Journal de Théorie des Nombres
de Bordeaux **10** (1998), 273–285

On traces of the Brandt-Eichler matrices

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Résumé. On calcule le nombre d'idéaux localement principaux de norme donnée dans une classe d'ordres quaternioniques définis, et la trace des matrices de Brandt–Eichler correspondant à ces ordres. Pour application, on calcule ainsi le nombre de représentations d'un entier algébrique comme la norme d'un ordre quaternionique défini de nombre de classes égale à un. On obtient aussi des relations sur le nombre de classes pour certains corps à multiplication complexe.

Abstract. We compute the numbers of locally principal ideals with given norm in a class of definite quaternion orders and the traces of the Brandt–Eichler matrices corresponding to these orders. As an application, we compute the numbers of representations of algebraic integers by the norm forms of definite quaternion orders with class number one as well as we obtain class number relations for some CM-fields.

0. Introduction

Let R be the ring of integers in a global field K and let Λ be an R-order in a quaternion algebra A over K. Recall that A is a central simple algebra of dimension four over K and Λ is a subring of A containing R, finitely generated as an R-module and such that $K\Lambda = A$. The paper is concerned with numbers of locally principal one-sided ideals in Λ with given reduced norm. These numbers play an important role in different arithmetical contexts. In particular, they are related to the traces of the Brandt–Eichler matrices corresponding to Λ and to the numbers of elements in Λ representing a given element in R by means of the reduced norm.

In Section 1, we recall some necessary notions related to orders, in particular, to Brandt–Eichler matrices. In Section 2, we show how to compute the traces of these matrices for a broad class of quaternion orders. In Section 3, we apply our computational results in two different ways.

First of all, if Λ is a totally definite order of class number one (for the definitions see Section 1), we get exact formulae for the numbers of elements

Manuscrit reçu le 2 avril 1998.

in Λ with given reduced norms. In particular, this gives a unified method of proving formulae for numbers of integral representations by quaternary quadratic forms when such formulae can be expected, for example, by sums of four squares (Jacobi) or by $x^2 + y^2 + 2z^2 + 2t^2$ (Liouville) in the case of the rational integers, and in similar cases over the integers in algebraic number fields. Non-analytical proofs of such formulae in different special cases over the rational integers can be found in [6], Chap. IX, [12] and [14].

The second application is related to Eichler's trace formula for traces of the Brandt–Eichler matrices corresponding to the order Λ. If Λ is totally definite of class number one, then the trace formula (which we explain in Section 1) relates the number of left ideals in Λ with given reduced norm to the class numbers of some maximal commutative suborders of it. When the number of left ideals with given norm is known, the trace formula gives a class number relation. In the case of the rational integers, the class number relations obtained in this way remind of the well-known class number relations proved by analytical means by Kronecker and generalized by Gierster (see [7, 10]). We show how to get such class number relations for rings of algebraic integers in CM-fields, that is, quadratic non–real extensions of totally real finite extensions of the rational numbers. They give a recursive method for computing class numbers of the rings involved in these relations. An interesting point is that in some cases the class numbers of all CM-fields containing a given totally real algebraic number field are involved in the class number relations resulting from the trace formula.

I express my thanks to Stefan Johansson for reading the manuscript and pointing out a computational mistake in [3] (see (3.3)).

1. Quaternion orders

Let R be a Dedekind ring with quotient field K. Assume that K is a global field and A is a quaternion algebra over K. We denote by Tr the reduced trace, and by Nr the reduced norm from A to K.

Recall that I is a locally principal left Λ-ideal in A if $I_\mathfrak{p} = \Lambda_\mathfrak{p}\alpha_\mathfrak{p}$, where $I_\mathfrak{p}$ and $\Lambda_\mathfrak{p}$ are completions of I and Λ at non-zero prime ideals $\mathfrak{p}$ in R and $\alpha_\mathfrak{p} \in A_\mathfrak{p}$, where $A_\mathfrak{p}$ is the completion of A at $\mathfrak{p}$. The right order of I is the R-order $O_r(I) = \{\alpha \in A : I\alpha \subseteq I\}$. Let $I_1 = \Lambda, \ldots, I_h$ represent all isomorphism classes of locally principal left Λ-ideals with corresponding right orders $\Lambda_1 = \Lambda, \ldots, \Lambda_h$. Here $h = h(\Lambda)$ is called the class number of Λ. Notice that the class number could be defined by means of locally principal right Λ-ideals giving the same value of $h(\Lambda)$.

If I is a Λ-ideal in A, then its norm $Nr(I)$ is the R-ideal generated by the reduced norms of the elements of I. Let $\mathfrak{a}$ be an ideal in R. Let us recall that the Brandt–Eichler matrix

$$M_\Lambda(\mathfrak{a}) = [\rho_{kl}(\mathfrak{a})]$$

has as its elements $\rho_{kl}(\mathfrak{a})$ the numbers of locally principal left ideals in Λ_k, which are isomorphic to $I_k I_l^{-1}$ and have norm equal to $\mathfrak{a}$ for $k, l = 1, \dots, h(\Lambda)$.

Let $\iota(\Lambda, \mathfrak{a})$ be the number of locally principal left ideals in Λ whose norm is equal to $\mathfrak{a}$. Observe that $\iota(\Lambda_k, \mathfrak{a}) = \rho_{kk}(\mathfrak{a})$. These numbers are finite, since K is a global field.

One of the main objectives of the paper is to compute the traces of the Brandt–Eichler matrices:

$$Tr_\Lambda(\mathfrak{a}) = \sum_{k=1}^{h(\Lambda)} \iota(\Lambda_k, \mathfrak{a}). \tag{1.1}$$

Notice that if $\mathfrak{a}$ is not principal, then the sum is equal to 0. If $\mathfrak{a} = Ra$, we shall also use notations $Tr_\Lambda(a)$ and $\iota(\Lambda, a)$ instead of $Tr_\Lambda(Ra)$ and $\iota(\Lambda, Ra)$.

Eichler's trace formula depends on finiteness assumptions on A. A will be called totally definite if for each R-order Λ in A, the group Λ^*/R^* is finite, where Λ^* and R^* denote the groups of units in Λ and R. It is not difficult to prove that if there is one order in A having this finiteness property, then A is totally definite. In fact, A is totally definite if and only if A does not satisfy the Eichler condition (see [4], p. 718 for the definition of the Eichler condition, and [9], Satz 2 for a proof of the equivalence).

Let

$$\mathcal{N}(\Lambda, a) = \{\lambda \in \Lambda : Nr(\lambda) = a\}$$

for $a \in R$. It is not difficult to prove that for a totally definite A this set is finite. Now let $\mathcal{P}(\Lambda, a)$ be the set of the minimal polynomials for all $\lambda \in \mathcal{N}(\Lambda, a)$. If f is the minimal polynomial of $\lambda \in \mathcal{N}(\Lambda, a)$, then let $S_f = R[X]/(f)$ and $L_f = S_f \otimes_R K$. Let $S_f \subseteq S \subset L_f$, where S is an R-order. An optimal embedding $\varphi : S \to \Lambda$, is an injective R-algebra homomorphism such that $\Lambda/\varphi(S)$ is R-projective. Let $e(S, \Lambda)$ be the number of orbits for the action by conjugation of Λ^* on the optimal embeddings: $\varphi \mapsto \lambda\varphi\lambda^{-1}$ for $\lambda \in \Lambda^*$. Now we are ready to formulate Eichler's trace formula for the Brandt–Eichler matrices (see [9], Satz 10):

Theorem 1.2. *Let Λ be an R-order in a central simple totally definite quaternion K-algebra A. Then*

$$Tr_\Lambda(a) = \delta_a M_\Lambda + \frac{1}{2} \sum_{S \neq R} \frac{1}{|S^*/R^*|} h(S) e_{U(\Lambda)}(S, \Lambda),$$

where the sum is over all S such that $S_f \subseteq S \subset L_f$ for $f \in \cup_{k=1}^{m} \mathcal{P}(\Lambda, a_k)$, $a_1 = a, \dots, a_m$ represent all orbits for the action by multiplication of R^{*2}

on the set R^*a, $\delta_a = 1$ *or* 0 *depending on whether* a *is a square in* R *or not,*

$$M_\Lambda = \sum_{k=1}^{h(\Lambda)} \frac{1}{|\Lambda_k^*/R^*|},$$

$h(S)$ *is the class number of the locally principal ideals in* S *and* $e_{U(\Lambda)}(S,\Lambda) = \prod_{\mathfrak{p}} e(S_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})$, $\mathfrak{p} \in SpecR$, $\mathfrak{p} \neq (0)$.

Notice that $a = 1$ in the last theorem gives an expression for the class number of Λ (see (1.1)).

2. Norms of locally principal ideals

In this section, we show how in some cases, it is possible to easily compute the left hand side in Eichler's trace formula. Throughout the whole section, we assume that R is a complete discrete valuation ring with maximal ideal $\mathfrak{m} = (\pi)$ and finite residue field $R/\pi R$. We also assume that A is a quaternion K-algebra. Let q be the number of elements in $R/\pi R$. For simplicity, the number of principal left ideals in Λ whose norm is equal to (π^m) will be denoted by $\iota(\Lambda, m)$ (later we return to the notation from Section 1: $\iota(\Lambda, \mathfrak{m}^m)$). The following result is well-known (see e.g. [9], §2):

Proposition 2.1. *Let* Λ *be a maximal* R*-order in* A*, and let* m *be a non-negative integer.*

(*a*) *If* A *is a matrix algebra, then* $\iota(\Lambda, m) = 1 + q + \ldots + q^m$.

(*b*) *If* A *is a skewfield, then* $\iota(\Lambda, m) = 1$.

It is well-known that if $J(\Lambda)$ denotes the Jacobson radical of Λ, and Λ is not a maximal order in a matrix algebra, then $\Lambda/J(\Lambda)$ is isomorphic to $R/\pi R \times R/\pi R$ or $R/\pi R$ or to a quadratic field extension of $R/\pi R$. We write $e(\Lambda) = 1$ or 0 or -1, respectively, to distinguish between these three cases. Let us start with the case $e(\Lambda) = 1$.

Proposition 2.2. *Let* Λ *be a quaternion* R*-order such that* $e(\Lambda) = 1$ *and* $d(\Lambda) = (\pi^n)$ *for* $n \geq 1$*. If* $0 \leq m < n$*, then*

$$\iota(\Lambda, m) = (m+1)q^m,$$

and if $m \geq n$*, then*

$$\iota(\Lambda, m) = \sum_{n-2}^{m} a_k q^k,$$

where $a_m = 2n$, $a_i = 2$ *for* $n-1 < i < m$, $a_{n-1} = (m-n+1)(1-n)+(2-n)$ *and* $a_{n-2} = (m-n+1)(n-1)$.

Proof. The result may be proved by the same method as its special case for $n = 1$ in [9] , §2. Such a proof is somewhat tedious but straightforward. □

Assume now that Λ is a non-maximal R-order in A such that $e(\Lambda) \neq 1$. Then $\Lambda/J(\Lambda)$ is a field and there is a unique minimal order $M(\Lambda)$ containing Λ (see [1], (1.12)). If $e(\Lambda) = -1$, then the number of principal (left) ideals in Λ with given norm can be computed in the following recursive way:

Proposition 2.3. *Let Λ be a non-maximal R-order such that $e(\Lambda) = -1$. If $m \geq 2$, then*

$$\iota(\Lambda, m) = [M(\Lambda)^* : \Lambda^*]\iota(M(\Lambda), m-2).$$

Moreover, $\iota(\Lambda, 0) = 1$, $\iota(\Lambda, 1) = 0$, and

$$[M(\Lambda)^* : \Lambda^*] = \begin{cases} q^2 - q & \text{if } M(\Lambda) \text{ is maximal in a split } A, \\ q^2 & \text{otherwise.} \end{cases}$$

Proof. Let the norm of $\Lambda\alpha$ be (π^m), where $m \geq 1$. It follows from (1.13) – (1.15) in [2] that $\alpha = \pi\alpha'$, where $\alpha' \in M(\Lambda)$. Thus $m \geq 2$ and $\Lambda\alpha \mapsto M(\Lambda)\alpha'$ is a surjective map on the set of principal left $M(\Lambda)$-ideals with norm (π^{m-2}). It remains to show that the inverse image in Λ of an $M(\Lambda)$-ideal $M(\Lambda)\alpha'$ consists of $[M(\Lambda)^* : \Lambda^*]$ elements. Let $I_1 = \Lambda\alpha'_1\pi$ and $I_2 = \Lambda\alpha'_2\pi$ be Λ-ideals such that $M(\Lambda)\alpha'_1 = M(\Lambda)\alpha'_2$. Then $\alpha'_2 = \varepsilon\alpha'_1$, where $\varepsilon \in M(\Lambda)^*$. Notice now that if $\Lambda\alpha'\pi \subseteq \Lambda$, then $\Lambda\varepsilon\alpha'\pi \subseteq M(\Lambda)\varepsilon\alpha'\pi \subseteq M(\Lambda)\pi \subseteq \Lambda$ according to (2.2) in [B1]. Thus $\Lambda\varepsilon\alpha'\pi$ are Λ-ideals for all $\varepsilon \in M(\Lambda)^*$ and a standard argument for counting the number of orbits under a group action (here $M(\Lambda)^*$) shows that the number of them equals $[M(\Lambda)^* : \Lambda^*]$. The last formula is very easy to check (see [2] , (3.3)). □

The case of quaternion orders with $e(\Lambda) = 0$ is, as usual, more involved. Recall that Λ is called a Bass order if each left Λ-ideal I in A is projective over its left order $O_l(I) = \{\alpha \in A : \alpha I \subseteq I\}$ (the right version of this definition gives the same notion – see [4], (37.8) and p.782). If Λ is maximal or $e(\Lambda) \neq 0$, then Λ is a Bass order (see e.g. [1]), so regarding the Bass orders, it remains to investigate the case $e(\Lambda) = 0$. Defining $M^2(\Lambda) = M(M(\Lambda))$, we have the following result:

Proposition 2.4. *Let Λ be a Bass R-order such that $e(\Lambda) = 0$. If $m \geq 2$, then*

$$\iota(\Lambda, m) = \begin{cases} [M^2(\Lambda)^* : \Lambda^*]\iota(M^2(\Lambda), m-2) & if \quad e(M(\Lambda)) = 0, \\ [M(\Lambda)^* : \Lambda^*]\iota(M(\Lambda), m-2) & if \quad e(M(\Lambda)) = -1, \\ [M(\Lambda)^* : \Lambda^*]\iota(M(\Lambda), m-1) & if \quad e(M(\Lambda)) = 1. \end{cases}$$

Moreover, $\iota(\Lambda, 1) = [M(\Lambda)^* : \Lambda^*] = q - e(M(\Lambda))$ and $[M^2(\Lambda)^* : \Lambda^*] = q^2 - e(M^2(\Lambda))q$.

Proof. If $e(M(\Lambda)) \neq 1$ and $m \geq 2$, then the arguments are similar to those in the proof of (2.3). The only difference is that if $I = \Lambda\alpha$ has norm (π^m), then (2.19) in [2] gives $\alpha = \pi\alpha'$, where $\alpha' \in M^2(\Lambda)$ for $e(M(\Lambda)) = 0$, and

$\alpha' \in M(\Lambda)$ for $e(M(\Lambda)) = -1$. In the first case, $\Lambda\varepsilon\alpha$ is a principal ideal in Λ for any $\varepsilon \in M^2(\Lambda)^*$, since $\pi M^2(\Lambda) \subseteq \Lambda$ by [1], (4.6). In the second case, similar statement follows for any $\varepsilon \in M(\Lambda)$, since $\pi M(\Lambda) \subseteq \Lambda$ by [1], (4.1). Now the final conclusion is exactly the same as in the proof of Proposition 2.3 with $M(\Lambda)^*$ replaced by $M^2(\Lambda)^*$ in the first case. (Notice that Λ^* need not be normal in $M^2(\Lambda)^*$.)

Assume now that $e(M(\Lambda)) = 1$ and $m \geq 2$. In this case, it follows from (1.1) and (4.2) in [1] that Λ is isomorphic to the order consisting of matrices:

$$\begin{bmatrix} a & b \\ \pi c & d \end{bmatrix}$$

such that $a, b, c, d \in R$ and $\pi | a - d$. If $I = \Lambda\alpha$ has norm (π^m), where $m \geq 1$, then it is easy to see that $\alpha \in J(\Lambda) = J(M(\Lambda))M(\Lambda)\sigma$, where

$$\sigma = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}.$$

Thus I defines an ideal $M(\Lambda)\alpha\sigma^{-1}$ in $M(\Lambda)$ with norm (π^{m-1}). The arguments now leading to the recursive relation are exactly the same as at the end of the proof of Proposition 2.3.

Now let $m = 1$, and let the norm of $I = \Lambda\alpha$ be (π). Let us show that $J(\Lambda) = M(\Lambda)\alpha$. Since $\Lambda/J(\Lambda)$ is a field and $M(\Lambda)J(\Lambda) = J(\Lambda)$ (see [1],(1.12)), we get $\alpha \in J(\Lambda)$ and $M(\Lambda)\alpha \subseteq J(\Lambda)$. But the orders of the (additive) groups $M(\Lambda)/M(\Lambda)\alpha$ and $M(\Lambda)/J(\Lambda)$ are equal (see [1], (4.1)), so $J(\Lambda) = M(\Lambda)\alpha$. We now see that the extensions $M(\Lambda)\alpha$ and $M(\Lambda)\alpha'$ of two principal ideals in Λ with norm (π) must be equal, so using similar arguments as before, we get that the number of them is $[M(\Lambda)^* : \Lambda^*]$. The indices of Λ^* in $M(\Lambda)^*$ and $M^2(\Lambda)^*$ can be calculated by elementary considerations (see [2] , (2.5) and (2.12)). $\square$

In order to apply the trace formula to all quaternion orders with class number 1, it is necessary to consider one more case.

Proposition 2.5. *If $\Lambda = R + \pi\Lambda'$, where Λ' is an R-order containing Λ, then*

$$\iota(\Lambda, m) = [\Lambda'^* : \Lambda^*]\iota(\Lambda', m-2),$$

for $m \geq 2$, and $\iota(\Lambda, 1) = 0$.

Proof. Assume that $I = \Lambda\alpha \subseteq \Lambda$ has norm (π^m), where $m \geq 1$. Expressing α with respect to an R-basis for Λ' and taking its norm, it is easy to see that $\alpha = \pi\alpha'$, where $\alpha' \in \Lambda'$. Thus, $m \geq 2$, and one can proceed as in the proof of Proposition 2.3 considering a surjection of the left principal ideals in Λ with norm (π^m) onto the principal left ideals in Λ' with norm (π^{m-2}). $\square$

3. Applications

As noted in the introduction, we consider two applications of the computations in Section 2. First, we show how to get formulae for numbers of integral representations by norm forms of quaternion orders in a unified way. As a special case, we get purely algebraic proofs of well-known classical results. The case of sums of four squares was studied by many authors using both algebraic and analytical methods ([12, 6, 10, 8, 5]). The case of forms corresponding to maximal orders over the rational integers was treated in [14]. The second application, which combines the computations in Section 2 with Eichler's trace formula for the Brandt–Eichler matrices, results in a series of class number relations for integers in CM-fields. These relations also need the knowledge of the embedding numbers, which appear on the right hand side of the trace formula.

Let R be a principal ideal ring whose quotient field K is global and let Λ be a totally definite R-order of class number $h(\Lambda) = 1$. If $a \in R$, then

$$Tr_\Lambda(a) = \iota(\Lambda, (a)) = \prod_{\mathfrak{p}} \iota_{\mathfrak{p}}(\Lambda, (a)), \tag{3.1}$$

where $\iota_{\mathfrak{p}}(\Lambda, (a))$ are the numbers of (principal) left ideals with norm (a) in the completions $\Lambda_{\mathfrak{p}}$ of Λ at all non-zero prime ideals $\mathfrak{p}$ of R (see [4], (37.8) and p. 782). Notice that $\iota_{\mathfrak{p}}(\Lambda, (a)) = 1$ if $a \notin \mathfrak{p}$, so the product above is well-defined. Let

$$r_{Nr}(a) = |\{\lambda \in \Lambda : Nr(\lambda) = a\}|$$

be the number of representations of a by the reduced norm from Λ to R. Assume that each totally positive unit in R is the norm of an element of Λ (according to Hasse–Schilling–Maass theorem such a unit is at least the norm of a unit in $K\Lambda$). Then it is easy to see that

$$r_{Nr}(a) = |\Lambda^1| Tr_\Lambda(a),$$

where Λ^1 denotes the group of units in Λ whose reduced norm is equal to 1. In fact, our assumption easily implies that if a is totally positive and $(a) = Nr(I)$ for an ideal I in Λ, then $I = (\lambda)$, where $a = Nr(\lambda)$. The number of different choices of λ (when a and I are fixed) is, of course, equal to $|\Lambda^1|$.

Let $\mathfrak{d}$ be an ideal in R. Denote by $\sigma_{\mathfrak{d}}(a)$ the sum of the norms of the ideals dividing (a), which are relatively prime to the ideal $\mathfrak{d}$. Observe that by the norm of an ideal $\mathfrak{I}$ in R, we mean here the number of elements in $R/\mathfrak{I}$. Using (2.1)(a) and (3.1), the last formula can be expressed as follows:

$$r_{Nr}(a) = |\Lambda^1|\sigma_{\mathfrak{d}(\Lambda)}(a) \prod_{\mathfrak{p}|\mathfrak{d}(\Lambda)} \iota_{\mathfrak{p}}(\Lambda, (a)), \tag{3.2}$$

where $\mathfrak{d}(\Lambda)$ is the discriminant of the order Λ. This general formula gives a very elementary proof of many classical results about the numbers of representations of algebraic integers by some quaternary quadratic forms whose coefficients are algebraic integers. The method for studying representations by some integral quaternary quadratic forms using quaternion orders over the integers is well-known and was applied by Hurwitz [12] for sums of four squares and Dickson [6], Kap. IX, in many other particular cases of such forms. Recently, it was also discussed in [14] in case of maximal orders over the integers. Let us consider some specific examples.

Examples 3.3. (a) According to [3], there are 24 isomorphism classes of definite quaternion orders of class number one over the integers[1]. Among them there are 5 maximal orders Λ with discriminants $\mathfrak{d}(\Lambda) = (p)$, where $p = 2, 3, 5, 7, 13$. Since according to Proposition 2.1 (b), $\iota_p(\Lambda, (a)) = 1$, (3.2) gives

$$r_{Nr}(a) = |\Lambda^1|\sigma_p(a),$$

where $\sigma_p(a)$ is simply the sum of positive divisors of a, $a > 0$, which are relatively prime to p, and $|\Lambda^1| = 24$ for $p = 2$ and 8 for the remaining values of p.

It is not difficult to prove that among the norm forms of the 24 isomorphism classes of orders with class number one, there are only 4 represented by diagonal forms: $x^2 + y^2 + cz^2 + ct^2$ for $c = 1, 2, 3$ and $x^2 + 2y^2 + 2z^2 + 4t^2$ (see [3], Theorem 1).

For example, if $c = 2$, then $f = x^2 + y^2 + 2z^2 + 2t^2$ corresponds to the order $\Lambda_f = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}(j + k) + \mathbb{Z}(j - k)$ with discriminant $\mathfrak{d}(\Lambda_f) = 8$, where $i^2 = j^2 = -1$ and $ij = -ji = k$. We have

$$\Lambda_f \subset \Lambda \subset \Gamma,$$

where $\Lambda = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ is the minimal overorder of Λ_f, which is contained in the unique maximal order $\Gamma = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\sigma$ with $\sigma = \frac{1+i+j+k}{2}$. Since $e(\Lambda_f \otimes \widehat{\mathbb{Z}}_2) = 0$, using Proposition 2.4, we get

$$\iota_2(\Lambda_f, a) = \begin{cases} 1 & \text{if } 2 \nmid a, \\ 2 & \text{if } 2|a \text{ and } 4 \nmid a, \\ 6 & \text{if } 4|a. \end{cases}$$

[1] There is a mistake in [3] – the quadratic form $x^2 + y^2 + 3z^2 + xy$ defines a quaternion order with class number two. Thus the number of isomorphism classes of definite quaternion orders over the integers is 24 and not 25 as claimed there.

Noting that $|\Lambda_f^1| = 4$, we get from (3.2) Liouville's formula (see [7] , p.227):

$$r_f(a) = \begin{cases} 4\sigma_{(2)}(a) & \text{if} \quad a \equiv 1 (\text{mod } 2), \\ 8\sigma_{(2)}(a) & \text{if} \quad a \equiv 2 (\text{mod } 4), \\ 24\sigma_{(2)}(a) & \text{if} \quad a \equiv 0 (\text{mod } 4). \end{cases}$$

If $c = 1$, then $\Lambda = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$. We have $\mathfrak{d}(\Lambda) = 4$ and $|\Lambda^1| = 8$. Λ is a Bass order with $e(\Lambda \otimes \widehat{\mathbb{Z}}_2) = 0$. Therefore applying (2.4), we get

$$\iota_2(\Lambda, a) = \begin{cases} 1 & \text{if } 2 \nmid a, \\ 3 & \text{if } 2 | a. \end{cases}$$

The formula (3.2) gives the well-known Jacobi theorem:

$$r_f(a) = \begin{cases} 8\sigma_{(2)}(a) & \text{if} \quad a \equiv 1 (\text{mod } 2), \\ 24\sigma_{(2)}(a) & \text{if} \quad a \equiv 0 (\text{mod } 2), \end{cases}$$

where $f = x^2 + y^2 + z^2 + t^2$.

Using the same methods, it is equally easy to get similar results for all quaternary forms corresponding to other orders with class number 1 in the case of rational integers (compare [6], Kap. IX for $x^2 + y^2 + cz^2 + ct^2$ with $c = 1, 3$, [14] for the five maximal orders and [5]).

(b) All definite quaternion orders with class number one over the rings of integers in totally real algebraic number fields are not known. However, all maximal orders (or even hereditary) with this property over the real quadratic extensions of the rational numbers are known (see [16], p. 155). The formula (3.2) in these cases is very simple like in the case of rational integers considered above. A more interesting case, which was studied intensively by analytical methods (see [10, 8]) is the case of quaternion orders whose norm forms are sums of four squares over the integers R in real quadratic fields. Let $\Lambda = R + Ri + Rj + Rk$, where as before $i^2 = j^2 = -1$, $k = ij = -ji$. We have $\mathfrak{d}(\Lambda) = (4)$, so Λ is never hereditary and its properties depend on the ramification of (2) in R. It is well-known that the genus of the quadratic form $x^2 + y^2 + z^2 + t^2$ consists of only one class if $R = \mathbb{Z}[\sqrt{2}]$ or $R = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ (see [8], Satz 25). It is not difficult to check that the class number $h(\Lambda) = 1$ only in the second case. This result may be proved using Eichler's trace formula (see (1.2)) or known results about the number of classes in the genus of the quadratic form $x^2 + y^2 + z^2 + t^2$ (see [8], Satz 25) in combination with a classical correspondence between proper equivalence classes of quaternary quadratic forms and submodules of quaternion algebras (see [13], Sec. 6).

If $R = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, then $\mathfrak{d}(\Lambda) = (2)^2$ is a square of a prime ideal in R. We have

$$\Lambda \subset \Lambda' \subset \Gamma,$$

where Γ is a maximal order with $\mathfrak{d}(\Gamma) = (1)$ and Λ' a hereditary order. Let $a = 2^m a'$, where $2 \nmid a'$. Using $e(\Lambda \otimes \widehat{R}_{(2)}) = 0$, (2.2) and (2.4), we get

$$\iota_2(\Lambda, a) = \begin{cases} 1 & \text{if } m = 0, \\ 2^{2m+1} - 5 & \text{if } m \geq 1. \end{cases}$$

Since $|\Lambda^1| = 8$, the formula (3.2) gives

$$r_f(a) = \begin{cases} 8\sigma_{(2)}(a) & \text{if } m = 0, \\ 8(2^{2m+1} - 5)\sigma_{(2)}(a) & \text{if } m \geq 1. \end{cases}$$

This formula was proved by Götzky (see [10]) and later by Dzewas (see [8], Satz 35) using analytical methods.

As we noted in the introduction, Eichler's trace formula can be considered as a relation between the class numbers of the maximal commutative suborders of a totally definite order. Such class number relations become explicit when the left hand side of the formula can be effectively computed. Let us consider some examples.

Examples 3.4. (a) Let $\Lambda = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$, where $i^2 = j^2 = -1$ and $ij = -ji = k$. In order to apply (1.2), we have to consider all S corresponding to the polynomials $g = x^2 + sx + N$ with negative discriminant. Moreover, if s is odd, then S_g can not be embedded into Λ, since the trace of any element in Λ is even. So let $s^2 - 4N = -2^{2r+2} f_s^2 d_s$, where f_s is odd, and d_s is square-free. Then the overorders S of $S_g = \mathbb{Z}[2^r f_s \sqrt{-d_s}]$, which have an optimal embedding into Λ are exactly $\mathbb{Z}[f\sqrt{-d_s}]$ with $f | f_s$ and $d_s \not\equiv 7 (\mathrm{mod}\ 8)$. Assume for simplicity that $N > 1$ is square-free. Using the embedding numbers $e_2(S, \Lambda)$ computed e.g. in [2], (3.10), and the trace formula, we get the following relations:

$$\sum_s \sum_{f | f_s} \varepsilon(f, d_s) h(\mathbb{Z}[f\sqrt{-d_s}]) = 2\sigma_{(2)}(N)\iota_2(\Lambda, N),$$

where

$$\varepsilon(f, d) = \begin{cases} 0 & \text{if} \quad d \equiv 7 (\mathrm{mod}\ 8), \\ 2 & \text{if} \quad d \equiv 3 (\mathrm{mod}\ 8), \\ 3 & \text{if} \quad d \equiv 1 \text{ or } 2 (\mathrm{mod}\ 4), \end{cases}$$

and $\varepsilon(1,1) = \frac{1}{2} e_2(\mathbb{Z}[\sqrt{-1}], \Lambda) = 3/2$. The right hand side is given by the Jacobi theorem (see above), that is, $\sigma_{(2)}(N)$ is the sum of the odd divisors of N, and $\iota_2(\Lambda, N) = 1$ if N is odd and 3 if N is even.

(b) If Λ is a maximal order (there are only 5 isomorphism classes of quaternion orders with class number 1 corresponding to $d(\Lambda) = 2, 3, 5, 7, 13$), then similar class number relations immediately follow from the original version of Eichler's trace formula and well-known results on norms of ideals in maximal orders gathered in Proposition 2.1. Let us formulate this general result in a form suitable for applications.

Let $d(\Lambda) = l$ be a prime and let $s^2 - 4N = l^{2r} f_s^2 \Delta_s$, where $SGD(l, f_s) = 1$ and Δ_s is the discriminant of $\mathbb{Q}(\sqrt{s^2 - 4N})$. Denote by χ_Δ the Kronecker character of the quadratic field extension of $\mathbb{Q}$ with discriminant Δ. Let $\mathbb{Z}[\omega_s]$ be the maximal order in the field $\mathbb{Q}(\sqrt{\Delta_s})$. Define $\sigma_{(l)}(N)$ as the sum of all divisors to N relatively prime to l. With these notations, we have:

$$\sum_s \sum_{f|f_s} \varepsilon_l(f, \Delta_s) h(\mathbb{Z}[f\omega_s]) = 2\sigma_{(l)}(N),$$

where $\varepsilon_l(f, \Delta_s) = e_l(\mathbb{Z}[\omega_s], \Lambda)$ for $(f, \Delta_s) \neq (1, -4), (1, -3)$, that is,

$$\varepsilon_l(f, \Delta_s) = \begin{cases} 0 & \text{if} \quad \chi_\Delta(l) = 1, \\ 1 & \text{if} \quad \chi_\Delta(l) = 0, \\ 2 & \text{if} \quad \chi_\Delta(l) = -1, \end{cases}$$

and $\varepsilon_l(1, -4)$ are the one half of the above values, while $\varepsilon_l(1, -3)$ are the one third of them.

It is interesting to note that comparing the relations for $\Lambda = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ in (a) with the relations for the maximal order containing it, one gets new relations when N is odd. Of course, the same is true for other discriminants, when the relations corresponding to maximal orders with discriminant l are compared with relations corresponding to its unique suborder of index l (see [1], (4.2)) for N not divisible by l.

(c) The class number relations are particularly simple when the quaternion algebra is unramified at all finite primes. For example, let $R = \mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{5}}{2}$ and let Λ be a maximal order in the quaternion algebra $A = K + Ki + Kj + Kk$, where $i^2 = j^2 = -1, ij = -ji = k$ and $K = \mathbb{Q}(\sqrt{5})$. One easily checks that $\mathfrak{d}(\Lambda) = (1)$. If a quadratic R–order S can be embedded into Λ, then it must be totally definite over R. If it is generated over R by a zero of $g(x) = x^2 + sx + a$, then $s^2 - 4a$ must be totally negative, and as a consequence, a must be totally positive. Let $s^2 - 4a = -f_s^2 \Delta_s$, where $f_s \in R$ and $-\Delta_s$ is the discriminant of $K(\sqrt{s^2 - 4a})$. Let $S = R[\omega_s]$ be the maximal order in the field $K(\sqrt{-\Delta_s})$. We have $e_{U(\Lambda)}(S, \Lambda) = 1$ according to the known results on the embedding numbers (see [2], but observe that in the present case all $\Lambda_\mathfrak{p}$ are maximal in matrix algebras, so the situation is very simple). Assume that a is not a square in R. Then the trace formula gives

$$\sum_s \sum_{(f)|(f_s)} \varepsilon(f, \Delta_s) h(R[f\omega_s]) = 2\sigma(a),$$

where $\sigma(a)$ is the sum of the norms of the ideal divisors of (a), and

$$\varepsilon(f,\Delta_s)=\begin{cases}\frac{1}{2} & \text{if} \quad (f)=R \text{ and } \Delta_s\in R^{*2},\\ \frac{1}{3} & \text{if} \quad (f)=R \text{ and } \Delta_s\in 3R^{*2},\\ \frac{1}{5} & \text{if} \quad (f)=R \text{ and } \Delta_s\in (2+\omega)R^{*2},\\ 1 & \text{otherwise.}\end{cases}$$

The numbers $\varepsilon(f,\Delta_s)$ correspond to different indices $|S^*/R^*|$. Let S^1 and $R^1=\{\pm 1\}$ be the groups of roots of unity in S and R. Then $S^*=S^1R^*$ according to [11], Satz 25, which shows that $|S^*/R^*|=|S^1/R^1|$. It is not difficult to check that $|S^1|=2r$, where $r=1,2,3,5$ and the last three cases correspond to three extensions of $\mathbb{Q}(\sqrt{5})$ by i, $i\sqrt{3}$ and $\sqrt{-(2+\omega)}$.

The relations obtained in (a) and (b), as well as all other relations of that type obtained by means of quaternion orders with class number one over the integers, are similar to the well-known class number relations proved by Kronecker and extended by Gierster (see [7], p. 108, and also [15]). The relations in (c), as well as in other cases related to totally definite orders of class number one over the rings of integers in totally real global fields may be used in numerical computations of class numbers for CM-fields.

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