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Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients

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RÉSUMÉ. Soit θ un nombre réel de développement en fraction continue

$$\theta = [a_0, a_1, a_2, \dots],$$

et soit

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

une matrice d'entiers tel que $\det M \neq 0$. Si θ est à quotients partiels bornés, alors $\frac{a\theta+b}{c\theta+d} = [a_0^*, a_1^*, a_2^*, \dots]$ est aussi à quotients partiels bornés. Plus précisément, si $a_j \leq K$ pour tout j suffisamment grand, alors $a_j^* \leq |\det(M)|(K+2)$ pour tout j suffisamment grand. Nous donnons aussi une borne plus faible qui est valable pour tout a_j^* avec $j \geq 1$. Les démonstrations utilisent la constante d'approximation diophantienne homogène $L_\infty(\theta) = \limsup_{q\to\infty}(q||q\theta||)^{-1}$. Nous montrons que

$$\frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty\left(\frac{a\theta+b}{c\theta+d}\right) \leq |\det(M)| L_\infty(\theta).$$

ABSTRACT. Let θ be a real number with continued fraction expansion

$$\theta = [a_0, a_1, a_2, \dots],$$

and let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a matrix with integer entries and nonzero determinant. If θ has bounded partial quotients, then $\frac{a\theta+b}{c\theta+d} = [a_0^*, a_1^*, a_2^*, \dots]$ also has bounded partial quotients. More precisely, if $a_j \leq K$ for all sufficiently large j, then $a_j^* \leq |\det(M)|(K+2)$ for all sufficiently large j. We also give a weaker bound valid for all a_j^* with $j \geq 1$. The proofs use the homogeneous Diophantine approximation constant $L_\infty(\theta) = \limsup_{q\to\infty}(q||q\theta||)^{-1}$. We show that

$$\frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty\left(\frac{a\theta+b}{c\theta+d}\right) \leq |\det(M)| L_\infty(\theta).$$

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1. Introduction.

Let θ be a real number whose expansion as a simple continued fraction is

$$\theta = [a_0, a_1, a_2, \dots] ,$$

and set

$$K(\theta) := \sup_{i \geq 1} \; a_i \; , \tag{1.1}$$

where we adopt the convention that $K(\theta) = +\infty$ if θ is rational. We say that θ has *bounded partial quotients* if $K(\theta)$ is finite. We also set

$$K_\infty(\theta) := \limsup_{i \geq 1} \; a_i \; , \tag{1.2}$$

with the convention that $K_\infty(\theta) = +\infty$ if θ is rational. Certainly $K_\infty(\theta) \leq K(\theta)$, and $K_\infty(\theta)$ is finite if and only if $K(\theta)$ is finite.

A survey of results about real numbers with bounded partial quotients is given in [17]. The property of having bounded partial quotients is equivalent to θ being a *badly approximable number*, which is a number θ such that

$$\liminf_{q \to \infty} \; q||q\theta|| > 0 \; ,$$

in which $||x|| = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$ denotes the distance from x to the nearest integer and q runs through integers.

This note proves two quantitative versions of the theorem that if θ has bounded partial quotients and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then $\psi = \frac{a\theta+b}{c\theta+d}$ also has bounded partial quotients.

The first result bounds $K_\infty(\frac{a\theta+b}{c\theta+d})$ in terms of $K_\infty(\theta)$ and depends only on $|\det(M)|$:

Theorem 1.1. *Let θ have a bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then*

$$\frac{1}{|\det M|} K_\infty(\theta) - 2 \leq K_\infty\left(\frac{a\theta+b}{c\theta+d}\right) \leq |\det M| (K_\infty(\theta) + 2) \; . \tag{1.3}$$

The second result upper bounds $K(\frac{a\theta+b}{c\theta+d})$ in terms of $K(\theta)$, and depends on the entries of M:

THEOREM 1.2. *Let θ have bounded partial quotients. If* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is an integer matrix with* $\det(M) \neq 0$, *then*

$$K\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det(M)|(K(\theta) + 2) + |c(c\theta + d)| \ . \tag{1.4}$$

The last term in (1.4) can be bounded in terms of the partial quotient a_0 of θ, since

$$|c\theta + d| \leq |c|(|a_0| + 1) + |d| \leq |ca_0| + |c| + |d| \ .$$

Theorem 1.2 gives no bound for the partial quotient $a_0^* := \lfloor \frac{a\theta+b}{c\theta+d} \rfloor$ of $\frac{a\theta+b}{c\theta+d}$.

Chowla [3] proved in 1931 that $K(\frac{a}{d}\theta) < 2ad(K(\theta)+1)^3$, a result rather weaker than Theorem 1.2.

We obtain Theorem 1.1 and Theorem 1.2 from stronger bounds that relate Diophantine approximation constants of θ and $\frac{a\theta+b}{c\theta+d}$, which appear below as Theorem 3.2 and Theorem 4.1, respectively. Theorem 3.2 is a simple consequence of a result of Cusick and Mendès France [5] concerning the Lagrange constant of θ (defined in Section 2).

The continued fraction of $\frac{a\theta+b}{c\theta+d}$ can be directly computed from that for θ, as was observed in 1894 by Hurwitz [9], who gave an explicit formula for the continued fraction of 2θ in terms of that of θ. In 1912 Châtelet [2] gave an algorithm for computing the continued fraction of $\frac{a\theta+b}{c\theta+d}$ from that of θ, and in 1947 Hall [7] also gave a method. Let $\mathcal{M}(n, \mathbb{Z})$ denote the set of $n \times n$ integer matrices. Raney [15] gave for each $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2, \mathbb{Z})$ with $\det(M) \neq 0$ an explicit finite automaton to compute the additive continued fraction of $\frac{a\theta+b}{c\theta+d}$ from the additive continued fraction of θ.

In connection with the bound of Theorem 1.1, Davenport [6] observed that for each irrational θ and prime p there exists some integer $0 \leq a < p$ such that $\theta' = \theta + \frac{a}{p}$ has infinitely many partial quotients $a_n(\theta') \geq p$. Mendès France [13] then showed that there exists some $\theta' = \theta + \frac{a}{p}$ having the property that a positive proportion of the partial quotients of θ' have $a_n(\theta') \geq p$.

Some other related results appear in Mendès France [11,12]. Basic facts on continued fractions appear in [1,8,10,18].

2. Badly Approximable Numbers

Recall that the continued fraction expansion of an irrational real number

$\theta = [a_0, a_1, \ldots]$ is determined by

$$\theta = a_0 + \theta_0 \ , \quad 0 < \theta_0 < 1 \ ,$$

and for $n \geq 1$ by the recursion

$$\frac{1}{\theta_{n-1}} = a_n + \theta_n \ , \quad 0 < \theta_n < 1 \ .$$

The *n-th complete quotient* α_n of θ is

$$\alpha_n := \frac{1}{\theta_n} = [a_n, a_{n+1}, a_{n+2}, \ldots] \ .$$

The *n-th convergent* $\frac{p_n}{q_n}$ of θ is

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots , a_n] \ ,$$

whose denominator is given by the recursion $q_{-1} = 0, q_0 = 1$, and $q_{n+1} = a_{n+1}q_n + q_{n-1}$. It is well known (see [8, §10.7]) that

$$||q_n\theta|| = |q_n\theta - p_n| = \frac{1}{q_n\alpha_{n+1} + q_{n-1}} \ . \tag{2.1}$$

Since $a_{n+1} \leq \alpha_{n+1} < a_{n+1} + 1$ and $q_{n-1} \leq q_n$, this implies that

$$\frac{1}{a_{n+1} + 2} < q_n||q_n\theta|| \leq \frac{1}{a_{n+1}} \ , \tag{2.2}$$

for $n \geq 0$.

We consider the following Diophantine approximation constants. For an irrational number θ define its *type* $L(\theta)$ by

$$L(\theta) = \sup_{q \geq 1} \ (q||q\theta||)^{-1} \ ,$$

and define the *homogeneous Diophantine approximation constant* or *Lagrange constant* $L_\infty(\theta)$ of θ by

$$L_\infty(\theta) = \limsup_{q \geq 1} \ (q||q\theta||)^{-1} \ .$$

We use the convention that if θ is rational, then $L(\theta) = L_\infty(\theta) = +\infty$. (N.B.: some authors study the reciprocal of what we have called the Lagrange constant.)

The best approximation properties of continued fraction convergents give

$$L(\theta) = \sup_{n \geq 0} \, (q_n ||q_n\theta||)^{-1} \tag{2.3}$$

and

$$L_\infty(\theta) = \limsup_{n \geq 0} \, (q_n ||q_n\theta||)^{-1} \; . \tag{2.4}$$

The set of values taken by $L_\infty(\theta)$ over all θ is called the *Lagrange spectrum* [4]. It is well known that $L_\infty(\theta) \geq \sqrt{5}$ for all θ. If $\theta = [a_0, a_1, a_2, \dots]$, then another formula for $L_\infty(\theta)$ is

$$L_\infty(\theta) = \limsup_{j \to \infty}([a_j, a_{j+1}, \dots] + [0, a_{j-1}, a_{j-2}, \dots , a_1]); \tag{2.5}$$

see [4, p. 1].

There are simple relations between these quantities and the partial quotient bounds $K(\theta)$ and $K_\infty(\theta)$, cf. [16, pp. 22–23].

Lemma 2.1. *For any irrational θ with bounded partial quotients, we have*

$$K(\theta) \leq L(\theta) \leq K(\theta) + 2 \; . \tag{2.6}$$

Proof. This is immediate from (2.2) and (2.3). $\square$

Lemma 2.2. *For any irrational θ with bounded partial quotients*

$$K_\infty(\theta) \leq L_\infty(\theta) \leq K_\infty(\theta) + 2 \; . \tag{2.7}$$

Proof. This is immediate from (2.2) and (2.4). $\square$

Although we do not use it in the sequel, we note that both inequalities in (2.7) can be slightly improved. Since $q_n \leq (a_n + 1)q_{n-1}$, (2.1) yields

$$q_n ||q_n\theta|| \leq \frac{1}{\alpha_{n+1} + \frac{q_{n-1}}{q_n}} \leq \frac{1}{a_{n+1} + 1/(a_n + 1)} \; .$$

Since $a_n \le K_\infty(\theta)$ from some point on, this and (2.4) yield

$$L_\infty(\theta) \ge K_\infty(\theta) + \frac{1}{K_\infty(\theta)+1} \ . \tag{2.8}$$

Next, from (2.1) we have

$$\begin{aligned} q_n||q_n\theta|| &= \frac{q_n}{\alpha_{n+1}q_n + q_{n-1}} \\ &= \frac{1}{a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}}. \end{aligned}$$

Hence

$$(q_n||q_n\theta||)^{-1} = a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}.$$

Let $K = K_\infty(\theta)$. Then for all n sufficiently large we have

$$\alpha_{n+2} \ge 1 + \frac{1}{K+1} = \frac{K+2}{K+1},$$

so

$$\begin{aligned} (q_n||q_n\theta||)^{-1} &\le K + \frac{K+1}{K+2} + 1 \\ &= K + 2 - \frac{1}{K+2}. \end{aligned}$$

We conclude that

$$L_\infty(\theta) \le K_\infty(\theta) + 2 - \frac{1}{K_\infty(\theta)+2} \ . \tag{2.9}$$

3. Lagrange Constants and Proof of Theorem 1.1.

An integer matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(M) \ne 0$, acts as a linear fractional transformation on a real number θ by

$$M(\theta) := \frac{a\theta + b}{c\theta + d} \ . \tag{3.1}$$

Note that $M_1(M_2(\theta)) = M_1M_2(\theta)$.

LEMMA 3.1. *If M is an integer matrix with* $\det(M) = \pm 1$, *then the Lagrange constants of* θ *and* $M(\theta)$ *are related by*

$$L_\infty(M(\theta)) = L_\infty(\theta) \ .$$

Proof. This is well-known, cf. [14] and [5, Lemma 1], and is deducible from (2.5). □

The main result of Cusick and Mendès France [5] yields:

THEOREM 3.2. *For any integer* $m \geq 1$, *let*

$$G_m = \{M \in \mathcal{M}(2, \mathbb{Z}) : |\det(M)| = m\} \ .$$

Then for any irrational number θ,

$$\sup_{M \in G_m} (L_\infty(M(\theta))) = mL_\infty(\theta) \ . \tag{3.2}$$

and

$$\inf_{M \in G_m} (L_\infty(M(\theta))) \geq \frac{1}{m} L_\infty(\theta) \ . \tag{3.3}$$

Proof. Theorem 1 of [5] states that

$$\max_{\substack{a,b,d \\ ad=m \\ 0 \leq b < d}} \left(L_\infty \left(\frac{a\theta + b}{d} \right) \right) = mL_\infty(\theta) \ . \tag{3.4}$$

Let $GL(2, \mathbb{Z})$ denote the group of 2×2 integer matrices with determinant ± 1. We need only observe that for any M in G_m there exists some $\tilde{M} \in GL(2, \mathbb{Z})$ such that $\tilde{M}M = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $a'd' = m$ and $0 \leq b' < d'$. For if so, and $\psi = \frac{a\theta + b}{c\theta + d}$, then Lemma 3.1 gives

$$L_\infty(\psi) = L_\infty(\tilde{M}(\psi)) = L_\infty(\tilde{M}M(\theta)) = L_\infty\left(\frac{a'\theta + b'}{d'} \right) \ ,$$

whence (3.4) implies (3.2). To construct $\tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we must have

$$Ca + Dc = 0 \ .$$

Take $C = \frac{\operatorname{lcm}(a,c)}{a}$ and $D = -\frac{\operatorname{lcm}(a,c)}{c}$. Then $\gcd(C,D) = 1$, so we may complete this row to a matrix $\tilde{\tilde{M}} \in GL(2,\mathbb{Z})$. Multiplying this by a suitable matrix $\begin{bmatrix} \pm 1 & c \\ 0 & \pm 1 \end{bmatrix}$ yields the desired $\tilde{M}$.

The lower bound (3.3) follows from the upper bound (3.2). We use the adjoint matrix

$$M' = \operatorname{adj}(M) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} ,$$

which has $M'M = \det(M)I = mI$ and $\det(M') = \det(M)$. If $\theta' = M(\theta)$, then

$$M'(\theta') = M'(M(\theta)) = M'M(\theta) = \theta .$$

We prove by contradiction. Suppose (3.3) were false, so that for some $M \in G_m$ and some θ we have

$$L_\infty(M(\theta)) < \frac{1}{m} L_\infty(\theta) .$$

This states that

$$mL_\infty(\theta') < L_\infty(M'(\theta')) ,$$

which contradicts (3.2) for θ', since $\det(M') = \det(M) = m$. $\square$

Remark. The lower bound (3.3) holds with equality for some values of θ and not for other values. If for given θ we choose an $M \in G_m$ which gives equality in (3.2), so that $L_\infty(M(\theta)) = mL_\infty(\theta)$, then equality holds in (3.3) for $\theta' = \operatorname{adj}(M)(\theta)$. However, if $L_\infty(\theta) = \sqrt{5}$, as occurs for $\theta = \frac{1+\sqrt{5}}{2}$, then $L_\infty(M(\theta)) \geq L_\infty(\theta)$ for all M; hence (3.3) does not hold with equality when $m \geq 2$.

Proof of Theorem 1.1. Theorem 3.2 gives $L_\infty(M(\theta)) \leq \det(M)L_\infty(\theta)$. Now apply Lemma 2.2 twice to get

$$\begin{aligned} K_\infty(M(\theta)) &\leq L_\infty(M(\theta)) \\ &\leq |\det(M)| L_\infty(\theta) \\ &\leq |\det(M)|(K_\infty(\theta) + 2) . \end{aligned} \tag{3.5}$$

To obtain the lower bound, we use the adjoint $M' = \operatorname{adj}(M) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$, and apply (3.5) with M' and $\theta' = M(\theta)$ to obtain

$$K_\infty(\theta) = K_\infty(M'(M(\theta))) \leq |\det(M')|(K_\infty(M(\theta))) + 2) .$$

Since $|\det(M)| = |\det(M')|$, this yields

$$K_\infty(M(\theta)) \geq \frac{1}{|\det(M)|} K_\infty(\theta) - 2 \ . \quad \square$$

4. Numbers of Bounded Type and Proof of Theorem 1.2

Recall that the *type* $L(\theta)$ of θ is the smallest real number such that $q||q\theta|| \geq \frac{1}{L(\theta)}$ for all $q \geq 1$.

Theorem 4.1. *Let θ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with* $\det(M) \neq 0$, *then*

$$L\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det(M)| L(\theta) + |c(c\theta + d)| \ . \tag{4.1}$$

Proof. Set $\psi = \frac{a\theta+b}{c\theta+d}$. Suppose first that $c = 0$ so that $|\det(M)| = |ad| > 0$. Then $L(\psi) \geq \frac{1}{x}$, where

$$x := q||q\psi|| = q||q\left(\frac{a\theta + b}{d}\right)|| = q|q\left(\frac{a\theta + b}{d}\right) - p| \ . \tag{4.2}$$

We have

$$\begin{aligned} |ad|x &= |aq| \ |aq\theta + (bq - dp)| \\ &\geq |aq| \ ||aq\theta|| \geq \frac{1}{L(\theta)} \ . \end{aligned} \tag{4.3}$$

For any $\epsilon > 0$ we may choose q in (4.2) so that $\frac{1}{x} \geq L(\psi) - \epsilon$. Then

$$|\det(M)| L(\theta) = |ad| L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon \ . \tag{4.4}$$

Letting $\epsilon \to 0$ yields (4.1) when $c = 0$.

Suppose now that $c \neq 0$. Again $L(\psi) \geq \frac{1}{x}$ where

$$x := q||q\psi|| = q|q\left(\frac{a\theta + b}{c\theta + d}\right) - p| \ .$$

We have

$$|c\theta + d|x = q|(qa - pc)\theta - (pd - qb)| \; , \tag{4.5}$$

so that

$$\begin{aligned} |c\theta + d| \left|\frac{qa-pc}{q}\right| x &= |qa - pc| \; |(qa-pc)\theta - (pd-qb)| \\ &\geq |qa - pc| \; ||(qa-pc)\theta|| \; . \end{aligned} \tag{4.6}$$

We first treat the case $qa - pc = 0$. Now

$$\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} qa - pc \\ pd - qb \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; ,$$

since $\det \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = \det(M) \neq 0$. Thus if $qa - pc = 0$ then $|pd - qb| \geq 1$, hence (4.5) gives

$$|c\theta + d|x = q|pd - qb| \geq 1 \; . \tag{4.7}$$

It follows that $qa - pc \neq 0$ provided that

$$\frac{1}{x} > |c\theta + d| \; . \tag{4.8}$$

We next treat the case when $qa - pc \neq 0$. Now from the definition of $L(\theta)$ we see

$$|qa - pc| \; ||(qa-pc)\theta|| \geq \frac{1}{L(\theta)} \; . \tag{4.9}$$

Given $\epsilon > 0$, we may choose q so that $\frac{1}{x} \geq L(\psi) - \epsilon$, and we obtain from (4.6) and (4.9) that

$$|c\theta + d| \; \left|\frac{qa-pc}{q}\right| L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon \; . \tag{4.10}$$

However, the bound

$$\left| q\left(\frac{a\theta + b}{c\theta + d}\right) - p \right| \leq \frac{1}{2}$$

implies that

$$\left|\frac{qa-pc}{c}\right| = \left|q\left(\frac{a}{c}\right)-p\right| \le \left|q\left(\frac{a\theta+b}{c\theta+d}\right)-q\left(\frac{a}{c}\right)\right| + \left|q\left(\frac{a}{c}\right)-p\right|$$
$$\le q|\det(M)|\left|\frac{1}{c(c\theta+d)}\right| + \frac{1}{2} .$$

Multiplying this by $\frac{c}{q}$ and applying it to the left side of (4.10) yields

$$L\left(\frac{a\theta+b}{c\theta+d}\right) - \epsilon \le |\det(M)|L(\theta) + \frac{1}{2}\frac{|c(c\theta+d)|}{q} . \tag{4.11}$$

Letting $\epsilon \to 0$ and using $q \ge 1$ yields

$$L\left(\frac{a\theta+b}{c\theta+d}\right) \le |\det(M)|L(\theta) + \frac{1}{2}|c(c\theta+d)| , \tag{4.12}$$

provided that (4.8) holds. Now (4.8) fails to hold only if

$$L\left(\frac{a\theta+b}{c\theta+d}\right) \le |c\theta+d| . \tag{4.13}$$

The last two inequalities imply (4.1) when $c \ne 0$. □

Proof of Theorem 1.2. Applying Theorem 4.1 and Lemma 2.1 gives

$$\begin{aligned} K\left(\frac{a\theta+b}{c\theta+d}\right) &\le L\left(\frac{a\theta+b}{c\theta+d}\right) \\ &\le |\det(M)|L(\theta) + |c(c\theta+d)| \\ &\le |\det(M)|(K(\theta)+2) + |c(c\theta+d)| , \end{aligned}$$

which is the desired bound. □

Remarks. (1). The proof method of Theorem 4.1 can also be used to directly prove the bounds

$$\frac{1}{|\det(M)|}L_\infty(\theta) \le L_\infty(M(\theta)) \le |\det(M)|L_\infty(\theta) , \tag{4.14}$$

of Theorem 3.2, from which Theorem 1.1 can be easily deduced. The lower bound in (4.14) follows from the upper bound as in the proof of Theorem 3.2. We sketch a proof of the upper bound in (4.14) for the case

$\psi = \frac{a\theta+b}{c\theta+d}$ with $c \neq 0$. For any $\epsilon^* > 0$ and all sufficiently large $q^* \geq q^*(\epsilon^*)$, we have

$$q^*||q^*\theta|| \geq \frac{1}{L_\infty(\theta) + \epsilon^*} \ . \tag{4.15}$$

We choose $q = q_n(\psi)$ for sufficiently large n, and note that

$$q^* = |q_n(\psi)a - p_n(\psi)c| \to \infty$$

as $n \to \infty$, since ψ is irrational. We can then replace (4.9) by (4.15), and then deduce (4.11) with $L(\theta)$ replaced by $L_\infty(\theta) + \epsilon^*$. Letting $q \to \infty$, $\epsilon \to 0$ and $\epsilon^* \to 0$ in that order yields the upper bound in (4.14).

(2). For a given matrix M consider the set of attainable ratios

$$\mathcal{V}(M) := \left\{ \frac{L_\infty(M(\theta))}{L_\infty(\theta)} : \theta \text{ has bounded partial quotients} \right\} \ . \tag{4.16}$$

By Lemma 3.1 the set $\mathcal{V}(M)$ depends only on its $SL(2,\mathbb{Z})$-double coset

$$[M] = \{N_1 M N_2 : N_1, N_2 \in SL(2,\mathbb{Z})\} \ .$$

Theorem 3.2 shows that

$$\mathcal{V}(M) \subseteq \left[\frac{1}{|\det(M)|} \ , \ |\det(M)| \right] \ . \tag{4.17}$$

It is an interesting open problem to determine the set $\mathcal{V}(M)$. Both $|\det(M)|$ and $\frac{1}{|\det(M)|}$ lie in $\mathcal{V}(M)$, as follows from Theorem 3.2 and the remark following it.

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