 2it)|^6 \, \mathrm{d}t \\ &\ll \int_1^U |\zeta(\frac{1}{2} + \frac{1}{\log X} + it)|^4 t^{-1-\frac{4}{\log X} + C_1(\log X)^{-3/2}} \log^{16/3} t \, \mathrm{d}t \\ &\ll \int_1^U |\zeta(\frac{1}{2} + \frac{1}{\log X} + it)|^4 t^{-1-\frac{3}{\log X}} \log^{16/3} t \, \mathrm{d}t. \end{aligned}$$

Let

$$W(t) := \int_1^t |\zeta(\frac{1}{2} + \frac{1}{\log X} + iv)|^4 \, \mathrm{d}v, \quad f(t) := t^{-1-\frac{3}{\log X}} \log^{16/3} t.$$

By integration by parts it follows that

$$I_2 \ll \int_1^U f(t) \, \mathrm{d}W(t) \ll W(U)f(U) + \left| \int_1^U W(t) f'(t) \, \mathrm{d}t \right|.$$

But we have

$$W(U)f(U) \ll U^{-3/\log X} \log^{28/3} U \ll e^{-30 \log\log X} (\log X \log\log X)^{28/3} \ll 1$$

and

$$\begin{aligned} \int_1^U W(t) f'(t) \, \mathrm{d}t &\ll \int_1^U t^{-1-3/\log X} \log^{28/3} t \, \mathrm{d}t \\ &= \log^{31/3} X \int_0^{10 \log\log X} e^{-3v} v^{28/3} \, \mathrm{d}v \ll \log^{31/3} X \end{aligned}$$

with the change of variable $v = \log t/\log X$. We also have

$$\int_M^{2M} \ll \int_M^{2M} |\zeta(\frac{1}{2} + \frac{1}{\log X} + it)|^4 t^{-1-\frac{4}{\log X}+C_1(\log X)^{-3/2}} \log^{16/3} t \, \mathrm{d}t$$
$$\ll M^{-2/\log X} \log^{28/3} M \le M^{-1/\log X}$$

for $M \ge U$, as given by (2.4). Hence ($M = 2^{j-1}U$)

$$I_3 = \int_U^\infty = \sum_{j=1}^\infty \int_{2^{j-1}U}^{2^j U} \ll \sum_{j=1}^\infty \exp(-\frac{j\log 2 + \log U}{\log X}) \ll 1.$$

Therefore

$$I_1 + I_2 + I_3 \ll (\log X)^{31/3},$$

and (2.3) gives

$$\int_X^{2X} \Delta^2(x)\,\mathrm{d}x \ll X^2(\log X)^{31/3}.$$

Replacing X by $X2^{-j}, j \in \mathbb{N}$ and summing the resulting estimates we obtain (1.7).

3. Proof of the omega-result

To prove the omega-result of Theorem 2 we shall use the method used in proving Theorem 2 of [5], with the necessary modifications. Namely in [5] the generating Dirichlet series was of the form

$$\zeta(a_1 s)\zeta(a_2 s)\cdots\zeta(a_k s), \tag{3.1}$$

where $1 \le a_1 \le a_2 \le \ldots \le a_k$ are integers, with k possibly infinite (e.g the generating series of the function $a(n)$, the number of non-isomorphic Abelian groups with n elements, is (3.1) with $a_j = j, k = \infty$). The Dirichlet series $H(s)$ (see (1.4)) is clearly not of the form (3.1), since it contains the factor $\zeta(2s-1)$. Writing

$$H(s) = \zeta(2s-1)V(s), \quad V(s) = \sum_{n=1}^\infty v(n)n^{-s} \quad (\Re\mathrm{e} > 1),$$

it is seen that $v(n) \ll_\varepsilon n^\varepsilon$, and consequently we obtain

$$t_2(n) = \sum_{k^2m=n} kv(m) \ll n^{\frac12+\frac{\varepsilon}{2}} \sum_{k^2m=n} 1 \ll n^{\frac12+\frac{\varepsilon}{2}} d(n) \ll n^{\frac12+\varepsilon}, \tag{3.2}$$

where $d(n)$ is the number of divisors of n. We remark that Bhowmik and Wu [3] proved the sharper bound $t_2(n) \ll n^{1/2}(\log\log n)^6$, but for our purposes (3.2) is more than sufficient. As on p. 82 of [5] we start from the Mellin inversion integral (see also p. 122 of [4])

$$e^{-U^h} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} U^{-w}\Gamma(1+\frac{w}{h})\frac{\mathrm{d}w}{w}, \tag{3.3}$$

where $h, U > 0$. We shall take $T^{1-\delta} \le t \le T, h = \log^2 T, s = \frac12 + it, Y = T^B$, where $\delta > 0$ is a sufficiently small constant and $B > 1$ is a suitable constant. Setting $U = n/Y$ we obtain from (3.3), by termwise integration,

$$\sum_{n=1}^{\infty} t_2(n)n^{-s}e^{-(n/Y)^h} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} H(s+w)Y^w\Gamma(1+\frac{w}{h})\frac{\mathrm{d}w}{w}.$$

We shift the line of integration to $\Re\mathrm{e}\, w = -1/4$ and apply the residue theorem. The pole $w = 0$ of the integrand gives the residue $H(s)$, while the poles of $H(s+w)$ give a total contribution which is $O(1)$ in view of Stirling's formula for the gamma-function. The integral along the line $\Re\mathrm{e}\, w = -1/4$ is bounded, and we obtain

$$\begin{aligned} H(s) &= \sum_{n=1}^{\infty} t_2(n)n^{-s}e^{-(n/Y)^h} + O(1) \\ &= \sum_{n\le T} t_2(n)n^{-s}e^{-(n/Y)^h} + \sum_{T<n\le 2Y} t_2(n)n^{-s}e^{-(n/Y)^h} + O(1). \end{aligned} \tag{3.4}$$

The idea of proof is as follows. We shall prove that

$$\int_{T^{1-\delta}}^{T} |H(\frac12 + it)|^2 t^{-2}\,\mathrm{d}t \gg \log^5 T, \tag{3.5}$$

and then use (3.4) to show that (3.5) gives a contradiction if we assume that (1.8) does not hold, namely that we have

$$\int_1^X \Delta^2(x)\,\mathrm{d}x = o(X^2\log^4 X) \qquad (X\to\infty). \tag{3.6}$$

To prove (3.5) it is enough to prove that

$$\int_M^{2M} |H(\frac12 + it)|^2 t^{-2}\,\mathrm{d}t \gg \log^4 M, \tag{3.7}$$

since (3.7) gives ($M = T2^{-j}, N = [\delta \log T/4]$)

$$\int_{T^{1-\delta}}^{T} |H(\frac{1}{2}+it)|^2 t^{-2}\ \mathrm{d}t \ge \sum_{j=1}^{N} \int_{T2^{-j-1}}^{T2^{-j}} |H(\frac{1}{2}+it)|^2 t^{-2}\ \mathrm{d}t$$
$$\gg \log T \cdot \log^4 T = \log^5 T.$$

Using (1.4) and (2.2) we have

$$\begin{aligned}&\int_{M}^{2M} |H(\frac{1}{2}+it)|^2 t^{-2}\ \mathrm{d}t\\ &\gg M^{-2}\int_{M}^{2M} |\zeta(\frac{1}{2}+it)|^4 |\zeta(1+2it)|^6 |\zeta(2it)|^2\ \mathrm{d}t\\ &\gg M^{-1}\int_{M}^{2M} |\zeta^2(\frac{1}{2}+it)\zeta^4(1+2it)|^2\ \mathrm{d}t.\end{aligned}$$

Now let $F(s) := \zeta^2(s)\zeta^4(2s)$ and use a general lower bound for mean values of Dirichlet series (see e.g. K. Ramachandra [8], [9]; note that the factor $1/n$ is missing in (4.2) of [5]):

(3.8)
$$\frac{1}{M}\int_{M}^{2M} |F(\frac{1}{2}+it)|^2\ \mathrm{d}t \gg \sum_{n\le M/100} \frac{|c(n)|^2}{n}\left(1-\frac{\log n}{\log M}+\frac{1}{\log\log M}\right),$$

where $F(s) = 1+\sum_{n=2}^{\infty} c(n)n^{-s}$ converges for $\Re\mathrm{e} = \sigma \ge \sigma_0, F(s)$ is regular for $\Re\mathrm{e} \ge 1/2, M \le t \le 2M$ and both $F(s) \ll e^{M^D}$ and $c(n) \ll M^D$ hold for some $D > 0$. In our case

$$c(n) \;=\; \sum_{km^2=n} d(k)d_4(m) \;\ge\; d(n),$$

where the divisor function $d_4(n)$ is generated by $\zeta^4(s)$. Hence (3.8) yields

$$\frac{1}{M}\int_{M}^{2M} |H(\frac{1}{2}+it)|^2 t^{-2}\ \mathrm{d}t \gg \sum_{n\le\sqrt{M}} \frac{c^2(n)}{n}\left(1-\frac{\log n}{\log M}+\frac{1}{\log\log M}\right)$$
$$\gg \sum_{n\le\sqrt{M}} \frac{d^2(n)}{n} \gg \log^4 M$$

by partial summation from $\sum_{n\le x} d^2(n) \sim Cx\log^3 x$ ($C > 0$). Thus (3.5) is proved, and it remains to see how it leads to the proof of Theorem 2.

To obtain the left-hand side of (3.5) from (3.4), we shall divide (3.4) by t, square and integrate over $T^{1-\delta} \le t \le T$. We use the mean value theorem for Dirichlet polynomials (see Theorem 5.2 of [4]) to deduce that

$$
\begin{aligned}
&\int_{T^{1-\delta}}^{T} \Big|\sum_{n\le T} t_2(n) n^{-1/2-it} e^{-(n/Y)^h}\Big|^2 t^{-2}\,\mathrm{d}t \\
&\ll T^{2\delta-2}\sum_{j\ge1} 2^{-2j}\int_{2^{j-1}T^{1-\delta}}^{2^jT^{1-\delta}} \Big|\sum_{n\le T} t_2(n) n^{-1/2-it} e^{-(n/Y)^h}\Big|^2\,\mathrm{d}t \\
&\ll T^{2\delta-2}\sum_{j\ge1} 2^{-2j}\sum_{n\le T} t_2^2(n) n^{-1}(n+2^jT^{1-\delta}) \\
&\ll T^{2\delta-3/2+\varepsilon/2}\sum_{j\ge1} 2^{-2j}\sum_{n\le T} t_2(n) n^{-1}(n+2^jT^{1-\delta}) \\
&\ll T^{2\delta+\varepsilon-1/2} \ll T^{-\varepsilon}
\end{aligned}
\tag{3.9}
$$

for δ sufficiently small, where we used the bound (3.2) and the trivial bound

$$\sum_{n\le x} t_2(n) \ll xH(1+\frac{1}{\log x}) \ll x\zeta^2(1+\frac{1}{\log x})\zeta(1+\frac{2}{\log x}) \ll x\log^3 x.$$

It remains to evaluate

$$I := \int_{T^{1-\delta}}^{T} \Big|\sum_{T<n\le 2Y} t_2(n) n^{-1/2-it} e^{-(n/Y)^h}\Big|^2 t^{-2}\,\mathrm{d}t. \tag{3.10}$$

This is done again by the use of the mean value theorem for Dirichlet polynomials. However first we integrate by parts and use (1.1) to obtain

$$
\begin{aligned}
&\sum_{T<n\le 2Y} t_2(n) n^{-1/2-it} e^{-(n/Y)^h} = \int_T^{2Y} x^{-1/2-it} e^{-(x/Y)^h}\,\mathrm{d}T(x) \\
&+ \int_T^{2Y} (K_1\log^2 x + (2K_1+K_2)\log x + K_2 + K_3) x^{-1/2-it} e^{-(x/Y)^h}\,\mathrm{d}x \\
&+ \int_T^{2Y} x^{-1/2-it} e^{-(x/Y)^h}\,\mathrm{d}\Delta(x) = I_1 + I_2,
\end{aligned}
$$

say. By the first derivative test (Lemma 2.1 of [4]) it is seen that $I_1 \ll YT^{-1/2}t^{-1}\log^2 T$. Hence the contribution of I_1 to I will be

$$\ll \int_{T^{1-\delta}}^{\infty} t^{-4}Y^2T^{-1}\log^4 T\,\mathrm{d}t \ll Y^2T^{3\delta-4}\log^4 T \ll T^{-\varepsilon}$$

for $B < 2 - \frac{3}{2}\delta$. In I_2 we use integration by parts and the bound $\Delta(x) \ll x^{2/3}$ (see (1.3)) to obtain

$$I_2 = O(T^{1/6}) + \int_T^{2Y} \Delta(x)e^{-(x/Y)^h}\left(h\Big(\frac{x}{Y}\Big)^h + \frac{1}{2} + it\right)x^{-3/2-it}\,\mathrm{d}x.$$

The contribution of the O–term to I will be negligible, and so will be also the contribution of $h(x/Y)^h + 1/2$ if $B < 3 - 3\delta$. The main contribution to I from I_2 will be from the term it. This is

$$\begin{aligned}
&\int_{T^{1-\delta}}^T \Big|\int_T^{2Y} \Delta(x)e^{-(x/Y)^h}x^{-3/2-it}\,\mathrm{d}x\Big|^2\,\mathrm{d}t\\
&= O(1) + \int_{T^{1-\delta}}^T \Big|\sum_{[T]\le n\le[2Y]}\int_n^{n+1}\Delta(x)e^{-(x/Y)^h}x^{-3/2-it}\,\mathrm{d}x\Big|^2\,\mathrm{d}t\\
&= O(1) + \int_{T^{1-\delta}}^T \Big|\int_0^1 \sum_{[T]\le n\le[2Y]}\Delta(v+n)e^{-((v+n)/Y)^h}(v+n)^{-3/2-it}\,\mathrm{d}v\Big|^2\,\mathrm{d}t.
\end{aligned}$$

By using the Cauchy-Schwarz inequality for integrals and inverting the order of integration it is seen that the last integral does not exceed

$$\begin{aligned}
&\int_0^1\int_{T^{1-\delta}}^T \Big|\sum_{[T]\le n\le[2Y]}\Delta(v+n)e^{-((v+n)/Y)^h}(v+n)^{-3/2-it}\Big|^2\,\mathrm{d}t\,\mathrm{d}v\\
&\ll \int_0^1 \sum_{[T]\le n\le[2Y]}\Delta^2(v+n)e^{-2((v+n)/Y)^h}(v+n)^{-3}(n+T)\,\mathrm{d}v\\
&\ll \int_{[T]}^{[2Y]+1}\Delta^2(x)x^{-2}\,\mathrm{d}x,
\end{aligned}$$

where we used again the mean value theorem for Dirichlet polynomials. If (3.6) holds, then obviously also

$$\int_M^{2M}\Delta^2(x)\,\mathrm{d}x \;=\; o(M^2\log^4 M) \qquad (M\to\infty),$$

consequently we finally obtain from (3.5) ($M = [T]2^{j-1}, Y = T^B$)

$$\begin{aligned}
&\log^5 T \ll \int_{T^{1-\delta}}^T |H(\tfrac{1}{2}+it)|^2t^{-2}\,\mathrm{d}t \ll 1 + \int_{[T]}^{[2Y]+1}\Delta^2(x)x^{-2}\,\mathrm{d}x\\
&\ll 1 + \sum_{j=1}^{O(\log T)}\int_{[T]2^{j-1}}^{[T]2^j}\Delta^2(x)x^{-2}\,\mathrm{d}x \ll 1 + \sum_{j=1,M=[T]2^{j-1}}^{O(\log T)} M^{-2}\int_M^{2M}\Delta^2(x)\,\mathrm{d}x\\
&\ll 1 + \sum_{j=1}^{O(\log T)} o(\log^4 T) \;=\; o(\log^5 T),
\end{aligned}$$

which is the contradiction that proves Theorem 2.

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