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Sets of block structure and discrepancy estimates

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Résumé. Soient $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ une suite d'éléments d'un ensemble fini M et $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ une suite d'applications $f_n : M \to M$. Quelle information sur $\mathbf{x}$ et $\mathbf{f}$ permet d'obtenir des estimations de la discrépance de la suite $\mathbf{f}(\mathbf{x}) = (f_n(x_n))_{n\in\mathbb{N}}$? Nous donnons dans cet article des réponses à cette question, en utilisant un résultat qualitatif récent.

Abstract. Given a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ on the finite set M and a sequence $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ of maps $f_n : M \to M$. Which information about $\mathbf{x}$ and $\mathbf{f}$ is suitable for getting estimates for the discrepancy of the sequence $\mathbf{f}(\mathbf{x}) = (f_n(x_n))_{n\in\mathbb{N}}$? The paper's object is, using a recent qualitative result, to give answers to this question.

1. Introduction

Let M be a finite set, w.l.o.g. $M = \{1, \dots, m\}$. For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ on M and a sequence $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ of transformations $f_n : M \to M$ we are going to study the distribution behaviour of the sequence $\mathbf{f}(\mathbf{x}) = \mathbf{y} = (y_n)_{n\in\mathbb{N}}$, $y_n = f_n(x_n)$. The qualitative question is: Which $\mathbf{f}$ have the property that $\mathbf{y}$ is uniformly distributed (u.d.) whenever $\mathbf{x}$ is u.d.? (As general references for the theory of u.d. sequences cf. [Kui-N], [H] or [D-T].) An answer is contained in [W] by means of a characterization of all those so-called u.d.p. (uniform distribution preserving) $\mathbf{f}$. Proposition 2 recalls the result adopted for our purposes. In this paper we show in which way one can derive quantitative versions which are necessary for applications similar to Monte Carlo and Quasi-Monte Carlo methods. More precise: We are looking for functions F_N such that

$$D_N(\mathbf{y}) \leq F_N(\mathbf{x}, \mathbf{f})$$

holds. $D_N(\mathbf{x})$ denotes the discrepancy which, for an arbitrary sequence $\mathbf{x}$ on a finite set M of cardinality $\#M = m$, is defined in the following way. For arbitrary $T \subseteq \mathbb{N}$ put

$$A(N, T, \mathbf{x}, a) = \#\{n \leq N \mid n \in T,\ x_n = a\},$$

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and (for $N \geq \min T$, $T \neq \emptyset$)

$$D_N(\mathbf{x},T) = \max_{a\in M} \left| \frac{A(N,T,\mathbf{x},a)}{\# T \cap (0,N]} - m^{-1} \right| .$$

If $T = \mathbb{N}$ we shortly write $A(N,\mathbf{x},a)$ resp. $D_N(\mathbf{x})$ for the discrepancy. $D_N(\mathbf{x})$ tends to 0 for $N \to \infty$ if and only if $\mathbf{x}$ is u.d. Similarly s-block discrepancy is defined by $D_N^{(s)}(\mathbf{x}) = D_N(\mathbf{x}^{(s)})$ where

$$\mathbf{x}^{(s)} = (x_n^{(s)})_{n\in\mathbb{N}}$$

is considered as a sequence on M^s consisting of the members

$$x_n^{(s)} = (x_n, x_{n+1}, \ldots , x_{n+s-1}).$$

A sequence $\mathbf{x}$ is called s-block u.d. if $D_N^{(s)}(\mathbf{x}) \to 0$, $\mathbf{x}$ is called completely u.d. if this holds for all $s \in \mathbb{N}$.

Section 2 presents Theorem 1 which shows why F_N has to depend on $\mathbf{x}$ in a more complicated way than just via $D_N(\mathbf{x})$. Interesting and much deeper results on similar complexity questions on uniform distribution can be found in [G] and [Ki-Li]. Section 3 is devoted to the notion of sets of block structure which has been introduced in [W] and is useful for our purposes. This notion is closely related to the concept of almost constant sequences which has been studied by Rindler and Losert, cf. for instance [Ri], [Ri-Lo] and [Lo]. They continued investigations due to Rauzy, cf. [Rau], who seems to be the first one who investigated questions on u.d.p. sequences in a similar way. For more references cf. [W]. In this paper (section 3) several simple properties of sets of block structure are proved which are relevant for us. Section 4 analyses how the qualitative characterization of u.d.p. $\mathbf{f}$ in Proposition 2 can be modified in such a way that it can be used for concrete and quantitative discrepancy estimates. The discussion is essentially lead by the results of Section 2 and 3. A discrepancy estimate of the desired type is carried out in the final section 5. The problem is discussed also for s-block discrepancy. One easily gets a characterization of those u.d.p. $\mathbf{f}$ which also preserve s-block or complete uniform distribution.

2. Discrepancy alone is not enough information

Our main question is: In which way should F_N depend on its arguments? If F_N is allowed to be an arbitrary function the setting of the question is too general, since $F_N(\mathbf{x},\mathbf{f}) = D_N(\mathbf{f}(\mathbf{x}))$, in a trivial way, is the optimal solution but not interesting for computations. The first restriction one thinks of is to require that F_N depends in the first argument not on all information about $\mathbf{x}$ but just on the discrepancy: $F_N = F_N(D_N(\mathbf{x}),\mathbf{f})$. But this cannot be done in a reasonable way as Theorem 1 explains.

THEOREM 1. *There is no family of functions* $F_N = F_N(D_N(\mathbf{x}), \mathbf{f})$, $N \in \mathbb{N}$, *each* F_N *depending on* $\mathbf{x}$ *only via* $D_N(\mathbf{x})$ *and on* $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$, $f_n : M \to M$, *in an arbitrary way with the following properties:*

$$D_N(\mathbf{f}(\mathbf{x})) \leq F_N(D_N(\mathbf{x}), \mathbf{f})$$

for all N *and*

$$\lim_{N\to\infty} F_N(D_N(\mathbf{x}), \mathbf{f}) = 0$$

whenever $\mathbf{x}$ *is uniformly distributed and* $\mathbf{f}$ *preserves uniform distribution.*

Proof: Suppose, by contradiction, that such a sequence of F_N exists. For technical convenience suppose $M = \{0, 1\}$ and define $g : M \to M$ by $g(i) \neq i$, $i = 0, 1$. Let $f_n = \mathrm{Id}_M$ if $a_j < n \leq b_j$ and $f_n = g$ if $b_j < n \leq a_{j+1}$, $j = 0, 1, \ldots$, where the sequence

$$0 = a_0 < 1 = b_0 < 2 = a_1 < 4 = b_1 < 6 = a_2 < 12 = b_2 < \ldots$$

is defined in such a way that $a_j = 3a_{j-1}$ and $b_j = 2a_j$ for $j \geq 2$. Then $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ is u.d.p. This follows from the main result in [W], here restated as Proposition 2 in section 4. Consider the sequence $\mathbf{x} = (x_n)_{n\geq 1}$ defined by $x_n = i \in M$ with $n \equiv i \bmod 2$. $\mathbf{x}$ is u.d. and, by assumption,

$$\lim_{N\to\infty} F_N(D_N(\mathbf{x}), \mathbf{f}) = 0.$$

Now define $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ in such a way that $y_n = 1$ if $f_n = \mathrm{Id}_M$ and $y_n = 0$ if $f_n = g$. Hence $f_n(y_n) = 1$ for all $n \in \mathbb{N}$ and $D_N(\mathbf{f}(\mathbf{y})) = \frac{1}{2}$ for all $N \in \mathbb{N}$. Note $D_{a_j}(\mathbf{y}) = D_{a_j}(\mathbf{x}) = 0$ for all $j \geq 1$. For arbitrary $\varepsilon > 0$ and sufficiently large j this yields

$$\frac{1}{2} = D_{a_j}(\mathbf{f}(\mathbf{y})) \leq F_{a_j}(D_{a_j}(\mathbf{y}), \mathbf{f}) = F_{a_j}(D_{a_j}(\mathbf{x}), \mathbf{f}) < \varepsilon,$$

contradiction. **q.e.d.**

Thus one has to make finer investigations of u.d. sequences $\mathbf{x}$ and use properties which fit to u.d.p. sequences $\mathbf{f}$ in the sense of Proposition 2 (cf. section 4). For that we have to turn to the concept of sets with block structure.

3. Sets of block structure and related concepts

For a better understanding of our problem we consider partitions of the set $\mathbb{N}$ of positive integers into blocks. For this reason we use the following notations. If

$$0 = a_0 < a_1 < \ldots$$

is a given sequence of integers let $I_k = (a_{k-1}, a_k] \cap \mathbb{N}$ denote the induced blocks (intervals) and call the family $\mathbf{I} = (I_k)_{k\in\mathbb{N}}$ the partition of $\mathbb{N}$ induced by $\mathbf{a} = (a_k)_{k\in\mathbb{N}}$. If $\#I_k/a_k \to 0$ (or, equivalently, $a_{k+1}/a_k \to 1$) for $k \to \infty$ we call $\mathbf{I} = (I_k)_{k\in\mathbb{N}}$ a short partition. We introduce the abbreviation

$$L(T, \mathbf{I}, N) = \frac{1}{a_{k(N)}} \sum_{k=1}^{k(N)} \min(\#I_k \cap T, \#I_k \setminus T),$$

where $k(N)$ is the greatest index such that $a_{k(N)} \le N$. If this quantity is small this means that almost each block I_k is almost contained either in T or in its complement $\mathbb{N}\setminus T$. Hence it is obvious why we say that T has block structure if $\lim_{N\to\infty} L(T, \mathbf{I}, N) = 0$ for all short partitions $\mathbf{I}$. It is easy to see that this is, for instance, the case if T is compatible with a sequence $\mathbf{a} = (a_k)_{k\in\mathbb{N}}$ with $a_{k+1}/a_k \ge q > 1$ (example: $\mathbf{a}^{(q)} = ([q^k])_{k\in\mathbb{N}}$, $q > 1$), in the following sense: A set $T \subseteq \mathbb{N}$ is called compatible with a sequence $\mathbf{a}$ (or with the induced partition $\mathbf{I} = \mathbf{I}(\mathbf{a}) = (I_k)_{k\in\mathbb{N}}$) if and only if each I_k is contained either in T or in its complement $\mathbb{N} \setminus T$. With $\mathcal{B}$ we denote the system of all sets $T \subseteq \mathbb{N}$ with block structure. Sets of block structure have been characterized in [W] in several ways.

A related topic which is important for us is the distribution of restricted sequences. For $T \subseteq \mathbb{N}$ define

$$\mathrm{dens}_N(T) = \frac{1}{N}\#\{n \le N \mid n \in T\}$$

and for $\mathbf{x}_T = (x_n)_{n\in T}$, $N \ge 1$, call the positive number

$$D_N(\mathbf{x}_T) = \max_{a\in M} \left| \frac{1}{N} A(N, T, \mathbf{x}, a) - \frac{1}{m}\mathrm{dens}_N(T) \right|$$

the restricted discrepancy. We say that $\mathbf{x}_T$ is u.d. w.r.t. T if $D_N(\mathbf{x}_T) \to 0$ for $N \to \infty$. Note that $D_N(\mathbf{x}_T) \le D_N(\mathbf{x}, T)$ where, in general, the inequality is strict.

The sequence of numbers $\mathrm{dens}_N(T) \in [0,1]$, $N \in \mathbb{N}$, has an upper limit $\overline{\mathrm{dens}}(T)$, called upper density of T. Similarly the lower density $\underline{\mathrm{dens}}(T)$ is defined as the lower limit. In case of coincidence of both the common value $\mathrm{dens}(T)$ is called density of T. It is easy to check that, for arbitrary sets $S, T \subseteq \mathbb{N}$ of natural numbers, $d(S,T) = \overline{\mathrm{dens}}(S \triangle T)$ defines a pseudometric d inducing a topology on the power set of $\mathbb{N}$. Here $S \triangle T = (S \setminus T) \cup (T \setminus S)$ denotes the symmetric difference of the involved sets. Note that the topological closure of the singleton $\{\emptyset\}$, for instance, contains exactly the sets T with $\mathrm{dens}(T) = 0$.

Here are descriptions from [W] of sets with block structure:

Proposition 1. *For $T \subseteq \mathbb{N}$ the following three conditions are equivalent:*

(1) $T \in \mathcal{B}$.

(2) *For all* $\varepsilon > 0$ *there is a* $q > 1$ *and an* S *compatible with* $\mathbf{a}^{(q)}$ *such that* $d(S,T) < \varepsilon$.

(3) *For all u.d. sequences* $\mathbf{x}$ *on a finite set* M *with* $\#M \geq 2$ *the restriction* $\mathbf{x}_T$ *is u.d. w.r.t.* T.

Proof: Follows from [W], Theorem 1. **q.e.d.**

The following theorem presents further useful observations on $\mathcal{B}$.

THEOREM 2. *$\mathcal{B}$ is a Boolean set algebra (not a σ-algebra) on $\mathbb{N}$, and, as a subset of the power set of $\mathbb{N}$, it is topologically closed w.r.t. d.*

Proof: $\mathcal{B}$ is not a σ-algebra since, for instance, $A_n = \{2n\} \in \mathcal{B}$, but $A = \bigcup_{n\in\mathbb{N}} A_n \notin \mathcal{B}$. Obviously $\mathcal{B}$ contains $\emptyset, \mathbb{N}$ and, immediately by definition, is closed under complements, i.e. $T \in \mathcal{B}$ implies $\mathbb{N} \setminus T \in \mathcal{B}$. To show that $\mathcal{B}$ is a Boolean set algebra on $\mathbb{N}$ it therefore suffices to prove that $S \cup T \in \mathcal{B}$ under the assumption that $S, T \in \mathcal{B}$:
For any given short partition $\mathbf{I}$ define

$$\begin{aligned} s_k &= \min\left(\#I_k \cap S, \#I_k \setminus S\right), \\ t_k &= \min\left(\#I_k \cap T, \#I_k \setminus T\right) \quad \text{and} \\ u_k &= \min\left(\#I_k \cap (S \cup T)), \#I_k \setminus (S \cup T)\right). \end{aligned}$$

If $s_k = \#I_k \cap S$ and $t_k = \#I_k \cap T$ then $u_k \leq \#(I_k \cap (S \cup T)) \leq s_k + t_k$. In the other case, $s_k = \#I_k \setminus S$ or $t_k = \#I_k \setminus T$, one also gets $u_k \leq \#I_k \setminus (S \cup T) \leq s_k + t_k$. This implies

$$\lim_{N\to\infty} L(S \cup T, \mathbf{I}, N) \leq \lim_{N\to\infty} L(S, \mathbf{I}, N) + \lim_{N\to\infty} L(T, \mathbf{I}, N) = 0,$$

since S and T are supposed to have block structure. Thus $S \cup T \in \mathcal{B}$.

Finally Proposition 1.2 says $\mathcal{B} = \overline{\mathcal{C}}$ for

$$\mathcal{C} = \{T \subseteq \mathbb{N} \mid T \text{ compatible with some } \mathbf{a}^{(q)}, q > 1\},$$

hence $\mathcal{B}$ is indeed a closed set. **q.e.d.**

From the results up to now it follows that, for $T \in \mathcal{B}$ and $\varepsilon > 0$, there is a set $S \in \mathcal{B}$ compatible with some $\mathbf{a} = (a_k)_{k\in\mathbb{N}}$ with $a_{k+1}/a_k \geq q$ for some $q > 1$ such that $d(S,T) < \varepsilon$. Since d is a pseudometric and not a metric the question arises whether one can even take $\varepsilon = 0$. That this is not the case follows from Theorem 3.

THEOREM 3. *There are sets T with block structure such that $d(S,T) > 0$ for all S compatible with some $\mathbf{I} = \mathbf{I}(\mathbf{a})$, $\mathbf{a} = (a_k)_{k\in\mathbb{N}}$ with $a_{k+1}/a_k \geq q > 1$ or, equivalently,*

$$\limsup_{N\to\infty} L(T, \mathbf{I}, N) > 0$$

for all such $\mathbf{I} = \mathbf{I}(\mathbf{a})$.

Proof: Construction of T: For $j, l \geq 1$ define the numbers $r(j,l) = 2^{2j+l} + 2^l$, $s(j,l) = 2^{2j+l} + 2 \cdot 2^l = 2^{2j+l} + 2^{l+1}$ and the sets

$$A_j = \mathbb{N} \cap \bigcup_{l=1}^{\infty} (r(j,l), s(j,l)] \quad \text{and} \quad T_i = \bigcup_{j=1}^{i} A_j.$$

We claim that the set $T = T_\infty = \bigcup_{j=1}^{\infty} A_j$ has the desired properties. (Note that $n \in A_j$ if and only if the binary representation of $n-1$ starts with a block $100\ldots001$ where the number of 0-digits between the first two 1-digits is $2j-1$.)

$T \in \mathcal{B}$: The set A_j is, by definition, compatible with the sequence

$$r(j,1) < s(j,1) < r(j,2) < s(j,2) < r(j,3) < s(j,3) < \ldots.$$

Check that the quotients of succeding members of this sequence are bounded below by $1 + (2^{2j}+1)^{-1} = q_j > 1$. Hence $A_j \in \mathcal{B}$ for all j and thus, by Theorem 2, $T_i \in \mathcal{B}$ for all $i \in \mathbb{N}$. Now observe

$$\begin{aligned} d(T, T_i) &\leq \limsup_{N\to\infty} \operatorname{dens}_N(T \setminus T_i) \leq \limsup_{N\to\infty} \sum_{j \geq i} \operatorname{dens}_N(A_j) \leq \\ &\leq \sum_{j=i}^{\infty} 2^{-2j} = \frac{4^{1-i}}{3} < \varepsilon \end{aligned}$$

for sufficiently large i. Thus $T \in \overline{\{T_i \mid i \in \mathbb{N}\}} \subseteq \overline{\mathcal{B}} = \mathcal{B}$ by Theorem 2.

It is obvious that the both remaining assertions of the theorem are indeed equivalent. We prefer to prove the second one. For that reason suppose now that $\mathbf{I} = \mathbf{I}(\mathbf{a})$, $\mathbf{a} = (a_k)_{k\in\mathbb{N}}$, is a partition with $a_{k+1}/a_k \geq q$ for some $q > 1$. If the quotients a_{k+1}/a_k are not bounded it is easy to see that

$$\limsup_{N\to\infty} L(T, \mathbf{I}, N) \geq \min(\underline{\operatorname{dens}}(T), \underline{\operatorname{dens}}(\mathbb{N} \setminus T)) > 0.$$

Hence we may suppose $a_{k+1}/a_k \leq Q$ for some fixed Q to show that $L(T, \mathbf{I}, N)$ has a positive upper limit.

For each n let k_n be chosen in such a way that $a_{k_n - 1} < 2^n \leq a_{k_n}$. For $q = 1 + \delta$ fix j such that $2^{-2j} < \delta/3$. For $n \geq 2j+2$ and $l = n - 2j$ we distinguish two cases:

First case: $a_{k_n} \leq r(j,l)$. With the notation

$$t(j,l) = s(j,l) + (s(j,l) - r(j,l)) = 2^{2j+l} + 3 \cdot 2^l = 2^{2j+l}(1 + 3 \cdot 2^{-2j})$$

and, using

$$a_k q \geq a_k(1 + 3 \cdot 2^{-2j}) \geq 2^{2j+l}(1 + 3 \cdot 2^{-2j}),$$

we get

$$2^{2j+l} \leq a_{k_n} \leq r(j,l) < s(j,l) < t(j,l) = 2^{2j+l}(1 + 3 \cdot 2^{-2j}) \leq a_{k_n} q \leq a_{k_n+1}.$$

From the definition of T follows

$$(r(j,l),s(j,l)]\cap\mathbb{N}\subseteq I_{k_n+1}\cap T \quad \text{and} \quad (s(j,l),t(j,l)]\cap\mathbb{N}\subseteq I_{k_n+1}\setminus T,$$

thus $\min(\# I_{k_n+1}\cap T, \# I_{k_n+1}\setminus T)\geq 2^l$, furthermore $a_{k_n+1}\leq Q^2 a_{k_n-1}\leq Q^2 2^n$, thus

$$L(T,\mathbf{I},a_{k_n+1})\geq \frac{2^l}{a_{k_n+1}}\geq\frac{2^l}{2^n Q^2}=\frac{1}{2^{2j}Q^2}.$$

Second case: $r(j,l)<a_{k_n}$. Similar to the first case, just replacing j by $j+1$, l by $l-2$ and k_n by k_{n-1} one gets

$$L(T,\mathbf{I},a_{k_n})\geq\frac{2^{l-2}}{a_{k_n}}\geq\frac{2^{l-2}}{2^n Q}=\frac{1}{2^{2j+2}Q}.$$

Combining both cases and choosing $\varepsilon=\min(\frac{1}{2^{2j}Q^2},\frac{1}{2^{2j+2}Q})$ one gets

$$\limsup_{N\to\infty} L(T,\mathbf{I},N)\geq\varepsilon>0. \quad \textbf{q.e.d.}$$

Theorem 3 and the following fact will also be relevant on the search for possible estimates, cf. the discussion in section 4.

THEOREM 4. *Let* $(\mathbf{I}^{(l)})_{l\in\mathbb{N}}$, $\mathbf{I}^{(l)}=\mathbf{I}(\mathbf{a}(l))=(I_k^{(l)})_{k\in\mathbb{N}}$, $\mathbf{a}(l)=(a_k^{(l)})_{k\in\mathbb{N}}$, *be an arbitrary countable family of short partitions. Then there is a set* $T\subseteq\mathbb{N}$ *with* $L(T,\mathbf{I}^{(l)},N)\to 0$ *for all* $l\in\mathbb{N}$ *but* $T\notin\mathcal{B}$.

Proof: Since all $\mathbf{I}^{(l)}$ are short partitions there is a sequence

$$0=N_0<N_1<N_2<\dots$$

such that $\frac{\# I_k^{(i)}}{a_k^{(i)}}<\frac{1}{l^3}$ whenever $i\leq l$ and $a_k\geq N_l$ and with $N_{l+1}/N_l\to\infty$. Furthermore there exists a short partition $J=(J_k)_{k\in\mathbb{N}}=J(b)$, $b=(b_k)_{k\in\mathbb{N}}$, such that sufficintly large l fulfill $\frac{1}{l^2}<\frac{\# J_k}{b_k}\leq\frac{2}{l^2}$ if $J_k\cap(N_l,N_{l+1})\neq\emptyset$. Define

$$T=\bigcup_{k\in\mathbb{N}} J_{2k}.$$

To estimate

$$L(T,\mathbf{I}^{(i)},N)=\frac{1}{a_{k(N)}}\sum_{k=1}^{k(N)}\min(\# I_k^{(i)}\cap T,\# I_k^{(i)}\setminus T),$$

note that only those k contribute a notvanishing term for which $b_{k'}\in I_k^{(i)}$ for some k'. Using the bounds for $\# I_k^{(i)}/a_k^{(i)}$ and $\# J_k/b_k$ we conclude

$$\limsup_{N\to\infty}L(T,\mathbf{I}^{(i)},N)\leq\limsup_{k'\to\infty}\frac{2\# I_k^{(i)}}{\# J_{k'}}\leq\limsup_{l\to\infty}\frac{2/l^3}{1/l^2}=0.$$

On the other hand let $\mathbf{I} = \mathbf{I}(\mathbf{a})$ be a short partition with $\frac{1}{l} < \frac{\# I_k}{a_k} \le \frac{2}{l}$ if $I_k \cap (N_l, N_{l+1}) \neq \emptyset$ for sufficiently large l. Since $\frac{\# J_{k'}}{\# I_k} \to 0$ for $J_{k'} \cap I_k \neq \emptyset$ and $k, k' \to \infty$ a further use of the bounds for $\# I_k^{(i)}/a_k^{(i)}$ and those for $\# I_k / \# a_k$ yields

$$\liminf_{k\to\infty} \min(\# I_k \cap T, \# I_k \setminus T) \ge \frac{\# I_k}{3},$$

implying $\liminf_{N\to\infty} L(T, \mathbf{I}, N) \ge \frac{1}{3}$. Thus T does not have block structure. **q.e.d.**

4. Reconsidering a qualitative result

For given $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$, $f_n : M \to M$, we consider the sets $T_\pi = \{n \in \mathbb{N} \mid f_n = \pi\}$ for every permutation $\pi \in S_M$ of the set M (S_M denotes the symmetric group acting on M), the set $C = \{n \in \mathbb{N} \mid f_n \equiv c_n\}$ of all n for which f_n is a constant function taking some value $c_n \in M$, and the remaining set $D = \mathbb{N} \setminus C \setminus T$ where $T = \bigcup_{\pi \in S_M} T_\pi$. Now we are ready to formulate the characterization of u.d.p. sequences $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ of maps which is the motivation for our quantitative results.

PROPOSITION 2. *The sequence $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$, $f_n : M \to M$, is u.d.p. if and only if the following three conditions hold:*
1. dens$(D) = 0$.
2. All sets C, D and T_π, $\pi \in S_M$, have block structure.
3. The sequence $c_C = (c_n)_{n\in C}$ is u.d. w.r.t. C.

Proof: Follows, by specialization, immediately from [W], Theorem 3. **q.e.d.**

If we say that $\mathbf{f}$ is compatible modulo constant maps with a partition $\mathcal{P}$ of $\mathbb{N}$ if on each $T \in \mathcal{P}$ either all f_n coincide or all f_n are constant maps then we can reformulate condition 2 of Proposition 2 as follows: $\mathbf{f}$ is compatible modulo constant maps with a finite partition of $\mathbb{N}$ into sets with block structure.

In order to use the conditions from Proposition 2 for discrepancy estimates we have to analyse how they can be made concrete by means of suitable quantities. For condition 1 and condition 3 this is clear: dens$(D) = 0$ if and only if the quantity dens$_N(D)$ tends to 0 and $c = (c_n)_{n\in C}$ is u.d. w.r.t. C if and only if the quantity $D_N(c_C)$ tends to 0.

Which quantity describes how good the block structure of some $T \in \mathcal{B}$ is? This question is not so trivial. The reason is that in the definition for $T \in \mathcal{B}$ the limit relation for $L(T, \mathbf{I}, N)$ must hold for all, even uncountably many, short partitions $\mathbf{I}$. Unfortunately the behaviour of one fixed $\mathbf{I}$ or even of countably many of them does not give enough information. This has been illustrated by Theorem 4.

Looking at certain special examples of sequences with block structure (as the sequences $\mathbf{a}^{(q)}$ for $q > 1$), one observes that $L(T, \mathbf{I}, N) \to 0$ for some suitably chosen "long" partition $\mathbf{I}$ with $a_{k+1}/a_k \geq q > 1$. The hope that this might happen for all $T \in \mathcal{B}$ has been destroyed by Theorem 3. But what we may expect is that $L(T, \mathbf{I}, N)$ gets small in dependence on N and δ if $a_{k+1}/a_k \leq 1 + \delta$. (Let as call partitions $\mathbf{I} = \mathbf{I}(\mathbf{a})$ with this property δ-partitions.) This approach indeed is possible as Theorem 5.1 will show. Theorem 5.2 is an application of Theorem 5.1 to u.d.p. $\mathbf{f}$.

But how to match a u.d. $\mathbf{x}$ with a u.d.p. $\mathbf{f}$? Condition 2 in Proposition 2 tells us that in a u.d.p. $\mathbf{f}$ the permutations $\pi : M \to M$ occur (with few exceptions) in long blocks. On such blocks the distribution behaviour of the x_n is the same as the behaviour of the $f_n(x_n) = \pi(x_n)$ (up to a fixed permutation π of the elements of M). Hence we have to guarantee, that the blocks are long enough such that the uniform distribution of $\mathbf{x}$ can be observed. This will be the object of Theorem 5.3.

THEOREM 5. (1) *Suppose that T has block structure and $\varepsilon > 0$ is given. Then there exist a positive $\delta = \delta_T(\varepsilon) > 0$ and a natural number $N_T(\varepsilon, \delta)$ such that*

$$L(T \setminus I_1, \mathbf{I}, N) < \varepsilon$$

holds for all $N \geq N_T(\varepsilon, \delta)$ and all δ-partitions $\mathbf{I} = (I_k)_{k \in \mathbb{N}}$.

(2) *Suppose that $\varepsilon > 0$ and $\mathbf{f}$ u.d.p. are given. Then there is a positive $\delta = \delta_{\mathbf{f}}(\varepsilon) > 0$ and a natural number $N_1 = N_{\mathbf{f}}(\varepsilon, \delta)$ such that for all $N \geq N_1$ and all δ-partitions $\mathbf{I}$ there is an $\mathbf{f}' = (f'_n)_{n \in \mathbb{N}}$ compatible with $\mathbf{I}$ modulo constant maps such that*

$$dens_N\left(\{n \leq N \mid f_n \neq f'_n\}\right) < \varepsilon + \frac{a_1}{N}.$$

Furthermore we may assume $D_N(\mathbf{f}(\mathbf{x})_C) < \varepsilon$ for such N.

(3) *Suppose that $\mathbf{x}$ is u.d. and $\varepsilon, \delta > 0$ are given. Then there is a positive integer $N_0 = N_{\mathbf{x}}(\varepsilon, \delta) \in \mathbb{N}$ such that for all $I = (N', N_1] \cap \mathbb{N}$ with $N' \geq N_0$ and $N_1 \geq (1 + \delta)N'$ and all $N \geq N_1$*

$$D_N(\mathbf{x}, I) < \varepsilon.$$

Proof:

(1) Suppose, by contradiction, that there is some $\varepsilon > 0$ such that for all $\delta > 0$ and all N there is a δ-partition $\mathbf{I}(\delta, N) = (I_k(\delta, N))_{k \in \mathbb{N}}$ and an $N' = N'(\delta, N) \geq N$ such that

$$L(T \setminus I_1(\delta, N), \mathbf{I}(\delta, N), N') \geq \varepsilon.$$

By a standard diagonal argument (similar as in the proof of Theorem 4) the partitions $\mathbf{I}^{(l)} = \mathbf{I}(\frac{1}{l}, N_l)$ with sufficiently rapidly increasing $N_l > lN_{l-1}$ and corresponding N_l' can be combined to a short partition $\mathbf{I}$ with

$$L(T, \mathbf{I}, N_l) \geq L(T, \mathbf{I}(\frac{1}{l}, N_l), N_l') - \frac{N_{l-1}}{N_l} \geq \varepsilon - \frac{1}{l},$$

implying

$$\lim_{N\to\infty} \sup L(T, \mathbf{I}, N) \geq \varepsilon > 0,$$

contradiction.

(2) Let $\mathcal{S} = \{C\} \cup \{T_\pi \mid \pi \in S_M\}$ where $C, T_\pi \in \mathcal{S}$ and $D \notin \mathcal{S}$ are as above. All $S \in \mathcal{S}$, by Proposition 2, have block structure. Thus part 1 of the theorem guarantees that there are, for each $S \in \mathcal{S}$ and for $\varepsilon_1 = \frac{\varepsilon}{2(m!+2)}$, numbers $\delta_S(\varepsilon_1)$ and $N_S(\varepsilon_1, \delta_S(\varepsilon_1))$ with the properties stated there. Let $\delta = \delta_{\mathbf{f}}(\varepsilon) > 0$ be the minimum of the $\delta_S(\varepsilon_1)$, $S \in \mathcal{S}$, w.l.o.g. $\delta_S(\varepsilon_1) < \varepsilon/2$. Furthermore, by Proposition 2, there is a natural number $N' = N_D(\varepsilon_1, \delta_D(\varepsilon_1))$, such that $\mathrm{dens}_N(D) < \varepsilon_1$ for all $N \geq N'$. Define

$$N_{\mathbf{f}}(\varepsilon, \delta) = \max_{S\in\mathcal{S}\cup\{D\}} N_S(\varepsilon_1, \delta_S(\varepsilon_1)).$$

Now choose any δ-partition $\mathbf{I} = \mathbf{I}(\mathbf{a})$. We have to find an $\mathbf{f}' = (f_n')_{n\in\mathbb{N}}$ with the properties stated in the theorem. For all $n \in I_1$ let f_n' be the same arbitrary but fixed mapping. For $n \in I_k$, $k \geq 2$ distinguish two cases:
Case 1: If $\#\{n \in I_k \mid f_n = \pi\} > \#I_k/2$ for some $\pi \in S_M$ put $f_n' = \pi$ for all $n \in I_k$.
Case 2: Otherwise define $f_n' \equiv c_n'$ for some constant c_n' where $c_n' = c_n$ for the case that $f_n \equiv c_n$ is constant.
By definition $\mathbf{f}'$ is compatible with $\mathbf{I}$ modulo constant maps. We have to estimate $\mathrm{dens}_N(X)$ for $X = \{n \in \mathbb{N} \mid f_n \neq f_n'\}$ and $N \geq N_{\mathbf{f}}(\varepsilon, \delta)$. Let $A_k = \#X \cap I_k$ and, for $S \in \mathcal{S}$, $A_{S,k} = \min(\#I_k \cap S, \#I_k \setminus S)$. In case 1 we have $A_k = A_{T_\pi,k}$ for some $\pi \in S_M$. In case 2 we have $A_k = \#(I_k \setminus C)$ and we distinguish two subcases: If $\#(I_k \cap C) \geq \#(I_k \setminus C)$ then $A_k = A_{C,k}$. In the other subcase we get

$$A_k \leq \sum_{S\in\mathcal{S}} A_{S,k} + \#(I_k \cap D).$$

Combining all cases and estimating very generously we conclude

$$\begin{aligned}\mathrm{dens}_N(X) &\le \tfrac{1}{N}\left(\textstyle\sum_{k\ge 2: a_k\le N} A_k + \delta N\right) + \tfrac{a_1}{N} \le \\ &\le \delta + \tfrac{a_1}{N} + \tfrac{1}{N}\textstyle\sum_{S\in\mathcal{S}\cup\{D\}}\sum_{k:a_k\le N} A_{S,k} + \mathrm{dens}_N(D) \le \\ &\le \delta + \tfrac{a_1}{N} + \mathrm{dens}_N(D) + \textstyle\sum_{S\in\mathcal{S}} L(S\setminus I_1, \mathbf{I}, N) < \\ &< \delta + \tfrac{a_1}{N} + (m!+2)\varepsilon_1 < \\ &< \varepsilon + \tfrac{a_1}{N},\end{aligned}$$

which is the assertion of the second part of the theorem.

(3) Follows with the same method as Lemma 1 in [W]. **q.e.d.**

5. Discrepancy estimates and s-blocks

Using Theorem 5 we are now in the position to deduce a discrepancy estimate for $\mathbf{f}(\mathbf{x})$ of the following type:

THEOREM 6. *Given a u.d. sequence* $\mathbf{x}$ *and a u.d.p. sequence* $\mathbf{f}$ *of maps. For arbirarily given* $\varepsilon > 0$ *let* $\delta = \delta_{\mathbf{f}}(\varepsilon)$ *(w.l.o.g.* $\delta < 1$*) and* $N_1 = N_{\mathbf{f}}(\varepsilon,\delta)$ *(w.l.o.g.* $N_1 \ge \frac{2}{\varepsilon\delta}$*) be as in Theorem 5.2. Furthermore let* $\delta_0 \in (0, \delta - \frac{2}{\varepsilon N_1})$ *and* $N_0 < N' < N_1$ *where* $N_0 = N_{\mathbf{x}}(\varepsilon, \delta_0)$ *is as in Theorem 5.3. Then for all* $N \ge N_1$ *the discrepancy of the sequence* $\mathbf{f}(\mathbf{x})$ *can be estimated by*

$$D_N(\mathbf{f}(\mathbf{x})) \le 6\varepsilon + 2\frac{N'}{N} + \delta.$$

Proof: Fix any partition $\mathbf{I} = \mathbf{I}(\mathbf{a})$ with $1 + \delta_0 \le a_{k+1}/a_k \le 1 + \delta$ and $\varepsilon N_1/2 \le a_1 \le \varepsilon N_1$. (Note that this is possible by the assumption on δ_0 since $(\delta - \delta_0)a_1 \ge 1$.) Let $\mathbf{f}'$ be as in Theorem 5.2 compatible with $\mathbf{I}$ modulo constant maps and let the sets C, D, T_π, T resp. C', D', T'_π, T' be associated to $\mathbf{f}$ resp. $\mathbf{f}'$ as in section 4. Define for arbitrary N

$$K = K(N) = \{k \mid I_k \subseteq T' \cap (N', N]\},$$

and the numbers k_0 and k_1 by $a_{k_0-1} < N' \le a_{k_0}$ resp. $a_{k_1} < N \le a_{k_1+1}$. Observe for $N \ge N_1$ the following relations.

$D_N(\mathbf{f}(\mathbf{x})) \le \mathrm{dens}_N(\{n \mid f_n \ne f'_n\}) + D_N(\mathbf{f}'(\mathbf{x}))$

$\mathrm{dens}_N(\{n \mid f_n \ne f'_n\}) < \varepsilon + \frac{a_1}{N}$ for $N \ge N_{\mathbf{f}}(\varepsilon,\delta)$ (Theorem 5.2),

$D_N(\mathbf{f}'(\mathbf{x})) \le \max_{k\in K} D_N(\mathbf{f}'(\mathbf{x}), I_k) + R$,

$R = D_N(\mathbf{f}'(\mathbf{x})_{C'\cup(0,a_{k_0}]\cup(a_{k_1},N]})$,

$D_N(\mathbf{f}'(\mathbf{x}), I_k) = D_N(\pi_k(\mathbf{x}), I_k) = D_N(\mathbf{x}, I_k) < \varepsilon$

if $k \in K$, $f'_n = \pi_k \in S_M$ and $n \in I_k$,

$R \le D_N(\mathbf{f}(\mathbf{x})_C) + \mathrm{dens}_N(\{n \mid f_n \ne f'_n\}) + \frac{a_{k_0}}{N} + \frac{N - a_{k_1}}{N}$,

$D_N(\mathbf{f}(\mathbf{x})_C) < \varepsilon$ (Theorem 5.2),

$\frac{a_{k_0}}{N} = \frac{N'}{N} + \frac{a_{k_0} - N'}{N'}\frac{N'}{N} \le \frac{N'}{N}(1+\delta)$ and

$\frac{N - a_{k_1}}{N} \le \delta.$

Combination of these estimates gives

$$D_N(\mathbf{f}(\mathbf{x})) \leq \varepsilon + \frac{a_1}{N} + \varepsilon + \varepsilon + \varepsilon + \frac{a_1}{N} + \frac{N'}{N}(1+\delta) + \delta < 6\varepsilon + 2\frac{N'}{N} + \delta. \quad \textbf{q.e.d.}$$

Since ε and δ can be chosen arbitrarily small, this result gives a method to determine how large N must be such that $D_N(\mathbf{f}(\mathbf{x}))$ gets smaller than any given positive value. For given u.d. $\mathbf{x}$ and u.d.p. $\mathbf{f}$ one has to know the integer functions $N_{\mathbf{x}}(\varepsilon, \delta)$ and $N_{\mathbf{f}}(\varepsilon, \delta)$ which exist by Theorem 5.

It is clear that the explicit bounds for $D_N(\mathbf{f}(\mathbf{x}))$ stemming from the proofs of Theorem 5 and Theorem 6 are not optimal. But a more careful estimation would have taken so much technical effort that the view to the main object would have been obscured by boring calculations. What we want to emphasize is that functions $N_{\mathbf{x}}(\varepsilon)$ and $N_{\mathbf{f}}(\varepsilon)$ depending only on one argument ε do not give enough information to guarantee $D_N(\mathbf{f}(\mathbf{x})) < \varepsilon$, but the functions $N_{\mathbf{x}}(\varepsilon, \delta)$ and $N_{\mathbf{f}}(\varepsilon, \delta)$ guaranteed by Theorem 5 and used in Theorem 6 do.

Finally we have to investigate how our methods apply to s-block discrepancy. If one assumes that $\mathbf{f}$ is u.d.p. then the generalizations are straight forward. The only modifications are as follows:

In estimating $D_N^{(s)}(\mathbf{f}(\mathbf{x}))$ one has to look at the blocks

$$y_n^{(s)} = (f_n(x_n), f_{n+1}(x_{n+1}), \ldots, f_{n+s-1}(x_{n+s-1})).$$

If only one of the x_i or f_i, $i = n, \ldots, n+s-1$, does not behave regularly, this may affect not only one but s members in the sequence $\mathbf{y}^{(s)} = (y_n^{(s)})_{n\in\mathbb{N}}$. Hence discrepancy estimates have to take this into account by a factor s. It is clear that, for every fixed $s \in \mathbb{N}$, the asymptotic definition of block structure causes that this problem does not affect the nature of the results. Of course we have to modify the definition of restricted discrepancy in the following way: For $T \subseteq \mathbb{N}$ and $a \in M^s$ define

$$A^{(s)}(N, T, \mathbf{x}, a) = \#\{n \leq N \mid x_n^{(s)} = a \text{ and } n, n+1, \ldots, n+s-1 \in T\}$$

and

$$D_N^{(s)}(\mathbf{x}_T) = \max_{a\in M^s} \left| \frac{1}{N} A^{(s)}(N, T, \mathbf{x}, a) - \frac{1}{m^s} \mathrm{dens}_N(T) \right|.$$

$\mathbf{x}$ is called s-block u.d. w.r.t. T if $D_N^{(s)}(\mathbf{x}_T) \to 0$ for $N \to \infty$. Similarly one calls $\mathbf{x}$ completely u.d. w.r.t. T if this is the case for all $s \in \mathbb{N}$.

In the natural way we call $\mathbf{f}$ s-block resp. complete u.d.p. if $\mathbf{f}(\mathbf{x})$ is s-block resp. completely u.d. whenever $\mathbf{x}$ is s-block resp. completely u.d. With the same notations as in Proposition 2 the qualitative result is the following:

THEOREM 7. *Let* $\mathbf{f}$ *be u.d.p. Then* $\mathbf{f}$ *is s-block resp. completely u.d.p. if and only if* $c_C = (c_n)_{n\in C}$ *is s-block resp. completely u.d. w.r.t.* C.

Proof: Carry out the program described above. We omit details. **q.e.d.**

Note that in Theorem 7 we have assumed that $\mathbf{f}$ is u.d.p. It is a conjecture but yet unproved whether s-block or complete u.d.p. always implies u.d.p.

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