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Continued Fractions, Multidimensional Diophantine Approximations and Applications

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Résumé. Cet article rassemble des résultats généraux d'approximation diophantienne, sur les meilleures approximations et leurs applications à la théorie de répartition uniforme.

Abstract. This paper is a brief review of some general Diophantine results, best approximations and their applications to the theory of uniform distribution.

1. Diophantine approximations.

1.1. One-dimensional approximations.

1.1.1. *Lagrange spectrum.* Let α be an irrational number. Dirichlet's theorem states that there are infinitely many positive integers q such that

$$||q\alpha|| < \frac{1}{q}$$

holds, where $||\cdot||$ denotes the distance to the nearest integer. Hurwitz obtained a more precise result: for any irrational number α, the inequality

$$||q\alpha|| < \frac{1}{\sqrt{5}q}$$

has infinitely many solutions in q. Moreover, there is a countable set of numbers α for which this inequality is an exact one, that is, for any positive ε there are only finitely positive integers q such that the inequality

$$||q\alpha|| < \left(\frac{1}{\sqrt{5}} - \varepsilon\right)\frac{1}{q}$$

holds.

We define the *Lagrange spectrum* to be the set of the real numbers λ for which there exists $\alpha = \alpha(\lambda)$ such that the inequality

$$||q\alpha|| < \lambda\frac{1}{q}$$

has infinitely many solutions, and for any positive ε the inequality

$$||q\alpha|| < (\lambda - \varepsilon)\,\frac{1}{q}$$

has only a finite number of solutions. It is well-known that Lagrange spectrum has a discrete part

$$\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \dots ,$$

and the minimal λ for which there are uncountably many $\alpha = \alpha(\lambda)$ is $\lambda = 1/3$. Also it is well-known that Lagrange spectrum contains an interval $[0, \lambda^*]$.

Moreover, for any decreasing function ψ satisfying $\psi(y) = o(y^{-1})$, as y tends to infinity, there is an uncountable set of real numbers α such that the inequality

$$||q\alpha|| < \psi(q)$$

has infinitely many solutions, but for any $\varepsilon > 0$, the stronger inequality

$$||q\alpha|| < (1 - \varepsilon)\psi(q)$$

has only a finite number of solutions.

One can find the above results in [5]. All of them can be obtained from the continued fraction expansion [14].

1.1.2. *Best approximations and continued fractions.* Any real number α may be written as

$$\alpha = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \dots}}}$$

where $b_0 \in \mathbb{Z}$ and, for $j > 0$, b_j are a nonnegative integers. For convenience, we use the notation

$$\alpha = [b_0; b_1, b_2, b_3, \dots].$$

This representation is infinite and unique when α is irrational. If α is rational, we have $\alpha = [b_0; b_1, b_2, b_3, \dots , b_t]$, and this representation is unique if we impose the condition $b_t \neq 0, 1$.

Convergents to α of the order ν are defined as

$$\frac{p_\nu}{q_\nu} = [b_0; b_1, b_2, b_3, \dots , b_\nu]$$

A simple theorem states that these fraction and only these form *the best approximations*, that is the relation

$$||q_\nu \alpha|| = \min_{q<q_\nu} ||q\alpha||$$

holds for the denominators q_ν and only for them (see [14]). We now give two other easy facts.

Theorem 1. *We have*

$$||q_\nu \alpha|| \asymp (q_{\nu+1})^{-1}, \qquad \textit{(in order of approximation).}$$

Proposition 2. *We have*

$$\Delta_\nu = \begin{vmatrix} p_\nu & q_\nu \\ p_{\nu+1} & q_{\nu+1} \end{vmatrix} = \pm 1.$$

1.1.3. *Klein polygons.* We now consider the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let $(q,a) \in \mathbb{Z}^2$ be a primitive point ($\gcd(q,a) = 1$) and $q, a > 0$. We define the two angles φ_+ and φ_- by

$$\varphi_+ = \left\{ Z = (x,y) \in \mathbb{R}^2 : \ x \geq 0, y \geq \frac{a}{q} x \right\},$$

$$\varphi_- = \left\{ Z = (x,y) \in \mathbb{R}^2 : \ y \geq 0, y \leq \frac{a}{q} x \right\},$$

Klein polygons $\mathcal{K}_+(a,q)$ and $\mathcal{K}_-(a,q)$ are defined to be respectively the following borders

$$\partial(\mathrm{conv}(\varphi_+ \cap (\mathbb{Z}^2 \backslash \{0\})))$$

and

$$\partial(\mathrm{conv}(\varphi_- \cap (\mathbb{Z}^2 \backslash \{0\})))$$

which consist of finite (nontrivial) intervals.

We now define $\Delta(a,q)$ to be the domain:

$$\Delta(a,q) = \Big\{ Z = (x,y) \in \mathbb{R}^2 : \ x > 0, y > 0,$$
$$Z \notin \mathrm{conv}(\varphi_+ \cap (\mathbb{Z}^2 \backslash \{0\})), \quad Z \notin \mathrm{conv}(\varphi_- \cap (\mathbb{Z}^2 \backslash \{0\})) \Big\}.$$

We have

Theorem 3 ([7, 9])**.** 1. *The vertices of $\mathcal{K}_-(a,q)$, (different from (q,a)) are integer points of the form $(q_{2\mu}, p_{2\mu})$, where $(p_{2\mu}/q_{2\mu})$ is the 2μ-th convergent to a/q.*
2. *The vertices of $\mathcal{K}_+(a,q)$ (different from (q,a)) are integer points of the form $(q_{2\nu+1}, p_{2\nu+1})$, where $p_{2\nu+1}/q_{2\nu+1}$ is the $(2\nu+1)$-th convergent to a/q.*
3. *If $(u,v) \in (\mathcal{K}_+(a,q) \cup \mathcal{K}_-(a,q)) \cap \mathbb{Z}^2$ is an integer point then v/u is a convergent to a/q or one of the intermediate fractions $(wp_\nu + p_{\nu-1})/(wq_\nu + q_{\nu-1})$, $1 \leq w \leq b_{\nu+1}$.*
4. *$\Delta(a,q) \cap \mathbb{Z}^2 = \emptyset$.*

One can easily verify the same results for infinite continued fractions (i.e. for irrational numbers).

Recently, several papers [1, 21, 45, 46, 27] devoted to multidimensional generalization of Klein polygons have appeared. Unfortunately one must notice that there is something incorrect in papers [45, 46].

1.1.4. *Representation of rationals.* The rationals a/q with bounded partial quotients b_j are of great interest (see [22, 23, 24, 11]).

Let $N(k,q)$ be the number of integers A, $1 \leq A < q$, $\gcd(A,q) = 1$ such that any component b_i of the continued fraction expansion

$$\frac{A}{q} = [0; b_1, \dots, b_{n(A)}]$$

is bounded by k: $b_i \leq k$, $i = 1, \dots, n(A)$. It is known ([22, 4, 52]) that if $k > \gamma \log q$ with γ sufficiently large, then $N(k,q) \geq 1$. Moreover we can show that for *almost all* positive integers q and A with $1 \leq A < q$, all partial quotients are bounded by $O(\log q)$.
By the way we may recall a famous and still open conjecture which asserts that for any $q \geq 1$, we have $N(6,q) \geq 1$. However it is known that the conjecture holds when $q = 2^\alpha$ or $q = 3^\alpha$ ([39]).

Sergei Konyagin (see [17]), by means of Farey fractions, proved the following upper bound for $N(k,q)$:

Theorem 4. *For any $\gamma < 1$ and for any $k \geq k(\gamma)$ we have*

$$N(k,q) \ll \varphi(q) q^{-\frac{\gamma}{k \log k}},$$

where φ denotes the Euler function.

We define the sequence $A_1 < A_2 < \dots < A_d$ to be *an almost arithmetic progression* if

$$\exists w > 1 : \ w \leq A_{j+1} - A_j \leq 3w, \ \ j = 1, \dots, d-1.$$

In [32], the author shows that numbers with bounded partial quotients cannot appear very regularly: they cannot form long almost arithmetic progressions. The following theorem improves the result from [32].

Theorem 5. *For $d \geq 3$, let $A_0, \dots, A_d$ be positive integers. Suppose*
(i) $0 < A_0 < \dots < A_d$ *form an almost arithmetic progression;*
(ii) $\gcd(A_i, q) = 1$, $i = 0, \dots, d$.
Let $A_\nu/q = [b_{\nu,1}, \dots, b_{\nu,s(\nu)}]$. Then there exist ν_0 and μ_0 such that

$$0 \leq \nu_0 \leq d \ , \ 1 \leq \mu_0 \leq s(\nu_0)$$

and

$$b_{\nu_0,\mu_0} \gg d^{1/2}.$$

Theorem 5 is proved by means of Klein polygons. The same result is true for real-valued (not integer) almost arithmetic progressions and in the last case S. Konyagin showed that the result for real-valued progressions is exact in order.

1.2. **Simultaneous approximations.** Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive and real-valued function. For given $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$, a positive integer p is said to be a *ψ-approximation* of α, if

$$\max_{j=1,\dots,s} ||p\alpha_j|| = \max_{j=1,\dots,s} \min_{a\in\mathbb{Z}} |p\alpha_j - a| \leq \psi(p).$$

1.2.1. *Dirichlet and Liouville's theorems.* Dirichlet's theorem states that for any $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$, where $1, \alpha_1, \dots, \alpha_s$ linearly independent over $\mathbb{Z}$, there are infinitely many ψ-approximations of α with $\psi(y) = y^{-1/s}$.

On the other hand, Liouville's theorem ([2], ch.5) shows that for any $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that $1, \alpha_1, \dots, \alpha_s$ form a basis of a real algebraic field of degree $s+1$, there exists $C(\alpha)$ such that

$$\max_{j=1,\dots,s} ||p\alpha_j|| \geq C(\alpha)p^{-1/s}, \qquad \forall p \in \mathbb{N}.$$

One can see that there are only *countably many* algebraic $\alpha = (\alpha_1, \dots, \alpha_s)$.

1.2.2. *Theorem by Cassels and Davenport and the result by Jarnik.* In [3, 6] the following result is obtained.

Theorem 6. *There exists a constant C_s for which there exists an uncountable set of elements $\alpha \in \mathbb{R}^s$ which do not have any ψ-approximation where $\psi(y) = C_s y^{-1/s}$.*

V. Jarnik [12, 13] proved another result:

Theorem 7. *Let ψ and λ be positive real-valued functions such that $\psi(y)y^{1/s}$ decreases as $y \to \infty$ and $\lambda(y) \to 0$ as $y \to \infty$. Then there exists an uncountable set of elements $\alpha \in \mathbb{R}^s$ for which there are infinitely many ψ-approximations, but only finitely many $\psi\lambda$-approximations.*

A review of other results can be found in [43, 10, 2].

1.2.3. *Exact results in terms of the order of approximation.* Generalizing the work [3] by means of chains of parallelepipeds [28, 50, 7, 8] we improve Jarnik's result.

Theorem 8. *For $y \geq 1$, let ψ and ω such that Let $\psi(y) = y^{-1/s}\omega(y)$, where $\omega(y)$ decreases as $y \to \infty$ and*

$$\omega(1) \leq 2^{-(s+1)(s+2)}(s!)^{-1/s}.$$

Then there exists a vector $\alpha = (\alpha_1, \dots, \alpha_s)$ which has infinitely many ψ-approximations but not any $2^{-(s+3)}\psi$-approximation.

Theorem 9. *Let ω and ψ be as in Theorem 8 and suppose that*

$$\omega(1) \le 2^{-(s+1)(s+3)}(s!)^{-1/s}.$$

Then there exists an uncountable set of vectors $\alpha = (\alpha_1, \dots, \alpha_s)$, each of them having infinitely many ψ-approximations but not any $2^{-(s+3)}\psi$-approximation.

It follows that in Cassels Theorem 6 we may put

$$C_s = 2^{-(s+2)(s+3)}(s!)^{-1/s}.$$

We say that $\alpha = (\alpha_1, \dots, \alpha_s)$ satisfies the ψ-condition if α has infinitely many ψ-approximations but not any $c\psi$-approximation for some $c = c(\alpha)$

Theorem 10. *Let ψ be defined by $\psi(y) = y^{-1/s}\omega(y)$ where ω is decreasing positive function. Then in any Jordan s-dimensional domain Ω with* Vol $\Omega > 0$, *there exists an uncountable set of $\alpha \in \mathbb{R}^s$ satisfying the ψ-condition.*

Theorems 8, 9, 10 are discussed in [33].

1.2.4. *Successive best approximations.* Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$. We define *a best simultaneous approximation* (b.a) of α to be any integer point $\zeta = (p, a_1, \dots, a_s) \in \mathbb{Z}^{s+1}$ such that $\forall q$, $\forall (b_1, ..., b_s) \in \mathbb{Z}^s$, $1 \le q \le p$, $(q, b_1, \dots, b_s) \ne (p, a_1, \dots, a_s)$, we have

$$D(\zeta) = \max_{j=1,\dots,s} |p\alpha_j - a_j| < \max_{j=1,\dots,s} |q\alpha_j - b_j|.$$

Let $\alpha_j \notin \mathbb{Q}$, $j = 1, \dots, s$. Then all b.a. of α form infinite sequences

$$\zeta^\nu = (p^\nu, a_1^\nu, \dots, a_s^\nu), \ \ \nu = 1, 2, \dots$$

where $p^1 < p^2 < \dots < p^\nu < p^{\nu+1} < \dots$ and

$$D(\zeta^1) > D(\zeta^2) > \dots > D(\zeta^\nu) > D(\zeta^{\nu+1}) > \dots \ .$$

Let

$$M_\nu[\alpha] = \begin{pmatrix} p^\nu & a_1^\nu & \dots & a_s^\nu \\ \dots & \dots & \dots & \dots \\ p^{\nu+s} & a_1^{\nu+s} & \dots & a_s^{\nu+s} \end{pmatrix}.$$

For $\alpha = (\alpha_1, \dots, \alpha_s)$ satisfying $\alpha_j \notin \mathbb{Q}$, $j = 1, \dots, s$, we define $R(\alpha) \in [2, s+1]$ to be the integer

$$R(\alpha) = \min \ \big\{n \ : \text{ there exist a lattice } \Lambda \subseteq \mathbb{Z}^{s+1} \text{ with } \dim \Lambda = n \text{ and a natural } \nu_0 \text{ such that } \zeta^\nu \in \Lambda, \ \forall \nu > \nu_0\big\}.$$

Proposition 11. *Let $s = 1$. Then for any $\nu \ge 1$ we have* $\det M_\nu[\alpha] = \pm 1$ *(*$\operatorname{rank} M_\nu[\alpha] = 2, \ \forall \nu$*).*

Proposition 12. *For any $s \ge 1$ we have $R(\alpha) = \dim_{\mathbb{Z}} (\alpha_1, \dots, \alpha_s, 1)$.*

Proposition 13. *Let $s = 2$ and α_1, α_2 such that $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Z}$. Then for infinitely many ν we have*

$$\operatorname{rank} M_\nu[\alpha] = 3 = \dim_{\mathbb{Z}}(\alpha_1, \alpha_2, 1).$$

Proposition 11 – 13 can be easily verified. The following result is proved in [36].

Theorem 14. *Let $s \geq 3$. There exists an uncountable set of elements $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that*
(i) $1, \alpha_1, \dots, \alpha_s$ *are linearly independent over* $\mathbb{Z}$,
(*thus* $\dim_{\mathbb{Z}}(\alpha_1, \dots, \alpha_s, 1) = s + 1$),
and
(ii) $\operatorname{rank} M_\nu[\alpha] \leq 3, \ \forall \nu \geq 1$,
(*Hence for all* $\nu \geq 1$ *we have* $\det M_\nu[\alpha] = 0$).

Theorem 14 represents a counterexample to the conjecture of J.S. Lagarias [26]. It shows that the successive b.a. have no such an asymptotic property as a reader can see in Proposition 12. The idea of the proof was suggested to the author by N.P. Dolbilin.

1.3. **Linear forms.** Again, let $\alpha_1, \dots, \alpha_s$ be real numbers such that $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Z}$, and put $\alpha = (\alpha_1, \dots, \alpha_s)$.
For $m = (m_0, m_1, \dots, m_s) \in \mathbb{Z}^{s+1} \setminus \{0\}$ we define

$$\zeta(m) = m_0 + m_1\alpha_1 + \dots + m_s\alpha_s, \qquad M = \max_{j=0,1,\dots,s} |m_j|.$$

A vector $m \in \mathbb{Z}^{s+1} \setminus \{0\}$ is *a best approximation of α in sense of linear form* if

$$\zeta(m) = \min_{\substack{n \in \mathbb{Z}^{s+1} \setminus \{0\} \\ \max_j |n_j| \leq M}} |\zeta(n)|.$$

All best approximations form sequences

$$\zeta_1 > \zeta_2 > \dots > \zeta_\nu > \zeta_{\nu+1} > \dots,$$
$$M_1 < M_2 < \dots < M_\nu < M_{\nu+1} < \dots$$

where $m_\nu = (m_{0,\nu}, \dots, m_{s,\nu})$ is the vector of the ν-th b.a., $\zeta_\nu = \zeta(m_\nu)$ and $M_\nu = \max_j |m_{j,\nu}|$.

By Minkowski's Theorem we have $\zeta_\nu M_{\nu+1}^s \leq 1$.

1.3.1. *Singular systems.* The theorem on the order of approximations from §1.1.2 does not admit multidimensional generalization in the sense of linear form.

Theorem 15 (see [29, 35]). *Let s be an integer ≥ 1 and ψ a function such that $\psi(y)$ decreases to zero when y tends to infinity. Then there exists an uncountable set of elements $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that*
(i) $1, \alpha_1, \dots, \alpha_s$ *are linearly independent over* $\mathbb{Z}$,

and
(ii) *the sequence of the best approximations of* α *satisfies*

$$\zeta_\nu \le \psi(M_{\nu+s-1}).$$

In the case $s = 1$ this theorem means that there are real numbers with any given order of the best approximations. In higher dimensions it gives something more.

Khinchin [15] defined a vector $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ to be a *ψ-singular system* if for any $T > 0$ the system

$$||m_1\alpha_1 + \cdots + m_s\alpha_s|| < \psi(T), \quad M = \max_{1\le j\le s} |m_j| < T$$

has a nontrivial solution $(m_1, m_2, \dots, m_s) \in \mathbb{Z}^s$.

Proposition 16. *System is ψ-singular* $\iff \quad \zeta_\nu < \psi(M_{\nu+1})\,,\ \forall\nu.$

1.3.2. *Successive best approximations for linear form.* Here we define Δ_ν^s to be the determinant of the successive best approximations

$$\Delta_\nu^s = \begin{vmatrix} m_{0,\nu} & m_{1,\nu} & \dots & m_{s,\nu} \\ \dots & \dots & \dots & \dots \\ m_{0,\nu+s} & m_{1,\nu+s} & \dots & m_{s,\nu+s} \end{vmatrix}.$$

The proposition below follows from Minkowski theorem on convex body. It seems to me that it is a well-known fact, but I could not find the corresponding reference.

Proposition 17. *Let $s = 2$. Then for infinitely many ν we have* $\Delta_\nu^2 \ne 0$.

The theorem below was proved by the author in [35] by means of singular systems.

Theorem 18. *Lets $\ge$ 3. Then there exists a uncountable set of vectors* $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ *such that*
(i) $1, \alpha_1, \dots, \alpha_s$ *are linearly independent over* $\mathbb{Z}$,
and
(ii) *there exists a linear subspace* $\mathcal{L}_\alpha \subset \mathbb{R}^{s+1}$, $\dim \mathcal{L}_\alpha = 3$ *satisfying the condition*

$$m_\nu \in \mathcal{L}_\alpha\,, \ \forall\nu > \nu_0.$$

We see that for $s \ge 3$ almost all best approximations may asymptotically lie in a three-dimensional plane but they cannot lie in two-dimensional plane. Of course these examples are degenerated in sense of measure. For almost all vectors $\alpha \in \mathbb{R}^s$ (in the sense of Lebesgue) best approximations are asymptotically $(s+1)$-dimensional.

2. Integrals from quasiperiodic functions

In the text below we discuss applications of diophantine results to certain problems of uniform distribution of irrational rotations on torus. A full review of the methods and results of the theory of uniform distribution is given in [25].

2.1. Uniform distribution. Let $\mathbf{T}$ be the one-dimensional torus and $f : \mathbf{T}^s \to \mathbb{R}$ defined by the series

$$f(x_1, \dots, x_s) = \sum_{\substack{m\in\mathbb{Z}^s \\ m\neq 0}} f_m \exp\bigl(2\pi i(m_1x_1 + \cdots + m_sx_s)\bigr).$$

We also define the integral

$$I(T,\varphi) = I_{f,\omega}(T,\varphi) = \int_0^T f(\omega_1 t + \varphi_1, \dots, \omega_s t + \varphi_s)\, dt.$$

where $\omega_1, \dots, \omega_s \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$, and $\varphi = (\varphi_1, \dots, \varphi_s) \in \mathbb{R}^s$.

H. Weyl [51] proved that if f is a continuous function, then for any φ we have $I(T,\varphi) = o(T), \quad T \to \infty$. This equality holds uniformly in φ if we suppose moreover that f is smooth.

V.V. Kozlov conjectured that the integral $I(T,\varphi)$ is *recurrent* that is the following condition holds:

$$(*) \qquad \forall\varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |I(T^*,\varphi)| < \varepsilon.$$

This conjecture is true when f is any trigonometric polynomial, and in this case $(*)$ holds uniformly in φ. This implies that for any trigonometric polynomial f of finite degree, J^∞ defined by

$$J^\infty(T) = J^\infty_{f,\omega}(T) = \sup_{\varphi\in\mathbb{R}^s} |I(T,\varphi)|,$$

is itself recurrent, that is

$$(\$) \qquad \forall\varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |J^\infty(T^*)| < \varepsilon.$$

2.2. Case $s = 2$. In the two-dimensional case, the conjecture above was proved by V.V. Kozlov himself for functions $f \in C^2(\mathbf{T}^2)$ in [18] (see also [19]). It is also easy to see that when f is a smooth function, then $(*)$ holds uniformly in φ, that is ($) is true. E.A. Sidorov [44] obtained a similar result for "absolutely" continuous f.

2.3. **The general result.** The author [34] proved the conjecture in the general case:

Theorem 19. *Suppose that*

$$f(x_1, ..., x_s) = \sum_{\substack{m\in\mathbb{Z}^s \\ m\neq 0}} f_m \exp\bigl(2\pi i(m_1x_1 + ... + m_sx_s)\bigr)$$

belongs to the class $C^d(\mathbf{T}^s)$, *where* $d > Cs^{s^3}$ *and* $\omega_1, ..., \omega_s$ *are linearly independent over* $\mathbb{Z}$. *Then for any* φ, *the integral* $I(T, \varphi)$ *satisfies* $(*)$.

The proof is based on consideration of best approximations in the sense of linear form (see §1.3.2.).

2.4. **Metric results.** It is known [49, 48] that for almost all (in the sense of Lebesgue) vectors $\omega = (\omega_1, \dots, \omega_s) \in \mathbb{R}^s$, if f is smooth enough, then the integral $I(T, \varphi)$ is bounded when $T \to \infty$ uniformly in φ. Hence the integral $I(T, \varphi)$ satisfies $(*)$, uniformly in φ. But even in the case $s \geq 3$, this result is not universal.

Let Φ be a decreasing function and assume that the series $\sum_{m\in\mathbb{Z}} \Phi(m)$ converges. We define a periodic function $\Theta : \mathbf{T}^s \to \mathbb{R}$ to be *of the type* Φ if, the coefficients $\Theta_{m_1,\dots,m_s}$ in the expansion

$$\Theta(x_1, \dots, x_s) = \sum \Theta_{m_1,\dots,m_s} e^{2\pi i m_1 x_1 + \dots + m_s x_s},$$

satisfy

$$|\Theta_{m_1,\dots,m_s}| \leq \Phi(M), \quad \text{where } M = \max_j |m_j|.$$

We consider

$$J^\infty(T) = J^\infty_{f,\omega}(T) = \max_{\varphi\in\mathbf{T}^s} \bigl|I_{f,\omega}(T, \varphi)\bigr|;$$

$$J^2(T) = J^2_{f,\omega}(T) = \left(\int_{T^s} \bigl|I_{f,\omega}(T, \varphi)\bigr|^2 \, d\varphi\right)^{1/2}.$$

The result below is proved in [29].

Theorem 20. *Let* $s \geq 3$. *Then for any function* Φ *which decreases to zero as* $y \to \infty$ *and for any function* ψ *with* $\psi(y) = o(1)$ *as* $y \to \infty$, *there exist* $\omega_1, \omega_2, \dots, \omega_s$ *which are linearly independent over* $\mathbb{Z}$, *and a function* f *of type* Φ *such that* $\int_{\mathbf{T}^s} f(x)\, dx = 0$ *and*

$$J^l(T) \gg T\psi(T) \quad \forall T, \quad l = 2, \infty.$$

We will reformulate Theorem 20 in the following way.

Theorem 21. *Let $s \geq 3$, and $f : \mathbf{T}^s \to \mathbb{R}$ be smooth with zero mean value. Assume that in the expansion*

$$f(x_1, \dots, x_s) = \sum_{(m_1,\dots,m_s)\neq 0} f_{m_1,\dots,m_s} e^{2\pi i(m_1x_1+\dots+m_sx_s)}$$

the coefficients $f_{m_1,\dots,m_s}$, where $(m_1, \dots, m_s) \neq 0$, are all different from zero.
Then there exist ω_1, ω_2, ... , ω_s which are linearly independent over $\mathbb{Z}$ such that

$$J^l(t) \gg t\psi(t) \quad \forall\, t; \qquad l = 2, \infty.$$

An improvement of the latter result was obtained recently by E.V. Kolomeikina [20].

One can see that the behaviour of integrals J_l in two-dimensional case radically differs from the case $s \geq 3$.

2.5. **Odd functions. Sergei Konyagin's result.** Recently, S. Konyagin [16] obtained the following result.

Theorem 22. *The Kozlov's conjecture is true (that is $(*)$ holds) for arbitrary $s \geq 1$ and any function f satisfying the condition*

$$f(-x_1, \dots, -x_s) = f(x_1, \dots, x_s), \ \ f \in C^\tau(\mathbf{T}^s), \ \ \tau \asymp s2^s.$$

2.6. **The smoothness.** In [42],[41] it is shown that we need some kind of smoothness conditions on f to insure that $(*)$ is true : indeed in the two-dimensional case ($s = 2$), there exists a function $f : \mathbf{T}^2 \to \mathbb{R}$ (with zero mean value) of the class $C \setminus C^1(\mathbf{T}^2)$ such that $I(T, 0)$ tends to infinity when $T \to \infty$ (with the choice $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$). On the other hand, in [44] it is shown that when $s = 2$, a sufficient condition on f for having $(*)$, is f to be absolutely continuous.

Developping an idea of D.V. Treshchev, the author, in [31], generalized Poincaré's example. He proved that for any real $\omega_1, \dots, \omega_s$ which form a basis of a real algebraic field, there exists a function $f \in C^{s-2}(T^s) \backslash C^{s-1}(T^s)$ such that $I(., 0)$ does not satisfy the property $(*)$ with $\varphi = 0$.

One may find some results on algebraic numbers in [40] and [38]. Recently, S.V. Konyagin [16] proved that for some Liouville transcendental numbers, there exists $f \in C^d(\mathbf{T}^s)$ with $d \asymp 2^s/s$ such that $(*)$ is not satisfied.

Some early results are reviewed in [37].

2.7. **Vector-functions: counterexample in dimension $s = 3$.** Let $f^j : \mathbf{T}^s \to \mathbb{R}$, $j = 1, 2$ be defined by

$$f^j(x_1, \dots, x_s) = \sum_{\substack{k\in\mathbb{Z}^s \\ k\neq 0}} f^j_k \exp(2\pi i(k_1x_1 + \dots + k_sx_s)).$$

For $\omega_1, \dots, \omega_s \in \mathbb{R}$ be linearly independent over $\mathbb{Z}$, we put

$$I^j(T) = \int_0^T f^j(\omega_1 t, \dots, \omega_s t)dt, \quad j = 1, 2.$$

The analogue of property $(*)$ for the vector-integral $I = (I^1, I^2) : \mathbb{R} \to \mathbb{R}^2$ becomes

$$(\%) \qquad \forall \varepsilon > 0 \ \forall T \ \exists T^* > T : \ |I^1(T^*)| + |I^2(T^*)| < \varepsilon.$$

Proposition 23. *In the case when $s = 2$ and f is a smooth vector-functions, then (%) holds.*

Proposition 24. *The analogue of Theorem 22 holds for vector-function, that is (%) is satisfied for any odd smooth vector-function f.*

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function such that $\sum_{k \in \mathbb{Z}^s} \Phi(\max_{1 \le j \le s} |k_j|)$ converges. A vector-function $f = (f^1, f^2) : \mathbf{T}^s \to \mathbb{R}^2$ is defined to be a function of type Φ if we have

$$|f^j_k| \le \Phi(\max_{1 \le j \le s} |k_j|) \ \ \forall k, \ \ j = 1, 2.$$

Recently, the author [34] constructed the following example.

Theorem 25. *For any given positive function Φ, there exist a vector-function $f = (f^1, f^2) : \mathbf{T}^3 \to \mathbb{R}^2$ of the type Φ with zero mean value ($\int_{\mathbf{T}^3} f^j(x)dx = 0, j = 1, 2$) and numbers $\omega_1, \omega_2, \omega_3$, which are linearly independent over $\mathbb{Z}$ such that*

$$|I^1(T)| + |I^2(T)| \to \infty, \ \ \textit{as } T \to +\infty.$$

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References

[1] A.D. Bruno & V.I. Parusnikov, *Klein's polyhedra for two cubic forms considered by Davenport.* Matemeticheskie Zametki **56** (1994), 9–27.

[2] J.W.S. Cassels, *An introduction to Diophantine approximation.* Cambridge Tracts in Mathematics and Mathematical Physics **45**. Cambridge University Press, New York, 1957, x+166 pp.

[3] J.W.S. Cassels, *Simultaneous Diophantine Approximations II.* Proc. London Math. Soc. (3) **5** (1955), 435–448.

[4] T.W. Cusick, *Zaremba's conjecture and sums of the divisor functions.* Math. Comp. **61** (1993), no. 203, 171–176.

[5] T. W. Cusick, M.E. Flahive, *The Markoff and Lagrange spectra.* AMS, Providence, Math. surveys and monographs **30**, 1989.

[6] H. Davenport, *Simultaneous Diophantine Approximations.* Mathematika **1** (1954), 51–72.

[7] B.N. Delone, *An algorithm for the "divided cells" of a lattice.* (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, (1947), 505–538.

[8] B.N. Delone, *The St. Petersburg school in the theory of numbers.* (Russian) Akad. Nauk SSSR Naučno-Popularnaya Seriya. Izdat Akad. Nauk SSSR, Moscow-Leningrad, 1947, 421 pp.

[9] P. Erdős, M. Gruber & J. Hammer, *Lattice Points.* Pitman Monographs and Surveys in Pure and Applied Mathematics, 39. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. viii+184 pp.

[10] M. Gruber & C.G. Lekkerkerker, *Geometry of Numbers.* Amsterdam, 2nd edition, 1987.

[11] Hua Loo Keng & Wang Yuan, *Application of Number Theory to Numerical Analysis.* Berlin; Heidelberg; N.Y.: Springer Verlag, 1981.

[12] V. Jarnik, *Uber die simultanen diophantischen Approximationen.* Math. Zeitschr. **33** (1933), 505–543.

[13] V. Jarnik, *Un théorème d'existence pour les approximations diophantiennes.* Enseign. Math. (2) **15** (1969), 171–175.

[14] A.Ya. Khinchin, *Continued fractions.* Moscow, 1978.

[15] A.Ya. Khinchin, *Über eine Klasse linearer Diophantischer Approximationen.* Rendiconti Circ. Mat. Palermo **50** (1926), 170–195.

[16] S.V. Konyagin, *On the recurrence of an integral of an odd conditionally periodic function.* (Russian) Mat. Zametki **61** (1997), no. 4, 570–577; translation in Math. Notes **61** (1997), no. 3-4, 473–479.

[17] S.V. Konyagin & N.G. Moshchevitin, *Representation of rationals by finite continuous fractions.* (Russian) Uspekhi Mat. Nauk **51** (1996), no. 4(310), 159–160.

[18] V.V. Kozlov, *The integrals of quasiperiodic functions.* (Russian) Vestnik Moskov. Univ. Ser. I Mat. Meh. 1978, no. 1, 106–115.

[19] V.V. Kozlov, *Methods of qualitative analysis in solid body motion.* Moscow, Moscow University, 1980.

[20] E.V. Kolomeikina, *Asymptotics of mean time values.* Recent problems in theory of numbers and applications. Conference in Tula (Russia) 09 – 14.09.1996., Thesis., S. 80.

[21] E.I. Korkina, *Two-dimensional continued fractions. The simplest examples.* (Russian) Trudy Mat. Inst. Steklov. **209** (1995), 143–166.

[22] N.M. Korobov, *Methods of theory of numbers in approximate analysis.* Moscow, 1963.

[23] N.M. Korobov, *On some problems in theory of Diophantine approximations.* (Russian) Uspekhi Mat. Nauk **22** (1967), 83–118.

[24] N.M. Korobov, *Trigonometric sums and their applications.* Moscow, 1989.

[25] L. Kuipers & H. Niederreiter, *Uniformly distributed sequences.* N.Y., 1974.

[26] J.S. Lagarias, *Best simultaneous Diophantine approximation II.* Pac. J. Math. **102** (1982), 61–88.

[27] G. Lachaud, *Polyèdre d'Arnold et voile d'un cône simplicial : analogues du théorème de Lagrange.* C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 8, 711–716.

[28] H. Minkowski, *Généralisation de la théorie des fractions continues.* Ann. Ecole norm. (3) **13** (1896), 41–60.

[29] N.G. Moshchevitin, *Distribution of values of linear functions and asymptotic behaviour of trajectories of some dynamical systems.* (Russian) Matematicheskie Zametki **58** (1995), 394–410.

[30] N.G. Moshchevitin, *On the recurrence of an integral of a smooth three-frequency conditionally periodic function.* (Russian) Mat. Zametki **58** (1995), no. 5, 723–735.

[31] N.G. Moshchevitin, *Nonrecurrence of an integral of a conditionally periodic function.* (Russian) Mat. Zametki **49** (1991), no. 6, 138–140.

[32] N.G. Moshchevitin, *On representation of rationals as continuous fractions and sets free from arithmetic progressions.* (Russian) Matematicheskie Zapiski **2** (1996), 99–109.

[33] N.G. Moshchevitin, *On simultaneous Diophantine approximations.* Matemticheskie Zametki **61** (1997), no. 5, 706–716.

[34] N.G. Moshchevitin, *On the recurrence of an integral of a smooth conditionally periodic function.* (Russian) Mat. Zametki **63** (1998), no. 5, 737–748.

[35] N.G. Moshchevitin, *On geometry of the best approximations.* Doklady Rossiiskoi Akademii Nauk **359** (1998), 587–589.

[36] N.G. Moshchevitin, *On best joint approximations.* (Russian) Uspekhi Mat. Nauk **51** (1996), no. 6(312), 213–214.

[37] N.G. Moshchevitin, *Recent results on asymptotic behaviour of integrals of quasiperiodic functions.* Amer. Math. Soc. Transl. (2) **168** (1995), 201–209.

[38] H. Niederreiter, *Application of Diophantine Approximation to Numerical Integration.* in book Dioph. Appr. and Appl. ed. Osgood C. N.Y., Academic Press, 1973, 129 – 199.

[39] H. Niederreiter, *Dyadic Fractions with Small Partial Quotients.* Monatsh. Math. **101** (1986), 309–315.

[40] L.G. Peck, *On Uniform Distribution of Algebraic Numbers.* Proc. Amer. Math. Soc. **4** (1953), 440–443.

[41] H. Poincaré, *Sur les séries trigonométriques.* Comptes Rendus **101** (1885), 1131–1134.

[42] H. Poincaré *On curves defined by differential equations.* (Russian transl.) Moscow-Leningrad, 1947.

[43] W. Schmidt, *Diophantine approximation.* Lecture Notes in Mathematics **785**. Springer, Berlin, 1980. x+299 pp.

[44] E.A. Sidorov, *On conditions of uniform Poisson stability of cylindrical systems.* Uspekhi Mat. Nauk **34** (1979), no. 6(210), 184–188.

[45] B.F. Skubenko, *Minima of decomposable cubic forms in 3 variables.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **168** (1988), 125–139.

[46] B.F. Skubenko, *Minima of decomposable forms of degree n in n variables.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 183 (1990), 142–154.

[47] V.G. Sprindzuk, *Metric theory of diophantine approximations.* Moscow, 1977.

[48] V.G. Sprindzuk, *Asymptotic behavior of integrals of quasiperiodic functions.* (Russian) Differential Equations **3** (1967), 862–868.

[49] I.L. Verbitzki, *Estimates of the remainder term in Kronecker theorem on approximations.* Theoriia Functzii, Funktzianal. Analiz i ih prilogeniia, Harkov, **5** (1967), 84–98.

[50] G.F. Voronoi, *On one generalization of continued fractions algorithm.* Selected works T.1, Kiev, 1952.

[51] G. Weyl, *Über die Gleichverteliung von Zahlen mod. Eins.* Math. Ann. **77** (1916), 313–352.

[52] S.K. Zaremba, *La méthode des "bons treillis" pour le calcul des intégrales multiples.* In : Appl. of Number Theory to Numerical Analysis. ed. S.K. Zaremba; N.Y., 1972, 39–119.

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