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Invariants of a quadratic form attached to a tame covering of schemes

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À Jacques Martinet

Résumé. Nous étendons des résultats de Serre, Esnault-Kahn-Viehweg et Kahn, et nous montrons une relation entre des invariants dans la cohomologie étale modulo 2, qui sont obtenus à partir d'un revêtement modérément ramifié de schémas à ramification impaire. Le premier type d'invariant est construit à l'aide d'une forme quadratique naturelle définie par le revêtement. Dans le cas d'un revêtement de schémas de Dedekind cette forme est donnée par la racine carrée de la codifférente avec la forme trace. Dans le cas d'un revêtement de surfaces de Riemann la forme provient de l'existence d'une caractéristique théta canonique. Le deuxième type d'invariant est défini à l'aide de la représentation du groupe fondamental modéré, qui est attachée au revêtement. Notre formule est valable sans restriction sur la dimension. Pour les revêtements non-ramifiés la formule est due aux auteurs précités.

Les deux contributions essentielles de notre travail sont de montrer (1) comment ramener la démonstration de la formule au cas non-ramifié en toute dimension et (2) comment maîtriser les difficultés provenant de la présence de points singuliers dans le lieu de ramification du revêtement, en utilisant ce que nous appelons "normalisation le long d'un diviseur". Notre approche toute entière est basée sur une analyse fine de la structure locale des revêtements modérément ramifiés.

Nous présentons aussi un survol des notions de la théorie des formes quadratiques sur les schémas et les techniques simpliciales de base nécessaires pour la compréhension de notre travail.

Abstract. We build on preceeding work of Serre, Esnault-Kahn-Viehweg and Kahn to establish a relation between invariants, in modulo 2 étale cohomology, attached to a tamely ramified covering of schemes with odd ramification indices. The first type of

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invariant is constructed using a natural quadratic form obtained from the covering. In the case of an extension of Dedekind domains, mains, this form is the square root of the inverse different equipped with the trace form. In the case of a covering of Riemann surfaces, it arises from a theta characteristic. The second type of invariant is constructed using the representation of the tame fundamental group, which corresponds to the covering. Our formula is valid in arbitrary dimension. For unramified coverings the result was proved by the above authors.

The two main contributions of our work consist in (1) showing how to eliminate ramification to reduce to the unramified case, in such a way that the reduction is possible in arbitrary dimension, and; (2) getting around the difficulties, caused by the presence of crossings in the ramification divisor, by introducing what we call "normalisation along a divisor". Our approach relies on a detailed analysis of the local structure of tame coverings.

We include a review of the relevant material from the theory of quadratic forms on schemes and of the basic simplicial techniques needed for our purposes.

Contents

Introduction

The theory of bilinear and quadratic forms over schemes has experienced a surge of activity in recent years. Even so, it still seems that little is known in general (see however the very recent work of Balmer [Ba1], [Ba2]). As an indication that this is the state of affairs, we note that–to the best of our knowledge–Knebusch's paper of the seventies [Kne] remains the only available source on the elements of the theory. Of course, since that time a lot of progress has been made on the theory of forms on rings, see *e.g.* [Knu], and some results and definitions on forms over rings are sufficiently functorial to work over schemes; but, for instance, the theory of bilinear forms over the rational integers, has not yet been related to the recent work in arithmetical algebraic geometry. (For the benefit of the reader we have collected most of the sources on the theory which are known to us in the bibliography–the article [Pa] contains a nice survey.) One explanation for this might be that there is not, as yet, a real need to develop a general theory, because few concrete examples and problems have been considered.

One of our aims here is to study a very special form, which arises from the consideration of coverings of connected, regular, proper $\mathbf{Z}[\frac{1}{2}]$-schemes. Namely, let

$$\pi : X \to Y$$

be a covering of such schemes, which is obtained as a sub-covering of a quotient $\tilde{\pi} : \widetilde{X} \to Y = \widetilde{X}/G$, where G is a finite group acting on the connected, projective, regular $\mathbf{Z}[\frac{1}{2}]$-scheme $\widetilde{X}$. That is, we let $X = \widetilde{X}/H$ for H a subgroup of G. Then, under suitable assumptions, among which is the requirement that the ramification be tame along a divisor b with normal crossings and that the ramification indices be odd, we are assured of the existence of a locally free sheaf $\mathcal{D}_{X/Y}^{-1/2}$ over X, whose square is the inverse different of X/Y (see Sect. 3.d). When viewed over Y and equipped with the trace form this sheaf, defines a *non-degenerate* symmetric bilinear form on Y, which we denote

$$E = (\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y}) \ ,$$

and which we shall call the *square root of the inverse different* (see the Example before Sect. 1.c.1).Note that for unramified coverings $\mathcal{D}_{X/Y}^{-1/2} = O_X$. In algebraic number theory this form has been studied by two of us in [E-T] (see also [E1]). The aim there was to compare this form with the natural form on the group algebra, which is a sum of squares. For coverings of Riemann surfaces it has been studied by Serre in [S2]. In [S1], Serre had previously given a formula for the Hasse-Witt invariant of the trace form for étale algebras over fields of characteristic different from 2 (no ramification and dim 0). The formula in [S2] is an analogue of that formula in the ramified case of dimension 1 (and characteristic 0). Esnault, Kahn and Viehweg then provided a result valid for all tamely ramified coverings of Dedekind schemes with odd ramification, which generalizes both of Serre's results (see [E-K-V] and [K]). In a different direction Lee and Weintraub have generalized Serre's formula to ramified coverings of higher dimensional manifolds, with smooth ramification locus, see [L-W]. It was puzzling for us to see that in higher dimension their formula looked just like Serre's and did not involve more terms. We wondered whether this was due to the smoothness assumption on the ramification locus. It turns out that the crossing points of the ramification locus do play a major role in our work, but they do not contribute to the final result.

We will keep to the algebraic set-up and provide a generalisation to higher dimensions of the work of Esnault-Kahn-Viehweg. We give an expression for invariants attached to the form E. Our strategy consists in reducing to the case of an étale cover. This case has been treated in full generality in [E-K-V] and [K]. (The proof in this case consists of a sophisticated cocycle computation generalizing that made by Serre to deal with the trace form of étale algebras, see Sect. 1.h.) The invariants we consider are generalized Stiefel-Whitney classes. They live in the étale cohomology groups modulo 2 of Y and are obtained by pulling back universal Hasse-Witt classes using a classifying map corresponding to E (as in [J1] and [J2], see Sects. 1.d and 1.e). Following Snaith and Jardine, we shall call the *i-th Hasse-Witt invariant* of E the class $w_i(E)$ in $H^i(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ attached to the form E in this way. Putting all these classes together we obtain the *total Hasse-Witt invariant* of E

$$w_t(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y}) = w_t(E) = 1 + w_1(E)t + w_2(E)t^2 + \cdots \ ,$$

which is a class in the (abelian) group $G^*(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ of invertible elements in $H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z})$. To determine this class we shall construct an étale covering $\pi_Z : T \to Z$ of Y-schemes, and a sequence of locally free sheaves $\mathcal{G}^{(h)}$ on Z, as follows. Let G_2 denote a 2-Sylow subgroup of G and put $Z = \widetilde{X}/G_2$, then T will be the normalization of the fiber product $T' = Z \times_Y X$. So we have the diagram.

$$\begin{array}{ccccc} T & \longrightarrow & T' = Z \times_Y X & \to & X \\ & \pi_Z \searrow & \downarrow & & \downarrow \pi \\ & & Z & \stackrel{\phi}{\to} & Y \end{array}$$

Our first task will be to show that with these definitions π_Z is indeed étale and that T is regular. We do this by using an explicit description of the local structure of the cover $\widetilde{X}/Y$ (see Sect. 2). The natural thing to try then is to compare the pull-back of E along ϕ with the form $F = (\pi_{Z*}(O_T), Tr_{T/Z})$. Note that by our choice of Z the pull-back map induced by ϕ in cohomology is injective. The two forms $\phi^*(E)$ and F coincide on the generic fiber of Z and we wish to compare their total Hasse-Witt invariants. For this, following [E-K-V], we try to show that the orthogonal sum $H := \phi^*(E) \perp F$, equipped with the difference of forms, contains an isotropic sub-bundle I, whose rank is half of the rank of H. With the terminology introduced in Sect. 1.c.1, we try to show that H is metabolic with lagrangian I. Note that hyperbolic forms are an example of metabolic forms and the reason for trying to show that H is metabolic is that, as with hyperbolic forms, it is expected that the quadratic/Hasse-Witt invariants of H should then be determined by the linear/Chern invariants of the maximal isotropic sub-bundle I. In fact this expectation is justified (see the Main Lemma 1.15). In dimension 1, which is the situation considered in [E-K-V], the intersection $\phi^*(E) \cap F$ gives the required sub-bundle I. In our general situation, problems appear due to the fact that the branch locus b of the covering $\widetilde{X}/Y$ is not smooth. We show that the form $\phi^*(E) \perp F$ is metabolic (in a generalized sense), but to exhibit a maximal isotropic sub-bundle explicitly we are led to introduce a sequence of forms $\Lambda^{(h)}$ which interpolate $\phi^*(E)$ and F. For this we decompose the normalization map $T \to T'$ into a sequence of Z-morphisms

$$T = T^{(m)} \to T^{(m-1)} \to \cdots \to T^{(0)} = T' \ ,$$

numbered by the m components of the branch locus b of the covering $\widetilde{X}/Y$ and which are obtained by "normalisation along a component of b" (see Sect. 3). The form $\Lambda^{(h)}$ is the square root of the inverse different for the cover $T^{(h)}/Z$ and $\Lambda^{(0)} = \phi^*(E)$, while $\Lambda^{(m)} = F$. For $0 \leq h \leq m-1$ we have short exact sequences of locally free O_Z-modules

$$0 \to I^{(h)} \to \Lambda^{(h)} \oplus \Lambda^{(h+1)} \to \mathcal{G}^{(h)} \to 0 \ ,$$

where the map on the right is the difference map (see Prop. 3.12). This shows that the sum $(\Lambda^{(h)}, Tr) \perp (\Lambda^{(h+1)}, -Tr)$ is metabolic.

We need to introduce one last notation before stating our first main result. For every h define an element of $H^*(Z_{et}, \mathbf{Z}/2\mathbf{Z})$ by

$$d_t(\mathcal{G}^{(h)}) = \sum_{i=0}^{n}(1+(-1)t)^{n-i}c_i(\mathcal{G}^{(h)})t^{2i} \ ,$$

where $c_i(\mathcal{G}^{(h)}) = c_i(I^{(h)})$ denotes the i-th Chern class of $\mathcal{G}^{(h)}$ and n denotes the rank of $\mathcal{G}^{(h)}$ (see the Main Lemma 1.15).

Theorem 0.1. *Under the above assumptions the following equality holds in* $H^*(Z_{et}, \mathbf{Z}/2\mathbf{Z})$

$$w_t(\phi^*(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y})) = w_t(\pi_{Z*}(O_T), (-1)^m Tr_{T/Z})^{(-1)^m} \prod_{0\le h\le m-1} d_t(\mathcal{G}^{(h)})^{(-1)^h}.$$

Remark. It should be noted that if we work with another divisor on Y which differs from b by the addition of a number of irreducible non-ramified divisors, then of course the number m of divisors will change and so the parity of the first term in the right hand side may change; however, it can be checked that the product of the two terms on the right remains constant (see the remark at the end of Sect. 4.a).

The proof of Thm. 0.1 is given in Sect. 4.a. The invariants of the form $(\pi_{Z*}(O_T), Tr_{T/Z})$ arising from the étale covering T/Z, have been related to other Stiefel-Whitney type invariants attached to the covering in [S1], [E-K-V] and [K]. These are what we call the *Galois theoretic classes* $w_i(\pi)$, which can be defined using Grothendieck's equivariant cohomology theory. Then, with some further work, in low degree Thm. 0.1 can be reformulated to give a simple expression for the difference between the second Hasse-Witt and Galois theoretic invariants. For this let us introduce an element $\rho(X/Y)$ in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ which only depends on the ramification data of the covering X/Y. We shall denote by the same symbol (the class of) a line bundle and its image in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ under the composition

$$H^1(Y_{et}, \mathbf{G}_m) = \mathrm{Pic}(Y) \to \mathrm{Pic}_2(Y) \to H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \ ,$$

where the second map is the coboundary induced by the Kummer sequence (note that under our assumptions we may identify μ_2 with $\mathbf{Z}/2\mathbf{Z}$). We define $\rho(X/Y)$ as a divisor class. Let ξ_h denote the generic point of the irreducible component b_h of the branch locus b and consider the divisor

$$\Gamma := \sum_{h,\xi_h^{'}\to\xi_h} \left(\frac{e(\xi_h^{'})^2-1}{8}\right)\overline{\{\xi_h^{'}\}} \ ,$$

where the sum runs over the points $\xi_h^{'}$ on X of codimension 1 above the ξ_h and where $e(\xi_h^{'})$ denotes the (odd) ramification index of $\xi_h^{'}$. The divisor

$\rho(X/Y)$ on Y is defined as the direct image

$$(0.1)\qquad \rho(X/Y) := \pi_*(\Gamma) = O_Y\left(\sum_h (\sum_{\xi_h^{'}\to\xi_h} (\frac{e(\xi_h^{'})^2-1}{8})f(\xi_h^{'}))\cdot b_h\right) ,$$

where $f(\xi_h^{'})$ denotes the residue class extension degree. Our second main result can then be formulated as follows.

Theorem 0.2.

a) *Let $d_{X/Y}$ denote the function field discriminant viewed as an element in $H^1(Y_{et}, \mathbf{Z}/2\mathbf{Z})$, then*

$$w_1(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y}) = (d_{X/Y}) = w_1(\pi) .$$

b) *The following equality holds in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$:*

$$w_2(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), Tr_{X/Y}) = w_2(\pi) + (2)\cup(d_{X/Y}) + \rho(X/Y) .$$

The main effort in proving this theorem, given the preceding work, goes into a determinantal computation, which again uses in an essential way the local description of the tame cover X/Y. More precisely in Thm. 4.2 we show that in $H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z})$:

$$\sum_{1\le h\le m}\left(c_1(\Lambda^{(h)}) + c_1(\mathcal{G}^{(h-1)})\right) = \phi^*(\rho(X/Y)) .$$

The origin of the coefficient $(e^2-1)/8$ is in the combinatorial Lemma 4.3. We do not have a conceptual definition of the ramification term $\rho(X/Y)$.

A word about invariants of forms over schemes might help clarify the significance of our results. Not all invariants of forms over a field extend to invariants of forms over schemes, except in degree 2 and smaller. Global obstructions appear. So for instance the Arason invariant, in degree 3, does not extend to schemes, see [E-K-L-V]. The Hasse-Witt invariants we study exist in arbitrary degree, but one should keep in mind that they are not independent one of the other. So for instance, as might be expected from the analogies with algebraic topology, if $w_1(E)$ and $w_2(E)$ are both zero, then $w_3(E)$ is zero as well. The formulæ in Thm. 0.2 can be viewed as expressing the difference between the "topological" Hasse-Witt invariant and the "discrete" Galois theoretic invariant (see [J1], p. 84, [J3] Sect. 3 or [J4] Intro.). They should also be viewed as "twisting formulae" following Fröhlich's work [F]. Yet another interpretation is given by Kahn in terms of equivariant cohomology (see [K]).

The invariants in degree smaller than 2 have been refined to give invariants in K-theory (see [G1], [O-P-S], [Sz], [G2] and [B-O]; the last two also deal with invariants in higher degree). These invariants are related to the invariants in étale cohomology *via* Chern class maps; we do not know if our formula can be lifted to K-theory.

The paper is structured as follows. In Sect.1 we fix notation and recall the necessary definitions and background material used in the rest of the paper. We offer a rather detailed review of the basic material on forms over schemes and their invariants as the literature on the subject is rather scarce and/or quite technical. In Sect. 1.h we state and sketch a proof of the result of [E-K-V] which we use in the proof of Thm. 0.2. Sect. 2, contains the material on tame covers on which our contribution is based. There we describe the normalization T locally (see Prop. 2.6). Next, we analyze the normalisation procedure and decompose it into steps, which we call "normalisation along a divisor" (see Sect. 3). Sect. 4 contains the proofs of Thm. 0.1 and Thm. 0.2. The first will follow easily from the preceding work. As explained above, to obtain Thm. 0.2, we have to make more explicit the right hand side in the equality of Thm. 0.1 as well as bring in the Galois theoretic invariants. We have collected in an Appendix, some material on homotopical algebra which we could not find presented in compact form anywhere in the literature.

To conclude this Introduction let us mention another related line of research which consists in studying forms on cohomology of (complexes of) sheaves. This has been pursued by Saito in [Sa] and Chinburg-Pappas-Taylor in [C-P-T1]. Saito's work also has as its starting point [S1], whereas [C-P-T1] deals with forms coming from Arakelov metrics. This work sheds new light on the role that the trace form plays in Galois module theory. We would also like to indicate that in the future we hope to relate the material in this paper to the programme developed in [CEPT2] and other papers of that consortium of authors (see [E2] and [C-P-T2] for a general overview of that programme). Other developments arising from Serre's original formula for étale algebras can be found in [Mo] and the references therein.

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1. Framework and notation

1.a. Notation and conventions.

Induced algebras. In our local description of the structure of tame coverings we will need the following construction. Let A be a ring and let B be an A-algebra equipped with an action of the finite group H, which preserves the algebra structure. For a group G and a subgroup H of G let $\mathrm{Map}_H(G,B)$ denote the B-algebra of all maps f from G to B such that for all g in

G and all h in H, $f(gh) = f(g)^h$. The group G acts on this algebra by $f^g(g') = f(gg')$.

Normalisation. For a ring A which injects into its total ring of fractions $Fr(A)$ we write $\widetilde{A}$ for its integral closure. We will also use a tilde superscript to denote certain objects related to the cover of schemes $\widetilde{X}/Y$. This should not cause any confusion.

Coefficients. We will work with mod 2 étale cohomology of schemes on which 2 is invertible, so we will always identify μ_2 with $\mathbf{Z}/2\mathbf{Z}$.

1.b. Tame coverings with odd ramification.

In what follows all schemes will be defined over $Spec(\mathbf{Z}[\frac{1}{2}])$. Let $\widetilde{X}$ be a connected, projective, regular scheme which is either defined over the spectrum $\mathrm{Spec}(\mathbf{F}_p)$ of the prime field of characteristic $p \neq 2$ or is flat over $\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$.

Tameness. Assume $\widetilde{X}$ is equipped with a tame action of G, in the sense of Grothendieck-Murre. In particular the quotient

$$\tilde{\pi} : \widetilde{X} \to Y = \widetilde{X}/G$$

exists and is a G-torsor outside a divisor b with normal crossings (see [Gr-M], [CEPT1], [C-E] Appendix and [CEPT2] 1.2 and Appendix).

Once and for all fix a subgroup H of G, and let

$$\lambda : \widetilde{X} \to X := \widetilde{X}/H$$

denote the quotient map. Let

$$\pi : X \to Y$$

be the induced map, so that $\tilde{\pi} = \pi \circ \lambda$. These maps are all finite.

Regularity and flatness. If $\tilde{\pi}$ is flat, then Y is regular (see [EGAIV] IV 6.5.2). Conversely a morphism between equidimensional regular schemes of the same dimension is flat (see [Ka-M], Notes to Ch. 4 or use [EGAIV] IV 6.1.5). We will assume that Y and X are regular, or equivalently that $\tilde{\pi}$ and λ are flat. Note that being quotients of a normal scheme by a finite group Y and X are certainly normal, however it is only under the above regularity assumptions that we will be able to give a precise description of the local structure of the quotient maps (see Lemma 2.3). It follows from these assumptions that π is flat as well.

Different; branch and ramification loci. The different $\mathcal{D}_{\widetilde{X}/Y}$ is the annihilator of $\Omega^1_{\widetilde{X}/Y}$ (see [Mi] Rem. I.3.7). The reduced closed subscheme of $\widetilde{X}$ defined by the different is the ramification locus of $\tilde{\pi}$, which we denote by

$\widetilde{B} = B(\widetilde{X}/Y)$. Then $b = b(\widetilde{X}/Y)$ is the reduced sub-scheme of Y defined by the image of $\widetilde{B}$ under $\tilde{\pi}$. There are decompositions

$$b(\widetilde{X}/Y) = \coprod_{1 \leq h \leq m} b_h \quad \text{and} \quad B(\widetilde{X}/Y) = \coprod_{h,k} \widetilde{B}_{h,k} \ ,$$

where the b_h are the irreducible components of b and where for any fixed integer h between 1 and m, the $\widetilde{B}_{h,k}$ are the irreducible components of $\widetilde{B}$ such that $\tilde{\pi}(\widetilde{B}_{h,k}) = b_h$.

The tameness assumption on the ramification implies that the ramification locus $B(\widetilde{X}/Y)$ on $\widetilde{X}$ is a divisor with normal crossings. For each irreducible component b_h of b, let ξ_h denote the generic point of b_h, and let $\xi_h^{''}$ denote a generic point of the branch locus on $\widetilde{X}$ above ξ_h. Put $\xi_h^{'} = \lambda(\xi_h^{''})$. Let $e(\xi_h^{'})$ (resp. $f(\xi_h^{'})$) denote the ramification index (resp. residue class extension degree) of $\xi_h^{'}$ over ξ_h.

Parity. We assume that the inertia group of each closed point of $\widetilde{X}$ has odd order, so every point has odd inertia and in particular the integers $e(\xi_h^{'})$ are odd.

The base change. Choose a 2-Sylow subgroup G_2 of G and consider the quotient $Z := \widetilde{X}/G_2$. We will show in Sect. 2, that by the parity assumption $\widetilde{X}$ is étale over Z, so Z is regular and flat over Y. We give a local description of Z/Y analogous to that of X/Y, without using the regularity of Z in (2.4). As mentioned in the Introduction our main effort will go into the construction of Z-schemes which will allow us to reduce our computation of the invariants to the case of an étale cover.

Memotechniques. For future reference and to help the reader with bookkeeping, we anticipate some of the notation to come. Components of the branch locus b will be numbered by h (from 1 to m). We fix points y in Y and $\widetilde{x}$ in $\widetilde{X}$ over y. The n components of b which are in the image of a component of the ramification divisor through $\widetilde{x}$ will be numbered by ℓ (see (2.2); of course there might be none of these). Let $I(\widetilde{x})$ denote the inertia group of $\widetilde{x}$ for the covering $\widetilde{X}/Y$, this is a group which is the direct product of n cyclic groups indexed by the components of b through y. Let e_ℓ denote the order of the ℓ-th component. In describing the local structure of our coverings we will consider the spectrum $S = \mathrm{Spec}(A_y)$ of a ring containing a regular sequence of parameters $a_1, \dots, a_n$, providing a sufficiently small étale neighbourhood of y in Y (see Lemma 2.3). Then, we will construct coverings

$$S \longleftarrow X_S \longleftarrow \widetilde{X}_S$$

corresponding to inclusions of algebras

$$A_y \longrightarrow \prod_i A_y[t_{i,1}, \dots, t_{i,n}] \xrightarrow{\text{diag}} \prod_i \left(\prod_{j(i)} A_y[t_1, \dots, t_n] \right) ,$$

where $t_{i,\ell} = t_\ell^{e_\ell/e_{i,\ell}}$ for a certain divisor $e_{i,\ell}$ of e_ℓ and where $t_\ell^{e_\ell} = a_\ell$ (see also the diagram just before Sect. 2.a). The index i arises from the double coset decomposition of G with respect to H and $I(\widetilde{x})$. Analogously, the index $j(i)$ arises from a coset decomposition $g_i I(\widetilde{x}) g_i^{-1} \cap H \backslash H$ (see (2.b)). Fixing i and going to the fields of fractions we obtain a sequence of abelian Galois extensions of the field $K_y = Fr(A_y)$. Namely let

$$M_{\widetilde{x}} = Fr(\prod_{j(i)} A_y[t_1, \dots, t_n]) \quad , \quad L_{i,x} = Fr(A_y[t_{i,1}, \dots, t_{i,n}])$$

and $H_i = I(\widetilde{x})/I(\widetilde{x}, i)$, then we have the diagram of fields

$$K_y = Fr(A_y) = \underbrace{M_{\widetilde{x}}^{I(\widetilde{x})} \longrightarrow L_{i,x}}_{H_i} = M_{\widetilde{x}}^{I(\widetilde{x},i)} \longrightarrow M_{\widetilde{x}} \ .$$

There is an analogous description of $S \leftarrow Z_S$ and quite naturally, in the local description of the normalisation T of $T' = Z \times_Y X$, we will consider algebras indexed by pairs (j, i) (see Prop. 2.6). Moreover the algebras appearing in the local descriptions will be decomposed according to the characters of the group H_i, which will be indexed by a set $A(i)$, whose elements will be α's (see Lemma 2.5).

1.c. Quadratic forms on schemes.

Quadratic forms on schemes are defined by generalizing the familiar definitions given for such forms over fields or rings (see e.g. [Kne] and [Knu] Chapt. VIII). The only definition which is not straightforward is that of a metabolic form (see below Sect. 1.c.1). However, for the definition of the invariants, it is also good to have an algebraic description of the set of isometry classes of forms of given rank over a scheme Y. We will recall the description in terms of non-abelian first cohomology sets, and the description in terms of homotopy classes of maps from Y to a certain classifying space. For completeness, we briefly go over the basic material on forms over schemes. As usual, since 2 is invertible over Y, then the theory of symmetric bilinear forms is equivalent to that of quadratic forms. Here we will concentrate on the former.

Let Y be a scheme. A *vector bundle* E on Y is a locally free O_Y-module of finite rank. The dual of a vector bundle E is the vector bundle $E^\vee$ such that, for any open subscheme Z of Y

$$E^\vee(Z) = Hom_{O_Z}(E|_Z, O_Z) \ .$$

There is a natural evaluation pairing $< \ , \ >$ between E and $E^\vee$ and one can identify E with the double dual $E^{\vee\vee}$ by

$$\kappa : E \cong E^{\vee\vee} \ ,$$

such that $<\alpha, \kappa(u)> = <u, \alpha>$. A *symmetric bilinear form* on X is a vector bundle E on Y equipped with a map of sheaves

$$B : E \times_Y E \to O_Y \ ,$$

which on sections over an open subscheme restricts to a symmetric bilinear form. This defines an *adjoint* map

$$\varphi = \varphi_B : E \to E^\vee \ ,$$

which because of the symmetry assumption equals its transpose:

$$\varphi = \varphi^t : E \xrightarrow{\kappa} E^{\vee\vee} \xrightarrow{\varphi^\vee} E^\vee \ .$$

We shall say that (E, B) is *non-degenerate* (or unimodular) if the adjoint φ is an isomorphism. (Note that some authors call non-degenerate forms spaces.) One can show that a form on a locally free module of rank n over a ring R in which 2 is invertible is non-degenerate if and only if locally in the étale topology it is isomorphic to

$$\begin{cases} x_1y_1 + \cdots + x_my_m & \text{if } n = 2m \text{ is even} \\ x_1y_1 + \cdots + x_my_m + az^2 \ , a \in R^\times & \text{if } n = 2m+1 \text{ is odd} \ , \end{cases}$$

see [Sw] Cor. 1.2, [Knu] IV (2.2.1), IV (3.2.1) or [M-R] 1.11.5 (In [M-H] I (3.4) it is shown how to diagonalize a non-degenerate form over a local ring in which 2 is invertible.) Morphisms among forms, orthogonal sums and tensor products are defined as expected.

Example 1.1. Let $\pi : X \to Y$ be an oddly, tamely ramified cover as in Sect. 1.b. We obtain a non-degenerate symmetric bilinear form on Y by letting

$$\begin{aligned} \mathcal{D}_{X/Y}^{-1/2} &:= O_X\left(\sum_{h, \xi_h' \to \xi_h} \left(\frac{e(\xi_h') - 1}{2}\right)\overline{\{\xi_h'\}}\right) \\ E &:= \pi_*(\mathcal{D}_{X/Y}^{-1/2}) \ , \end{aligned}$$

and by considering the bilinear form $B : E \times_Y E \to O_Y$ given by $B(x, y) = \mathrm{trace}_{X/Y}(x \cdot y)$ (see [EGAIV] IV 18.2.1). Note that twice the divisor defining E is isomorphic to the (relative) canonical divisor of X/Y and that if π is étale, then $\mathcal{D}_{X/Y}^{-1/2} = O_X$.

There is a further useful way to view a form over a scheme. For E a vector bundle over Y, let $\mathbf{P}(E) \to Y$ denote the associated projective scheme. Then, giving a symmetric bilinear form on E is equivalent to giving a section of the invertible sheaf $O(2)_{\mathbf{P}(E)}$ over $\mathbf{P}(E)$ (a section of $S_2(E)$) (see [Sw] Lemma 2.1 and [Del]).

1.c.1. *Metabolic forms.* The notion of a metabolic form was introduced by Knebusch for his definition of the Witt group of a scheme in [Kne]. Hyperbolic forms are metabolic and the greater generality of the latter notion is the right one in the global situation of non-affine schemes (see below). More recently in [Ba1], [Ba2] Balmer has extended the notion of a metabolic form in the context of triangulated categories with duality and he has shown its usefulness. In fact the interpolation procedure which we will set-up in Sect. 3 shows that even in our very special situation it is quite natural to consider forms as self-dual complexes (see the remark after Prop. 3.12).

Let E be a vector bundle over Y. A sub-O_Y-module V of E is a *sub-bundle* of E if it is *locally* a direct summand, *i.e.* for any y in Y, there is an open Z containing y such that $V|_Z$ has a direct summand in $E|_Z$. If a V is a sub-bundle of E, then V and the quotient E/V are both vector bundles. Let now (E, B) be a form. For a sub-module $i : V \subset E$ one defines an orthogonal complement, which is the sub-O_Y-module $V^\perp$, whose sections over the open Z consist of those sections of E which are orthogonal to all sections of V over any open subset of Z. Alternatively:

$$V^\perp = \ker(E \overset{\varphi_B}{\to} E^\vee \overset{i^\vee}{\to} V^\vee) \ .$$

If furthermore V is a sub-bundle of E, then $i^\vee$ is an epimorphism, and if (E, B) is non-degenerate, then φ_B is an isomorphism (by definition). So under these assumptions we have an isomorphism

$$\alpha : E/V^\perp \cong V^\vee \ ,$$

$E/V^\perp$ is locally free and $V^\perp$ is also a sub-bundle. There also is an isomorphism

$$\beta : V^\perp \cong (E/V)^\vee \ .$$

(The form $(E/V)^\vee$ can be identified with the sub-O_Y-module of E whose sections over $Z \subset Y$ are the linear forms $\lambda : E|_Z \to O_Z$ which vanish on V, so $V^\perp = \varphi_B^{-1}((E/V)^\vee)$.) The maps α and β correspond to bilinear maps which are perfect dualities, obtained by “restriction” from B:

$$V \times_Y E/V^\perp \to O_Y$$

$$V^\perp \times_Y E/V \to O_Y \ .$$

A further duality is

$$V^{\perp\perp} \times_Y E/V^\perp \to O_Y \ ,$$

which gives $V = V^{\perp\perp}$. We have the following commutative diagrams

$$\begin{array}{ccccc} E & \to & E/V^{\perp} & \to & 0 \\ \varphi_B \downarrow & & \downarrow \alpha & & \\ E^{\vee} & \to & V^{\vee} & \to & 0 \end{array} ,$$

and

$$\begin{array}{ccccc} 0 & \to & V^{\perp} & \to & E \\ & & \beta \downarrow & & \downarrow \varphi_B \\ 0 & \to & (E/V)^{\vee} & \to & E^{\vee} \end{array} .$$

A sub-bundle V of a non-degenerate bilinear form (E, B) is a *totally isotropic sub-bundle* (also called a *sub-lagrangian*) if $V \subset V^{\perp}$. If V is a sub-lagrangian of (E, B), then $V^{\perp}/V$ is a sub-bundle of E/V and the form on $V^{\perp}/V$ obtained by reducing B modulo V is non-degenerate. A sub-bundle V of a non-degenerate form (E, B) is called a *lagrangian* if it is such that $V = V^{\perp}$. The form (E, B) is called *metabolic* if it contains a lagrangian. If V is a lagrangian in (E, B), then $\mathrm{rank}(E) = 2 \cdot \mathrm{rank}(V)$ and V is in a sense a maximal totally isotropic sub-bundle. Using the above, we see that V is a lagrangian in E if and only if one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & V & \to & E & \to & V^{\vee} & \to & 0 \\ & & id \downarrow & & \downarrow \varphi_B & & \downarrow id & & \\ 0 & \to & V \cong V^{\vee\vee} & \to & E^{\vee} & \to & V^{\vee} & \to & 0 \end{array} .$$

That is a metabolic form is given by a symmetric/self-dual short exact sequence (see [Kne] Chapt. 3).

Example 1.2. A special case of the notion of a metabolic form is that of a so-called *split metabolic* form, which corresponds to the case of a split exact sequence. Let (U, C) be a form with adjoint φ. The split metabolic space associated with (U, C) is the form $M(U, C)$, which has as underlying bundle $U \oplus U^{\vee}$ and form B the form with adjoint

$$\begin{pmatrix} \varphi & id \\ id & 0 \end{pmatrix} .$$

The form $H(U) := M(U, 0)$ is called *hyperbolic.* For a form (E, B), the form $(E \perp E, B \perp -B)$ is metabolic. It is shown in [Kne] Prop. 3.1 and 3.2, that $M(U, 2C) \cong H(U)$, and that

$$M(U, C) \perp M(U, -C) \cong H(U) \perp M(U, -C) .$$

Since 2 is invertible, a split metabolic form is isomorphic to a hyperbolic form: just consider the equality

$$\begin{pmatrix} id & -\frac{1}{2}\varphi \\ 0 & id \end{pmatrix} \begin{pmatrix} \varphi & id \\ id & 0 \end{pmatrix} \begin{pmatrix} id & 0 \\ -\frac{1}{2}\varphi & id \end{pmatrix} = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix} .$$

Moreover, for an affine scheme Y, one shows that any metabolic space is split metabolic (*loc. cit.* Cor. 3.1) and that for any non-degenerate form (E, B) over Y, there is a splitting analogous to the classical Witt decomposition. Namely (E, B) is isometric to the orthogonal sum of a form which does not contain any sub-lagrangian (*i.e.* is anisotropic) and a form which is split metabolic (*loc. cit.* Prop. 3.3). (One should be aware that such a decomposition does not determine either summand. One obtains a better analogue of the Witt decomposition by working inside the Grothendieck-Witt ring of Y [Kne] Thm. 4.3.) Over a complex elliptic curve there exist infinitely many non-isometric metabolic forms which are not split metabolic (see [Knu] VIII (1.1.1)).

Example 1.3. In our set-up, metabolic forms will be useful in comparing two forms over an irreducible scheme Z which are isometric when restricted to the generic fiber of Z. Namely if (E, B_E) and (F, B_F) are defined over Z and agree on the generic fiber

$$E|_{\eta_Z} = F|_{\eta_Z} \ ,$$

then *under suitable assumptions* $(E \perp F, B_E \perp - B_F)$ is metabolic (see *e.g.* Prop. 3.12). A natural approach is to consider the sub-sheaf $\mathcal{G}$ of $E|_{\eta_Z} = F|_{\eta_Z}$ defined as

$$\mathcal{G} := < e - f | e \in E \ , f \in F > \ ,$$

and the exact sequence

$$0 \to E \cap F \to E \perp F \to \mathcal{G} \to 0 \ ,$$

where the maps are obtained by restricting to $E \perp F$ the maps defined at the level of the generic fibers given by the diagonal map and the map sending (x, y) to $(x - y)/2$. Then, if $\mathcal{G}$ is locally free, it follows that $E \cap F$ is a sub-bundle of $E \perp F$, which is a lagrangian and

$$E \perp F / E \cap F \simeq (E \cap F)^\vee \simeq \mathcal{G} \ .$$

The point is that $\mathcal{G}$ *might not be locally free* in general; on the other hand $\mathcal{G}$ is certainly locally free in the case of Dedekind schemes, which is the case which was considered in [E-K-V].

1.d. Classifications.

As with forms over fields, non-degenerate, symmetric, bilinear forms of a given rank n over a scheme Y can be considered as torsors under an orthogonal group and hence are classified by the non-abelian cohomology set

$$H^1(Y, \mathbf{O}(n)) \ .$$

Jardine shows in [J1] how to identify this set with the set of morphisms

$$[Y, \mathbf{BO}(n)]$$

from Y to the "classifying space" $\mathbf{BO}(n)$ inside the homotopy category of (étale) simplicial sheaves on Y. This leads to one approach to the invariants of forms on schemes which is very close to the approach to characteristic classes following Borel (see [St] App. 10 and Rem. 1.d.2 below). We think it is worthwhile to go over Jardine's description in some detail. The powerful techniques and ideas he exploits are seldom presented in a form suitable to the non-specialists. Going over this material also allows us to see how simplicial schemes arise quite naturally in our set-up (see the Appendix).

1.d.1. *Cohomology sets.* We start by giving precise references for the classification in terms of first cohomology sets. The *orthogonal group* of a given form is a group *scheme*. It is, however, most easily defined in terms of its associated functor of points. Let R be a ring in which 2 is invertible. *Models* over R are R-algebras. The functor we are interested in is a functor from models to sets which satisfies some special properties, making it a sheaf for the (Grothendieck) (fppf)-topology. Let (M, B_M), (N, B_N) be two quadratic forms over R. Let $\mathbf{Isom}(B_M, B_N)$ denote the functor from models over R to sets, which sends an extension R' of R to the set of isometries between the forms B_M and B_N extended to R'. If $B_M = B_N$, this defines a sheaf

$$\mathbf{O}(B_M) := \mathbf{Isom}(B_M, B_M) \ ,$$

which is the *orthogonal group* (functor) of (M, B_M) ([D-G] III, Sect. 5, n.2). This sheaf is representable by an affine scheme, but this property does not make the discussion any easier. Let us fix a quadratic form (M, B_M) over R. We say that the quadratic form (N, B_N) is a *twisted form* of (M, B_M) if there is a (fppf)-covering R-algebra R' such that (N, B_N) becomes isomorphic to (M, B_M) over R'. Recall that a *torsor* X under a (sheaf of) group(s) G over a sheaf Y (*e.g.* a scheme) is a sheaf X covering Y, which is equipped with an action of G such that

$$X \times G \cong X \times_Y X$$

under the map given on sets of sections by $(x, g) \mapsto (x, xg)$. (We are essentially saying that the action of G on X is simply transitive.) In *loc. cit.* one finds a proof that the map

$$(N, B_N) \mapsto \mathbf{Isom}(B_M, B_N) \ ,$$

is a bijection between the set of isometry classes of twisted forms of (M, B_M) and the set $\widetilde{H}^1(R, \mathbf{O}(B_M))$ of isomorphism classes of torsors under $\mathbf{O}(B_M)$ over R. This is a pointed set, pointed by the class of the trivial torsor. All goes through for a scheme Y in place of the ring R and in our set-up the torsors X are schemes.

Example 1.4. In view of what we have recalled about non-degenerate forms, we see that non-degenerate forms of rank n, over a scheme Y over which 2 is invertible are classified by $\widetilde{H}^1(Y, \mathbf{O}(n))$.

Example 1.5. One can give a cohomological description of metabolic forms over a ring R (see [K-O2] and [Knu] IV (2.4.2)). If $\mathbf{O}(2n,n)$ denotes the group functor which sends an extension R'/R to the group of isometries of the hyperbolic space $H(R'^n) = R'^n \perp R'^{n\vee}$ which stabilize $R'^{n\vee}$, then metabolic spaces (with a distinguished lagrangian) are classified by $\widetilde{H}^1(R, \mathbf{O}(2n,n))$.

By the Theorem of Torsor Descent recalled below, one can "compute" the cohomology set $\widetilde{H}^1$ in terms of a set (of orbits) of cocycles, and more precisely as the cohomology of a cosimplicial group, called the Amitsur complex. In fact one first identifies the set $\widetilde{H}^1(Y, \mathbf{O}(B_M))$ with the direct limit of sets of torsors which are trivialized over a given covering $Y' \to Y$ (*i.e.* a sheaf epimorphism). More generally for any (sheaf of) group(s) G one has

$$\widetilde{H}^1(Y,G) = \lim_{\to} \widetilde{H}^1(Y'/Y,G) \ ,$$

where the limit is taken over the coverings Y'/Y. The sets on the right are defined by

$$\widetilde{H}^1(Y'/Y,G) := \ker(\widetilde{H}^1(Y,G) \to \widetilde{H}^1(Y',G)) \ .$$

(Here ker consists of elements in the inverse image of the distinguished point.) We recall the definition of the Amitsur complex and of the first cohomology set $H^1(Y'/Y,G)$ in the Appendix, see (5.4). The Theorem of Torsor Descent is the following statement (see [D-G] III, Sect. 4, n. 6.5).

Theorem 1.6. *Let $Y' \to Y$ be a sheaf epimorphism, and let G be a sheaf of groups. There is a canonical bijection*

$$H^1(Y'/Y,G) \leftrightarrow \widetilde{H}^1(Y'/Y,G) \ .$$

1.d.2. *Morphisms in the homotopy category.* Let now $[Y, \mathbf{B}G]$ denote the set of morphisms between the simplicial sheaves Y and $\mathbf{B}G$ *in the homotopy category* (see Appendix 5.c for definitions). The next result is due to Jardine [J1] Cor. 1.4. We sketch a proof of this in Appendix 5.d.

Proposition 1.7. *For any sheaf of groups G on Y, there is a bijection*

$$H^1(Y,G) \leftrightarrow [Y, \mathbf{B}G] \ .$$

Remark 1.8. The proposition is an algebraic/combinatorial version of a result which is quite typical in geometry, where principal G-bundles on a space Y are classified by maps into a classifying space BG, so that the bundles are obtained by pull-back from a universal bundle EG on BG (see [St] Sect. 19). For certain groups G, one has a very explicit description of the space BG. For instance the classifying space for the real orthogonal group $\mathbf{O}(n, \mathbf{R})$ is given by the Grassmann manifold G_n (see [M-S] 5.6).

In the first two sections of [Mac-Mo] Chapt. VII, the reader will find how to identify principal G-bundles over a space Y with G-torsors over Y, in the case of a discrete group G. For the simplicial version of this see [Cu] Sect. 6. A version of the proposition for simplicial sets can be found in [Go-Ja] Thm. 3.9.

1.e. Cohomological invariants I: Hasse-Witt.
Let (E, B) denote a non-degenerate form over a scheme Y on which 2 is invertible. We recall the definition of the Hasse-Witt invariants $w_i(E)$ attached to (E, B) in $H^i(Y_{et}, \mathbf{Z}/2\mathbf{Z})$, following Jardine's work and [E-K-V]. We begin by defining the invariants in low degree. Then, we give two definitions of the higher degree Hasse-Witt invariants: the first is by pulling back universal classes, the second is via the decomposition of the cohomology of the complement of the quadric defined by B inside the projective bundle associated to E.

1.e.1. *Low degree.* To a form (E, B) of rank 1 over Y, there corresponds a class

$$w_1(E) = w_1(E, B) := [(E, B)] \text{ in } H^1(Y_{et}, \mathbf{O}(1)) = H^1(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \ .$$

Let $\det(-)$ denote the top exterior power operation. Then the *first Hasse-Witt invariant* of a form (E, B) of arbitrary rank, is

$$w_1(E) := w_1(\det(E), \det(B)) \ ,$$

where the right hand term is the Hasse-Witt invariant of the rank one form $(\det(E), \det(B))$ as defined above. Thus, under the map induced in cohomology by the map $\mathbf{Z}/2\mathbf{Z} = \mu_2 \to \mathbf{G}_m$, the first Hasse-Witt invariant is sent to the first Chern class $c_1(E)$:

$$\begin{array}{ccc} H^1(Y_{et}, \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & \mathrm{Pic}(Y) = H^1(Y_{et}, \mathbf{G}_m) \\ & & \\ w_1(E) & \longmapsto & c_1(E) = [\det(E)] \ , \end{array}$$

(see [Knu] III.4.2). Below we shall view c_1 as taking values in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$, via the coboundary map coming from the Kummer sequence.

In degree 2 one Let $\pi : X \to Y$ can define an invariant by a construction, which generalizes that of the Clifford algebra. Namely, to a form (E, B) over Y, one associates a sheaf $\mathcal{C}(E)$ of algebras over Y, called the *Clifford algebra* of (E, B) (see [K-O1], [Knu] IV (2.2.3); in odd rank one only considers the even part of the full Clifford algebra). The algebra $\mathcal{C}(E)$ is an Azumaya algebra, that is an O_Y-algebra which locally in the étale topology is isomorphic to the O_Y-algebra $\mathrm{End}_{O_Y}(W)$ of endomorphisms of some O_Y-bundle W. The class of this algebra in the Brauer group of Y is 2-torsion and is called the *Clifford invariant* of (E, B)

$$C(E) := [\mathcal{C}(E)] \text{ in } {}_2H^1(Y_{et}, \mathbf{PGL}(2^{n'})) \subset {}_2Br(Y) \ ,$$

where $n' = n/2$, if the rank n of E is even, and $n' = (n-1)/2$, if the rank is odd. The Clifford invariant can be viewed in ${}_2H^2(Y_{et}, \mathbf{G}_m)$ (see *e.g.* [Mi] Chapt. 4, Thm. 2.5). The invariant we will be interested in is a lift of this invariant under the map

$${}_2H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \to {}_2H^2(Y_{et}, \mathbf{G}_m) \ .$$

In [F], Fröhlich has shown how to construct an extension

$$\alpha_B : 1 \to \mathbf{Z}/2\mathbf{Z} \to \widetilde{\mathbf{O}}(B) \to \mathbf{O}(B) \to 1 \ ,$$

which, in the case B is the standard form, gives rise to a map

$$\Delta_\alpha : H^1(Y_{et}, \mathbf{O}(n)) \to H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \ .$$

The *second Hasse-Witt invariant* of (E, B) is

$$w_2(E) = w_2(E, B) := \Delta_\alpha([(E, B)]) \ ,$$

see also [E-K-V] Sect. 1.9, and [J2] Appendix. (Note that the Pinor extension constructed in [A-B-S] gives $w_2 + w_1^2$.) Further approaches to invariants in degree 2 are given in [Pat] and [Pa-S].

1.e.2. *Universal classes.* To obtain invariants in higher degree one has to work harder. The approach outlined here is that of Jardine [J1]. It relies on his computation of the cohomology algebra of the classifying simplicial scheme $\mathbf{BO}(n)$ (over the big étale site of $\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$. The computation shows that this algebra is a polynomial algebra over a finite number of elements. These elements are the *universal Hasse-Witt invariants.* We will use these universal invariants to give one definition of the Hasse-Witt invariants of forms, but also, in Sect. 1.f below, to define invariants of orthogonal representations. As before let Y be a scheme over $\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$.

Theorem 1.9. *The cohomology ring $H^*(\mathbf{BO}(n)_{et}, \mathbf{Z}/2\mathbf{Z})$ is a polynomial algebra over the algebra $H^*(\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])_{et}, \mathbf{Z}/2\mathbf{Z})$, with n distinguished generators HW_i, where for $1 \le i \le n$, $deg(HW_i) = i$.*

This allows one to define the Hasse-Witt invariants of a form (E, B) over Y as follows. Recall from Prop. 1.7 that if (E, B) has rank n, then it corresponds to an element $[E]$ in the set $[Y, \mathbf{BO}(n)]$ of homotopy classes of maps from Y to $\mathbf{BO}(n)$. One sets

$$w_i(E) = w_i(E, B) := [E]^*(HW_i) \ .$$

It can be shown that in low degree these classes agree with those defined in the previous paragraph (see [E-K-V] Sect. 1.9 and Sect. 1.h below).

1.e.3. *Projective bundles.* A further, equivalent, way of defining higher degree invariants proceeds along the lines indicated by Grothendieck's definition of Chern classes in [Gr]. Following Delzant, Laborde has shown how to use this approach to define invariants in étale cohomology for quadratic forms over schemes (see [Dz] [La]). His work has been detailed in [E-K-V] and proceeds as follows.

Given a non-degenerate form (E, B) over Y of rank n, one considers the complement U of the quadric associated to B in the projective bundle $p : \mathbf{P}(E) \to Y$. By identifying the form with a section $s : O_Y \to S_2(E)$ (as in the end of Sect. 1.c), one obtains a *non-degenerate* form on the invertible sheaf $O(1) := O_{\mathbf{P}(E)}(1)|_U$. This form defines a class η in $H^1(U_{et}, \mathbf{Z}/2\mathbf{Z})$. The following result is Claim 5.2 in [E-K-V].

Theorem 1.10. *Let p' denote the restriction to U of the bundle map p. For any integer $i \geq 0$, one has the decomposition*

$$H^i(U_{et}, \mathbf{Z}/2\mathbf{Z}) = \bigoplus_{0 \leq j \leq \inf(i, n-1)} p'^*(H^{i-j}(Y_{et}, \mathbf{Z}/2\mathbf{Z}))\eta^j \ .$$

Corollary 1.11. *There exist unique elements $w_1(E), w_2(E), \dots, w_n(E)$ in $H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ such that*

$$\eta^n = \sum_{0 \leq j \leq n-1} p'^*(w_{n-j}(E))\eta^j \ .$$

The equivalence of this definition of the Hasse-Witt invariants $w_i(E)$ with the previous one is shown in [E-K-V] Prop. 1.4.

1.e.4. *Axiomatic approach.* We give a list of properties characterizing the Hasse-Witt invariants $w_i(E) = w_i(E, B)$ in $H^i(Y_{et}, \mathbf{Z}/2\mathbf{Z})$.

(HW0) *Normalization.* $w_0(E) = 1$, $w_1(E) = [(\det(E), \det(B_E))]$ and $w_i(E) = 0$ for $i \geq \operatorname{rank}(E) + 1$.

(HW1) *Functoriality.* For any morphism $f : Z \to Y$,

$$f^*(w_i(E)) = w_i(f^*(E)) \ .$$

(HW2) *Whitney formula.* Let $w_t(E) = \sum_i w_i(E)t^i$ be viewed as a formal power series. Then for non-degenerate forms (E, B_E) and (F, B_F) one has the identity

$$w_t(E \perp F) = w_t(E) \cdot w_t(F) \ .$$

Remark 1.12. The topological Stiefel-Whitney classes satisfy a list of properties analogous to the ones above (see [M-S] Chapt. 4). The two kinds of invariants can be related as follows (see [Ca]). Let Y be a compact, connected, locally connected topological space and write $\mathbf{R}(Y)$ for the ring of continuous real valued functions on Y. One can compare the Hasse-Witt invariants and the topological Stiefel-Whitney classes in the context of the correspondence between bundles over Y and modules over the ring $\mathbf{R}(Y)$. There is a homomorphism

$$H^i(\mathrm{Spec}(\mathbf{R}(Y))_{et}, \mathbf{Z}/2\mathbf{Z}) \to H^i(Y, \mathbf{Z}/2\mathbf{Z}) \ .$$

A non-degenerate form (E, B) over $\mathbf{R}(Y)$ determines an orthogonal bundle E over Y. This defines a homomorphism from the Grothendieck-Witt group $L(\mathbf{R}(Y))$ to the Grothendieck group $KO(Y)$ of orthogonal bundles over Y. Let $G^*(-, \mathbf{Z}/2\mathbf{Z})$ denote the group of invertible elements in the cohomology ring $H^*(-, \mathbf{Z}/2\mathbf{Z})$. Then, as shown in [Ca] Thm. 5.18, the Hasse-Witt invariants w_i and the topological Stiefel-Whitney classes sw_i give rise to a commutative diagram

$$\begin{array}{ccc} L(\mathbf{R}(Y)) & \rightarrow & KO(Y) \\ w_t \downarrow & & \downarrow sw_t \\ G^*(\mathrm{Spec}(\mathbf{R}(Y)), \mathbf{Z}/2\mathbf{Z}) & \rightarrow & G^*(Y, \mathbf{Z}/2\mathbf{Z}) \end{array}$$

1.f. Cohomological invariants II: Galois theoretic.
Let $\pi : X \to Y$ be a tamely and oddly ramified cover of degree n as in Sect. 1.b, and assume Y is connected. The classes $w_i(\pi)$ we are going to define in this section will depend on the natural representation of the tame fundamental group of Y defined by π. In fact we will only need the class w_2, but since it does not cost more effort we give the general definition.

The cover π is determined by a homomorphism e_π from the tame-odd fundamental group $\pi_1(Y)^{t,o}$ into the symmetric group S_n. For étale covers this is well known: in this case $\pi_1(Y)^{t,o}$ equals the fundamental group of Y and the homomorphism is described in–say–[Mu] Remark at the end of Sect. 12. For general tame covers see [Gr-M] 2.4.4. In principle we should indicate the dependence of $\pi_1(Y)^{t,o}$ upon the choice of a base point and the fixed branch locus, but we do not need to be so precise.

Let $\overline{K}$ denote the algebraic closure of the residue field of some point on Y. By embedding S_n into the orthogonal group $\mathbf{O}(n)(\overline{K})$ by permutation matrices we obtain an orthogonal representation of the discrete group $\pi_1(Y)^{t,o}$, which we denote by ρ_π. We want to use this representation to pull back universal classes to the cohomology of $\pi_1(Y)^{t,o}$. Consider the composition c of the map

$$H^i((\mathbf{BO}(n)/Y)_{et}, \mathbf{Z}/2\mathbf{Z}) \to H^i((\mathbf{BO}(n)/\overline{K})_{et}, \mathbf{Z}/2\mathbf{Z})$$

induced by the change of sites given by $\mathrm{Spec}(\overline{K}) \to Y$ and of the map

$$H^i((\mathbf{BO}(n)/\overline{K})_{et}, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\epsilon^*} H^i(\mathbf{BO}(n)(\overline{K}), \mathbf{Z}/2\mathbf{Z}) \ ,$$

which is induced by a map ϵ on simplicial sheaves defined as follows. View the simplicial scheme $\mathbf{BO}(n)$ as the simplicial sheaf it represents and let

$\Gamma^*\mathbf{BO}(n)(\overline{K})$ denote the constant simplicial sheaf attached to the simplicial set $\mathbf{BO}(n)(\overline{K})$ of global sections of the simplicial sheaf $\mathbf{BO}(n)$. Here Γ^* is adjoint to the global section functor and ϵ is the counit of the adjunction: a map from $\Gamma^*\mathbf{BO}(n)(\overline{K})$ to $\mathbf{BO}(n)$. To obtain ϵ^* one has to use the isomorphisms

$$H^i(\mathbf{BO}(n)(\overline{K}), \mathbf{Z}/2\mathbf{Z}) \cong [\Gamma^*\mathbf{BO}(n)(\overline{K}), K(\mathbf{Z}/2\mathbf{Z}, i)]$$

and

$$H^i((\mathbf{BO}(n)/\overline{K})_{et}, \mathbf{Z}/2\mathbf{Z}) \cong [\mathbf{BO}(n), K(\mathbf{Z}/2\mathbf{Z}, i)] \ ,$$

where $K(\mathbf{Z}/2\mathbf{Z}, i)$ is the Eilenberg-MacLane presheaf of simplicial abelian groups, and then precompose with ϵ (see [J3] Sect. 3 and Appendix 5.c). Next, from the orthogonal representation ρ_π we have the map

$$\rho_\pi^* : H^*(\mathbf{BO}(n)(\overline{K}), \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^*(\pi_1(Y)^{t,o}, \mathbf{Z}/2\mathbf{Z}) \ ,$$

which is obtained by identifying the first group with the cohomology of the discrete group $\mathbf{O}(n)(\overline{K})$. Thus, using Thm 1.9, we let

$$w_i(\rho_\pi) := \rho_\pi^*(c(HW_i)) \quad \text{in} \quad H^i(\pi_1(Y)^{t,o}, \mathbf{Z}/2\mathbf{Z}) \ .$$

Note that by a result of Friedlander and Parshall in [F-P] one has an isomorphism

$$H^i(\mathbf{BO}(n)/\overline{K}, \mathbf{Z}/2\mathbf{Z}) \cong H^i(\mathbf{BO}(n)/\mathbf{C}, \mathbf{Z}/2\mathbf{Z})$$

and this last group can be identified with the i-th singular cohomology group with modulo 2 coefficients of the group $\mathbf{O}(n)(\mathbf{C})$ (this is a simplicial version of the classical comparison theorem between étale and singular cohomology for algebraic varieties over $\mathbf{C}$, see *e.g.* [Mi] Thm. 3.12). Hence the algebra $H^*(\mathbf{BO}(n)/\overline{K}, \mathbf{Z}/2\mathbf{Z})$ is generated by Stiefel-Whitney classes SW_i over $\mathbf{Z}/2\mathbf{Z}$. The image under the first map in the composition above sends HW_i to SW_i.

We now want to push the element $w_i(\rho_\pi)$ forward to $H^i(Y_{et}, \mathbf{Z}/2\mathbf{Z})$. This is achieved by the analysis of the spectral sequences converging to the equivariant cohomology $H^*(Y_{et}, \pi_1(Y)^{t,o}; \mathbf{Z}/2\mathbf{Z})$ (see [E-K-V] Sect. 3.1). Because we are working with oddly ramified coverings, we obtain the canonical map

$$\mathrm{can} : H^i(\pi_1(Y)^{t,o}, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^i(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \ .$$

We let

$$w_i(\pi) := \mathrm{can}(w_i(\rho_\pi)) \ .$$

Note that there is also a canonical map

$$\mathrm{can} : H^i(\pi_1(Y), -) \longrightarrow H^i(Y_{et}, -) \ ,$$

compatible with the above map, whose origin is explained for instance in [K] Proof of Prop. 6.1 (see also [Gil] Lemma 1.16; in [Go-Ja] III Rem. 1.3 a map from a space X to the classifying space of its fundamental group is constructed). We shall use a version of this map for the non-abelian H^1 in Sect. 1.h.

Remark 1.13. In low degree the Galois theoretic invariants we have defined have interpretations in terms of the properties of the map π. So $w_1(\pi)$ is (also) the function field discriminant, and $w_2(\pi)$ can be interpreted as giving an obstruction to an embedding problem (see [J2]).

1.g. The main lemma: comparing Chern and Hasse-Witt invariants.

Here we state and sketch a proof of a result from [E-K-V] which is one of the main tools in calculating the Hasse-Witt invariants. The result can be viewed as answering the natural question of determining whether these invariants vanish on metabolic forms. As we have recalled above, such forms are the right generalization of hyperbolic forms to the setting of forms over (global) schemes and so one would expect their properties to be determined by the (linear) properties of the lagrangians they contain. It turns out that the invariants *do not* vanish, that is they do not define invariants on the Witt group. However, they are indeed determined by the Chern classes of a lagrangian. Note that the Clifford invariant of a metabolic form is trivial, as can be seen by using the cohomological description of the second example in Sect. 1.d.1 (see [K-O2] and [Knu] IV (2.4)).

As a preparation for the general statement we observe the following. Let R be a ring in which 2 is a unit. Then, the second Hasse-Witt invariant of a hyperbolic form $H(I)$ over an invertible R-module I equals the image, under the coboundary map induced by the Kummer sequence, of the class of I in the Picard group: that is the first Chern class (see [O] Thm. 19).

For a bundle V of rank n over Y let $c_i(V)$ denote the i-th Chern class of V in $H^{2i}(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ (see [Gr]). The following equality holds

$$c_i(V) = c_i(V^\vee) \ .$$

Define an element of $H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ by

$$d_t(V) = \sum_{i=0}^{n} (1 + (-1)t)^{n-i} c_i(V) t^{2i} \ .$$

Lemma 1.14. *Let (E, B) be a metabolic form with lagrangian V. Then*

$$w_t(E) = d_t(V) \ .$$

In particular

$$1 + w_1(E)t + w_2(E)t^2 = 1 + n(-1)t + \left(c_1(V) + \binom{n}{2} (-1) \cup (-1) \right) t^2 \ .$$

This is Prop. 5.5 in [E-K-V]. We shall use the result in the following form (see *loc. cit.* Thm. 6.2, Cor. 6.3).

Corollary 1.15. *Let Y be an irreducible scheme with generic point η. Let (E, B_E) and (F, B_F) be non-degenerate forms which are isometric when restricted to η.*

a) $w_1(E) = w_1(F)$ *and* $c_1(E) = c_1(F)$.

b) *As in the example at the end of Sect. 1.c.1 consider the exact sequence*

$$0 \to E \cap F \to E \perp F \to \mathcal{G} \to 0 \ .$$

Assume that $\mathcal{G}$ (and $E \cap F$) is locally free, and that $(E \perp F, B_E \perp - B_F)$ is metabolic with lagrangian $E \cap F = \mathcal{G}^\vee$. Then

$$w_t(E, B_E) \cdot w_t(F, -B_F) = d_t(\mathcal{G}) \ .$$

c) $w_2(E) + w_2(F) = c_1(\mathcal{G}) + c_1(E)$. *In particular this sum belongs to the image of the Picard group modulo 2 in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$.*

d) $w_t(E, B_E) \cdot w_t(F, B_F) \cdot w_t(E, -B_E)^2 = \sum_i (1 + (-1)t)^{n-i} c_i(\mathcal{G} \perp E) t^{2i}$.

Part (a) is proved independently (see Sect. 4.b). Part (b) follows directly from the lemma and (d) follows from the lemma applied to the form

$$(E \perp F \perp E \perp E, B_E \perp B_F \perp - B_E \perp - B_E) \ .$$

One proves (c) along the lines of [E-K-V] p. 185 and one uses that $w_1(E, -q) = n \cdot (-1) + w_1(E, q)$.

We will apply the Main Lemma to the metabolic forms of Prop. 3.12.

1.h. The étale case.

As mentioned in the introduction, the case of étale covers has been considered in full generality in [E-K-V] Thm. 2.3. The result is the following.

Theorem 1.16. *Let $\pi : T \to Z$ be an étale cover, with Z connected over* $\mathrm{Spec}\mathbf{Z}[\frac{1}{2}]$. *Consider the non-degenerate form $(E, B_E) = (\pi_*(O_T), Tr_{T/Z})$ over Z. Then*

a) $w_1(E) = w_1(\pi)$ *and this also equals the function field discriminant* $(d_{T/Z})$

b) $w_2(E) = w_2(\pi) + (2) \cup (d_{T/Z})$.

Kahn has extended this result to the higher degree Hasse-Witt invariants in [K] Thm. 6.1. The formulation of this more general result requires the definition of another set of classes attached to π, and so we do not go into this. (Note that the Conjecture 2.4 of [E-K-V] turns out to be slightly false as stated.)

Since this theorem is one of the main ingredients in the proof of our Thm. 0.2, we summarize the main steps for a proof. The strategy for (b) consists in computing the difference of the classes $w_2(E)$ and $w_2(\pi)$. The computation is made possible by the fact that these classes are the image of the same element under two coboundary maps which correspond to locally isomorphic short exact sequences. More precisely:

a) Let S_n denote the symmetric group on n letters. Up to isomorphism, the degree n cover $\pi : T \to Z$ corresponds to an element $[e]$ in $H^1(\pi_1(Z), S_n)$ and hence under the canonical map of Sect. 1.f, to a class, again denoted $[e]$ in the first étale cohomology set $H^1(Z_{et}, S_n)$ (see *e.g.* [Mi] Chap. I, Thm. 5.3).

b) There are two embeddings/sheaf homomorphisms of S_n into orthogonal groups:

$$\begin{tikzcd} & \mathbf{O}(n) \\ S_n \arrow[ur] \arrow[dr] & \\ & \mathbf{O}(n)(\overline{K}) \end{tikzcd}$$

where $\mathbf{O}(n)$ is the orthogonal group of rank n and $\mathbf{O}(n)(\overline{K})$ is the group of points of $\mathbf{O}(n)$ with values in a geometric point $\mathrm{Spec}(\overline{K})$ of Z (whose choice is irrelevant).

c) Corresponding to these embeddings, $[e]$ defines classes in the cohomology of the orthogonal groups:

$$\begin{tikzcd} & {[\rho'_\pi] \in H^1(\pi_1(Z), \mathbf{O}(n))} \arrow[r, "\mathrm{can}"] & {[E] \in H^1(Z_{et}, \mathbf{O}(n))} \\ {[e] \in H^1(\pi_1(Z), S_n)} \arrow[ur] \arrow[dr] \arrow[r, "\mathrm{can}"] & {[e] \in H^1(Z_{et}, S_n)} \arrow[ur] \arrow[dr] & \\ & {[\rho_\pi] \in H^1(\pi_1(Z), \mathbf{O}(n)(\overline{K}))} \arrow[r, "\mathrm{can}"] & {[\pi] \in H^1(Z_{et}, \mathbf{O}(n)(\overline{K}))} \end{tikzcd}$$

One checks that $[E]$ is the class corresponding to (E, B_E) by the classification recalled in Sect. 1.d.1 (see [S1] 1.4 or [W] Sect. 6). Because the signature of a permutation equals the determinant of the corresponding permutation matrix one deduces that $w_1(E) = \det[E]$. Also $w_1(\pi) = \det[\pi]$.

d) There are two exact sequences of étale sheaves:

$$(\alpha) \quad 0 \to \mathbf{Z}/2\mathbf{Z} \to \widetilde{\mathbf{O}(n)} \to \mathbf{O}(n) \to 0$$

and

$$(\beta) \quad 0 \to \mathbf{Z}/2\mathbf{Z} \to \widetilde{\mathbf{O}(n)(\overline{K})} \to \mathbf{O}(n)(\overline{K}) \to 0\ ,$$

where the second is the $\overline{K}$-points of the first (see [F] Appendix 1, [E-K-V] Sect. 1.9 and [J2] Appendix). By pulling back along the embedding of S_n into the orthogonal groups we get central extensions:

$$(\alpha_S) \quad 0 \to \mathbf{Z}/2\mathbf{Z} \to \widetilde{S_n}' \to S_n \to 0$$

and

$$(\beta_S) \quad 0 \to \mathbf{Z}/2\mathbf{Z} \to \widetilde{S_n} \to S_n \to 0\ .$$

Note that in the first of these $\widetilde{S_n}'$ is not a constant group (see [E-K-V] Lemma 2.11). However these two extensions are locally isomorphic (see [E-K-V] Lemma 2.10).

e) The above four exact sequences give rise to coboundary maps Δ which fit into the following diagram:

$$\begin{tikzcd}
 & H^1(Z_{et}, \mathbf{O}(n)) \arrow[dr, "\Delta_\alpha"] & \\
H^1(Z_{et}, S_n) \arrow[ur] \arrow[dr] \arrow[rr, shift left, "\Delta_{\alpha_S}"] \arrow[rr, shift right, "\Delta_{\beta_S}"'] & & H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z}) \\
 & H^1(Z_{et}, \mathbf{O}(n)(\overline{K})) \arrow[ur, "\Delta_\beta"'] &
\end{tikzcd}$$

f) Consider the diagram

$$\begin{tikzcd}
H^1(Z_{et}, \mathbf{O}(n)) \arrow[dr, "\Delta_\alpha"] & & H^2((\mathbf{BO}(n)/Z)_{et}, \mathbf{Z}/2\mathbf{Z}) \ni HW_2 \arrow[dl, "{[E]^*}"'] \arrow[dd, "c"] \\
 & H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z}) & \\
H^1(Z_{et}, \mathbf{O}(n)(\overline{K})) \arrow[ur, "\Delta_\beta"'] & & H^2(\mathbf{BO}(n)(\overline{K}), \mathbf{Z}/2\mathbf{Z}) \arrow[ul, "{[\pi]^*}"]
\end{tikzcd}$$

where the map c was defined in Sect. 1.f. The class of HW_2 corresponds to the sequence α and that of $c(HW_2)$ to β. So from [Gir] VIII 6.2.10 (ii), we deduce that

$$\Delta_\alpha([E]) = [E]^*(HW_2) =: w_2(E)$$

and that

$$\Delta_\beta([\pi]) = [\pi]^*(c(HW_2)) =: w_2(\pi) \ .$$

Here we use the fact that HW_2 corresponds to the extension α and that $c(HW_2)$ corresponds to β. So $w_2(E) - w_2(\pi) = \Delta_{\alpha_S}([e]) - \Delta_{\beta_S}([e])$.

g) Now $\Delta_{\alpha_S} - \Delta_{\beta_S} = \Delta_{\alpha_S - \beta_S}$, and the central extensions α_S and β_S are locally isomorphic, so the difference of their classes in the equivariant cohomology group $H^2_Z(S_n, \mathbf{Z}/2\mathbf{Z})$ is locally split and hence defines a unique element (f) in $H^1(Z_{et}, \mathbf{Z}/2\mathbf{Z})$ such that

$$\Delta_{\alpha_S - \beta_S}(-) = (f) \cup \det(-) \ ,$$

see [E-K-V] Prop. 2.8.

h) One computes that $(f) = (2)$, so the theorem follows by using again the fact that $(d_{T/Z}) = \det([e])$.

Remark 1.17. We give some indications to show how little is known about invariants of étale algebras over a field F of *characteristic* 2 (for more details see [W] and [Deg]). Essentially one only knows how to deal with the first invariant of such an algebra. An étale algebra E over a field F corresponds to a class $[e]$ in $H^1(G_F, S_n)$, where $G_F = \mathrm{Gal}(F_s/F)$ denotes the absolute Galois group of F and n denotes the degree of E over F. In characteristic different from 2 the image of $[e]$ in $H^1(G_F, \mathbf{Z}/2\mathbf{Z}) \simeq F^*/F^{*2}$ given by the signature homomorphism equals the discriminant $d(E/F)$. Moreover in characteristic different from 2, the discriminant can either be defined as the discriminant of the bilinear trace form on E or, in case E is defined by a polynomial f, in terms of the roots of f. In characteristic 2 one has $H^1(G_F, \mathbf{Z}/2\mathbf{Z}) \simeq F/\mathcal{P}(F)$, where $\mathcal{P}(x) = x^2 + x$. The image of $[e]$ in this group can be identified with an *additive discriminant* $d^+(E/F)$, which in the case of an algebra defined by a polynomial has first been considered by Berlekamp (see *loc. cit.*, [Be] and [Ber-M]). In characteristic 2 one does not consider the bilinear trace form, given by the first coefficient in the characteristic polynomial, but rather naturally the *second trace form*, which is the *quadratic* form defined by the second coefficient. This leads to the definition of another invariant with values in $F/\mathcal{P}(F)$, namely the Arf invariant $Arf(E/F)$ of this quadratic form. It turns out that this invariant differs from the additive discriminant (by a small amount): indeed, $d^+(E/F) - Arf(E/F)$ equals 0 in case the degree E over F is congruent to 0, 1, 2 or 7 modulo 8 and equals 1 if not (see *loc. cit.*).

2. Construction of the étale cover T/Z

Let $\pi : X \to Y$ denote a tamely and oddly ramified cover of projective, regular schemes over $\mathrm{Spec}(\mathbf{Z}[\frac{1}{2}])$, which as in Sect. 1.b is obtained as a quotient cover from a cover $\tilde{\pi} : \widetilde{X} \to Y$ corresponding to a group action $(\widetilde{X}, G)$, with $\widetilde{X}$ regular and connected and $X = \widetilde{X}/H$ for some subgroup H of G. Let G_2 denote a 2-Sylow subgroup of G and let $Z := \widetilde{X}/G_2$.

Lemma 2.1. a) *The morphism $\psi : \widetilde{X} \to Z$ is étale. In particular Z is regular.*
b) *The base change $\phi : Z \to Y$ is flat.*

Proof. For (a) we have to show that $\widetilde{X}/Z$ is unramified and flat. Let us begin by showing that Z is normal. This follows from the fact that $\widetilde{X}$ is normal together with the fact that for any open U of Z we have that

$$O_Z(U) = O_{\widetilde{X}}(\psi^{-1}(U))^{G_2} \ . \tag{2.1}$$

By the parity assumption on the ramification, ψ is unramified and because Z is normal, to show that ψ is flat, it is sufficient to show that for any $\widetilde{x}$ in $\widetilde{X}$ and $z = \psi(\widetilde{x})$ the morphism $O_{Z,z} \to O_{\widetilde{X},\widetilde{x}}$ is an injection (see [Mi] Chapt. I, Thm. 3-20). But this is clear, since for any (affine) open U of Z containing z we have the inclusion $O_Z(U) \to O_{\widetilde{X}}(\psi^{-1}(U))$ (see (2.1)).

Part (b) follows along the same lines used in Sect. 1.b.

Put $T' := Z \times_Y X$ and let T be the normalisation of T'. Our aim is to show the following.

Theorem 2.2.

a) *The pull-back map* $\phi^* : H^*(Y_{et}, \mathbf{Z}/2\mathbf{Z}) \to H^*(Z_{et}, \mathbf{Z}/2\mathbf{Z})$ *is an injection.*
b) *The natural map* $\pi_Z : T \to Z$ *is étale.*
c) *The scheme* T *is regular.*

Proof. Part (a) is a consequence of the fact that the degree of the cover Z/Y is odd and the fact that the composition $\phi_* \circ \phi^*$ is multiplication by the degree [SGA4] IX 5.1.4. Part (c) follows from (b) and Lemma 2.1 (a). The proof of (b) will occupy the rest of this section.

Our strategy consists in exhibiting for any closed point z of Z an étale neighbourhood $\mathcal{U}$ of z, such that $T_{\mathcal{U}} := T \times_Z \mathcal{U} \to \mathcal{U}$ is étale. This will do, because $T_{\mathcal{U}}$ coincides with the normalisation of $T'_{\mathcal{U}} := T' \times_Z \mathcal{U}$. Our choice of $\mathcal{U}$ allows us to exhibit $T_{\mathcal{U}}$ as an étale Kummer cover of $\mathcal{U}$ in the sense of [Gr-M]. In fact $\mathcal{U}$ will be defined as $Z \times_Y S$, where S is a neighbourhood of the image y of z in Y. The result will then follow from a study of the local structure of the tame cover $\widetilde{X}/Y$. The key result is Prop. 2.6 below. The picture is the following.

$$
\begin{tikzcd}
 & & & & & & \widetilde{X}_S \arrow[dl, dashed] \arrow[dd, dashed] & \\
 & & & & & \widetilde{X} \arrow[dd, "\lambda"] & & 1 \arrow[dd, no head] \\
 & T_{\mathcal{U}} \arrow[rr, dashed] \arrow[dl, dashed] \arrow[ddrrr, dashed] & & T'_S \arrow[rrr, dashed] \arrow[dl, dashed] & & & X_S \arrow[dl, dashed, "q_X"] \arrow[dd, dashed, "\pi_S"] & \\
T \arrow[rr, "\text{norm.}"] \arrow[ddrr, "\pi_Z"'] & & T' \arrow[rrr, "\phi_X"] \arrow[dd, "\pi_0"] \arrow[ddr, phantom, "="] & & & X \arrow[dd, "\pi"] & & H \arrow[dd, no head] \\
 & & & & \mathcal{U} = Z_S \arrow[rr, dashed, "\phi_S"] \arrow[dll, dashed, "q_Z"] & & S \arrow[dl, dashed, "{q,\ \text{étale}}"] & \\
 & & Z \arrow[rrr, "\phi"'] & & & Y & G_2 \arrow[r, no head] & G
\end{tikzcd}
$$

2.a. Local structure of the covering $\widetilde{X}/Y$.
Let us introduce some notation. Consider a point $\widetilde{x}$ in $\widetilde{X}$. We need to describe the inertia group $I(\widetilde{x})$ at $\widetilde{x}$ for the cover $\widetilde{X}/Y$. Because of the tameness and regularity assumptions, $I(\widetilde{x})$ is the product of the inertia groups of the components of the ramification locus, which pass through $\widetilde{x}$. Namely, recall from Sect. 1.b the decompositions

$$b(\widetilde{X}/Y) = \coprod_{1\leq h\leq m} b_h \quad \text{and} \quad B(\widetilde{X}/Y) = \coprod_{h,k} \widetilde{B}_{h,k} \ ,$$

where the b_h are the irreducible components of b and where for any fixed integer h between 1 and m, the $\widetilde{B}_{h,k}$ are the irreducible components of $\widetilde{B}$ such that $\tilde{\pi}(\widetilde{B}_{h,k}) = b_h$. For any h let I_h denote the inertia group of a generic point of $\widetilde{B}_{h,k}$. This only depends on h up to conjugacy, because the action of G conjugates the inertia groups. Write e_h for the order of I_h. Let

$$\begin{aligned} J(\widetilde{x}) \ &:= \ \{\ell \mid 1\leq \ell \leq m, \exists k : \widetilde{x} \in \widetilde{B}_{\ell,k}\} \\ &= \ \{1,2,\dots,n\} \ , \end{aligned} \tag{2.2}$$

(after reordering if necessary). Because $\widetilde{X}/Y$ is tame, the group I_ℓ is cyclic and we shall often identify I_ℓ with $\mathbf{Z}/e_\ell\mathbf{Z}$. Then

$$I(\widetilde{x}) \cong \prod_{\ell\in J(\widetilde{x})} I_\ell \cong \prod_{\ell\in J(\widetilde{x})} \mathbf{Z}/e_\ell\mathbf{Z} \ .$$

From the results in the appendices of [CEPT2] and [C-E] we deduce our first auxiliary result for which it is crucial that the schemes $\widetilde{X}$ and Y be regular. Note that the result is stated there for X flat over $\mathbf{Z}$, but the proof goes through *mutatis mutandis* for X defined over a finite field $\mathbf{F}_p$. Recall the notation $\mathrm{Map}_H(G,B)$ from Sect. 1.a.

Lemma 2.3. *Let y be a closed point of Y and let $\widetilde{x}$ be a closed point such that $\pi(\widetilde{x}) = y$. There exists an affine étale neighbourhood of y*

$$S = \mathrm{Spec}(A_y) \ ,$$

where A_y is an algebra containing a sequence $a_1, a_2, \dots, a_n$ of regular parameters, such that

$$\widetilde{X} \times_Y S \cong \mathrm{Spec}(O_{\widetilde{X}}(S))$$

is an isomorphism of schemes with a G-action, where

$$O_{\widetilde{X}}(S) := Map_{I(\widetilde{x})}(G, B_{\widetilde{x}})$$

and

$$B_{\widetilde{x}} := A_y[t_1,\dots,t_n] = A_y[T_1,\dots,T_n]/(T_1^{e_1} - a_1, \dots, T_n^{e_n} - a_n) \ ,$$

with each e_ℓ prime to the residue characteristic of y and $t_\ell = \overline{T_\ell}$, so that $t_\ell^{e_\ell} = a_\ell$. The e_ℓ are the orders of inertia and the a_ℓ are local equations for *the b_ℓ. Moreover the algebra A_y is a domain and it contains enough roots of unity of order prime to the residue characteristic of y. The inertia group $I(\widetilde{x})$ acts on $B_{\widetilde{x}}$ through its action on the cotangent space. More precisely, if the group I_ℓ acts on the cotangent space of the ℓ-th component by the character χ_ℓ, then I_ℓ acts on t_ℓ by χ_ℓ.*

2.b. Local structure of the coverings X/Y and Z/Y.

For a Y-scheme V let us write V_S for $V \times_Y S$. Since by definition $X = \widetilde{X}/H$, we deduce from the result of the previous section that

$$X_S = \mathrm{Spec}\left(\mathrm{Map}_{I(\widetilde{x})}(H\backslash G, B_{\widetilde{x}})\right) .$$

A similar result holds for Z_S with G_2 in place of H, since $Z = \widetilde{X}/G_2$. We now want to obtain an alternative description of X_S and Z_S (see (2.4) and (2.4) below). To this end let $\{g_i\}_{1\leq i\leq r}$ denote a system of representatives of $H\backslash G/I(\widetilde{x})$. For g an element of G, let $\overline{g}$ denote the class of g in $H\backslash G$. The group $I(\widetilde{x})$ acts from the right on $H\backslash G$, and the $\overline{g_i}$ form a system of representatives for the orbits in $H\backslash G$ under this action. Let $H^{g_i} := g_i^{-1}Hg_i$. The stabilizer of $\overline{g_i}$ is $I(\widetilde{x}, i) := I(\widetilde{x}) \cap H^{g_i}$ and the map

$$Map_{I(\widetilde{x})}(H\backslash G, B_{\widetilde{x}}) \longrightarrow \prod_{1\leq i\leq r} B_{\widetilde{x}}^{I(\widetilde{x},i)} ,$$

which sends f to $(f(\overline{g_i}))_i$ is an algebra isomorphism. For fixed i, let $\{h_{j(i)}^{(i)}\}$ denote a system of representatives of $(g_i I(\widetilde{x}) g_i^{-1}) \cap H\backslash H$, where $1 \leq j(i) \leq s(i)$. Then, the $h_{j(i)}^{(i)} g_i$ for $1 \leq i \leq r$ and $1 \leq j(i) \leq s(i)$ form a system of representatives of $G/I(\widetilde{x})$. The map

$$Map_{I(\widetilde{x})}(G, B_{\widetilde{x}}) \longrightarrow \prod_{1\leq i\leq r} \prod_{1\leq j(i)\leq s(i)} B_{\widetilde{x}} ,$$

which sends f to $(f(h_{j(i)}^{(i)} g_i))_{i,j(i)}$ is an algebra isomorphism. These two isomorphisms fit into the commutative diagram

$$\begin{array}{ccc} Map_{I(\widetilde{x})}(H\backslash G, B_{\widetilde{x}}) & \xrightarrow{\sim} & \prod_{1\leq i\leq r} B_{\widetilde{x}}^{I(\widetilde{x},i)} \\ (1)\downarrow & & \downarrow(2) \\ Map_{I(\widetilde{x})}(G, B_{\widetilde{x}}) & \xrightarrow{\sim} & \prod_{1\leq i\leq r}\left(\prod_{1\leq j(i)\leq s(i)} B_{\widetilde{x}}\right) \end{array}$$

The vertical map (1) sends the map f to the map $g \mapsto f(Hg)$. The vertical map (2) is given by the diagonal embedding of $B_{\widetilde{x}}^{I(\widetilde{x},i)}$ into $\prod_{1\leq j(i)\leq s(i)} B_{\widetilde{x}}$.

Again from the results in the appendix of [CEPT2], we deduce the second auxiliary result. For this result it is vital that X is regular.

Lemma 2.4. *For any (i,ℓ), with $1 \leq i \leq r$ and $1 \leq \ell \leq n$, there exists a divisor $e_{i,\ell}$ of e_ℓ such that*

$$I(\widetilde{x}, i) = \prod_{\ell \in J(\widetilde{x})} \mathbf{Z}e_{i,\ell}/\mathbf{Z}e_\ell \ .$$

So for any i we can write

$$B_{x,i} := B_{\widetilde{x}}^{I(\widetilde{x},i)} = A_y[t_{i,1}, \dots, t_{i,n}] \ ,$$

where $t_{i,\ell} := t_\ell^{e_\ell/e_{i,\ell}}$. Finally

$$X_S = \mathrm{Spec}(O_X(S)) \quad \text{with} \quad O_X(S) = \prod_{1 \leq i \leq r} B_{x,i} \ . \tag{2.4}$$

We obtain an analogous description of Z_S by considering a system of representatives $\{\gamma_j\}_{1 \leq j \leq s}$ for $G_2 \backslash G / I(\widetilde{x})$. However, since $I(\widetilde{x})$ has odd order we see that for any j, the intersection $I(\widetilde{x}) \cap G_2^{\gamma_j}$ is reduced to the identity. So we do not need the previous lemma and the analogue of (2.4) is more simply

$$Z_S = \mathrm{Spec}(O_Z(S)) \quad \text{with} \quad O_Z(S) = \prod_{1 \leq j \leq s} B_{j,x} \ , \tag{2.4}$$

which is a product of s copies of $B_{j,x} = B_{\widetilde{x}}$.

Remark. The two notations $B_{x,i}$ and $B_{j,x}$ are used in order to emphasize that the first appears in the decomposition of $O_X(S)$ and the second in that of $O_Z(S)$, and that, as in our big diagram above, "X is on the right" and "Z is on the left."

2.c. Local structure of the covering T/Z; integral closure.
From the isomorphism of Y-schemes $(Z \times_Y X) \times_Y S \cong Z_S \times_S X_S$ and using (2.4) and (2.4) we obtain

$$T'_S = \mathrm{Spec}(O_{T'}(S)) \quad \text{with} \quad O_{T'}(S) = \prod_{1 \leq i \leq r, 1 \leq j \leq s} B_{j,x} \otimes_{A_y} B_{x,i} \ .$$

Let $Fr(R)$ denote the field of fractions of a domain R. In particular write $K_y = Fr(A_y)$.

We now determine an explicit description of the *integral closure* $C_{j,x,i}$ of $B_{j,x} \otimes_{A_y} B_{x,i}$ inside $Fr(B_{j,x}) \otimes_{K_y} Fr(B_{x,i})$. Note that the latter is a product of fields. Indeed it follows from Lemma 2.3 that $Fr(B_{\widetilde{x}})/K_y$ is separable, since by tameness the polynomials $T_\ell^{e_\ell} - a_\ell$ are separable (e_ℓ is prime to the residue characteristic). For i as above, let

$$H_i = H_i(x) := I(\widetilde{x})/I(\widetilde{x}, i) \ .$$

The group H_i is the Galois group of the extension $Fr(B_{x,i})/K_y$. From Lemma 2.4 we deduce a group isomorphism

$$\delta : H_i \cong \prod_{\ell \in J(\widetilde{x})} \mathbf{Z}/e_{i,\ell}\mathbf{Z} \ .$$

Let $\widehat{H_i}$ denote the group of characters of H_i and let $A(i)$ be the set of sequences $\alpha = (\alpha_{i,\ell})$ such that for every ℓ in $J(\widetilde{x})$, we have $0 \le \alpha_{i,\ell} < e_{i,\ell}$. Define a character $\chi_\ell : \mathbf{Z}/e_{i,\ell}\mathbf{Z} \to A_y$ by the requirement that for all u in $\mathbf{Z}/e_{i,\ell}\mathbf{Z}$

$$\chi_\ell(u) := t_{i,\ell}^{\delta^{-1}(u)}/t_{i,\ell} \ .$$

The next result is clear.

Lemma 2.5. *Every character χ in $\widehat{H_i}$ can be written uniquely in the form*

$$\chi = \prod_{\ell \in J(\widetilde{x})} \chi_\ell^{\alpha_{i,\ell}} \ ,$$

with $0 \le \alpha_{i,\ell} < e_{i,\ell}$. This establishes a bijection between $\widehat{H_i}$ and $A(i)$.

Proposition 2.6. *The integral closure $C_{j,x,i}$ of $B_{j,x} \otimes_{A_y} B_{x,i}$ inside the product of fields $Fr(B_{j,x}) \otimes_{K_y} Fr(B_{x,i})$ admits the following description as a $B_{j,x}$-algebra:*

$$C_{j,x,i} = B_{j,x}[\{t_{i,\ell}^{-1} \otimes t_{i,\ell} \ : \ \ell \in J(\widetilde{x})\}] \ .$$

Proof. Let us consider the morphism of $Fr(B_{\widetilde{x}})$-algebras

$$\iota : Fr(B_{j,x}) \otimes_{K_y} Fr(B_{x,i}) \longrightarrow Map(H_i, Fr(B_{j,x})) \ ,$$

which sends $a \otimes b$ to the map $h \mapsto ab^h$. This is an isomorphism. To see this, it is sufficient to show that it is an injection, because the dimension of the range and the source over $Fr(B_{j,x})$ both equal the order of H_i. The injectivity is a standard result (see *e.g.* [F1] p. 132).

Let us now pass to the integral level. We know that $B_{j,x} = B_{\widetilde{x}}$ and hence $Map(H_i, B_{j,x})$ are integrally closed, because $\widetilde{X}$ is regular and $\widetilde{X}_S/\widetilde{X}$ is étale, so that $\widetilde{X}_S$ is also regular (hence normal). So ι restricts to an injection $C_{j,x,i} \hookrightarrow Map(H_i, B_{j,x})$. For all i and ℓ, we have that $(t_{i,\ell}^{-1} \otimes t_{i,\ell})^{e_{i,\ell}} = 1$, so we obtain one inclusion for the proposition:

$$B_{j,x}[\{t_{i,\ell}^{-1} \otimes t_{i,\ell} \ : \ \ell \in J(\widetilde{x})\}] \subset C_{j,x,i} \ .$$

It will thus be sufficient to show that

$$\iota\left(B_{j,x}[\{t_{i,\ell}^{-1} \otimes t_{i,\ell}\} \ : \ell \in J(\widetilde{x})\}]\right) = Map(H_i, B_{j,x}) \ . \tag{2.6}$$

We first show that the characters χ form a basis of $Map(H_i, B_{j,x})$. We know that $Map(H_i, B_{j,x})$ is a free $B_{j,x}[H_i]$-module of rank one, with a basis given by the map d such that $d(g) = \delta_{g,1}$ (Kronecker delta). Since

$B_{j,x}$ contains the $e_{i,\ell}$-st roots of unity and since the order of H_i is prime to residue characteristic of x, it is known that the idempotents e_χ, for χ in $\widehat{H_i}$ form a basis of $B_{j,x}[H_i]$ over $B_{j,x}$. So the set of $(d|\chi) = \sum_{g\in H_i} \chi(g) d^{g^{-1}}$ for χ in $\widehat{H_i}$, is a basis of $Map(H_i, B_{j,x})$ over $B_{j,x}$. Now, $(d|\chi) = \chi$, so the characters χ in $\widehat{H_i}$ indeed form a basis of $Map(H_i, B_{j,x})$. Hence to prove (2.6), it suffices to show that for ℓ in $J(\widetilde{x})$, the character χ_ℓ belongs to $\iota\left(B_{j,x}[\{t_{i,\ell}^{-1} \otimes t_{i,\ell}\} \ : \ \ell \in J(\widetilde{x})\}]\right)$. This is clear, since by definition, we have

$$\iota(t_{i,\ell}^{-1} \otimes t_{i,\ell}) = \chi_\ell \ .$$

This concludes the proof of the proposition.

Remark 2.7. Prop. 2.6 also shows that for every (i,j) the algebra $C_{j,x,i}$ (obtained by normalisation) is a $B_{j,x}$-algebra and a free $B_{j,x}$-module of finite type with a basis given by the set of $\theta_i(\alpha)$, for $\alpha = (\alpha_{i,\ell})$ sequence in $A(i)$, defined by

$$\theta_i(\alpha) = \prod_{\ell\in J(\widetilde{x})} t_{i,\ell}^{-\alpha_{i,\ell}} \otimes t_{i,\ell}^{\alpha_{i,\ell}} = \prod_{\ell\in J(\widetilde{x})} (t_{i,\ell}^{-1} \otimes t_{i,\ell})^{\alpha_{i,\ell}} \ .$$

We shall write $C_{j,x,i} = \bigoplus_{\alpha\in A(i)} B_{j,x}\theta_i(\alpha)$. This will be generalised below, see Prop. 3.6.

2.d. Proof that T/Z is étale.

We now come back to the proof Thm.2.2 (b). Let t be a closed point of T, such that $\pi_Z(t) = z$, with z lying over y. Let S be defined as in Lemma 2.3. Put $\mathcal{U} = Z_S = Z \times_Y S$. Then $\mathcal{U}$ is an étale neighbourhood of z and $T_{\mathcal{U}} = T_S$ is an étale neighbourhood of t. We have shown that

$$\mathcal{U} = \operatorname{Spec}(\prod_{1\le j\le s} B_{j,x}) \quad \text{and} \quad T_{\mathcal{U}} = \operatorname{Spec}(\prod_{1\le i\le r, 1\le j\le s} C_{j,x,i}) \ .$$

Prop. 2.6 shows that $C_{j,x,i}$ is a polynomial algebra over $B_{j,x}$ and that the variables raised to a power prime to the residue characteristic are units in the ground ring, hence $T_{\mathcal{U}} \to \mathcal{U} = Z_S$ is étale, and this concludes the proof of the theorem. (Note that indeed $(t_{i,\ell}^{-1} \otimes t_{i,\ell})^{e_{i,\ell}} = 1$.)

3. Normalisation along a branch divisor

Recall the notation introduced in Sect. 1 and the diagram used in the proof of Thm. 2.2. The aim of this section is to first construct a sequence of schemes $T^{(h)}$ over T' indexed by the m components of the branch locus of $\pi : X \to Y$, such that $T^{(0)} = T'$ and such that $T^{(m)} = T$, the normalisation of $T^{(0)}$ (see the definition after Prop. 3.3). Then we define forms $(\Lambda^{(h)}, Tr)$ over Z, where $\Lambda^{(h)} := (\pi_h)_*(\mathcal{D}_{T^{(h)}/Z}^{-1/2})$ is equipped with the trace form (see

Lemma 3.8). These forms all agree on the generic fiber of Z and are such that

$$\Lambda^{(m)} = (\pi_Z)_*(O_T) \quad \text{and} \quad \Lambda^{(0)} = \phi^*(\pi_*(\mathcal{D}_{X/Y}^{-1/2})) \ .$$

Moreover for $0 \leq h \leq m-1$ there are short exact sequences of locally free O_Z-modules

$$0 \to I^{(h)} \to \Lambda^{(h)} \oplus \Lambda^{(h+1)} \to \mathcal{G}^{(h)} \to 0 \ , \tag{3.0}$$

and as we will see in Prop. 3.12 the form

$$(\Lambda^{(h)}, (-1)^h Tr) \oplus (\Lambda^{(h+1)}, (-1)^{h+1} Tr)$$

is metabolic, with lagrangian $I^{(h)}$. The point is that the maps in the sequence (3.0) are explicitly given. This will allow us in the next section to apply the Main Lemma (Cor. 1.15) and complete the proof of Thm. 0.1. One could try to obtain a sequence analogous to (3.0) for $\Lambda^{(0)} \oplus \Lambda^{(m)}$, however the resulting "kernel sheaf" would not be locally free in general. To deduce the properties of the forms $(\Lambda^{(h)}, Tr)$ we analyze the structure of the schemes $T^{(h)}$ over an étale neighbourhood, generalizing the results of the previous section (see Sects. 3.c and 3.d).

One can view the normalisation T as the "normalisation along (the inverse image of) $b_1 \coprod \ldots \coprod b_m$" (see below). For $1 \leq h \leq m$ we will define $T^{(h)}$ to be the normalisation along $b_1 \coprod \ldots \coprod b_h$. More precisely $T^{(h)}$ will be defined as $\mathbf{Spec}(N^{(h)})$ for a certain coherent O_Y-algebra $N^{(h)}$ obtained by partial normalisation, namely by "gluing" algebras $N^{(h)}(U)$, for U open in Y, which are the intersection of $O_Y(U)$-algebras indexed by points ξ of codimension 1 in U. (In particular, this interpolation procedure depends on the choice of an order on the branches of b, but the final result is independent of this choice.) We will have the following diagram of schemes and morphisms

$$\begin{tikzcd}
T^{(m)} \arrow[r] \arrow[drrr, "\pi_m = \pi_Z"'] & T^{(m-1)} \arrow[r] \arrow[drr, "\pi_{m-1}"] & \cdots \arrow[r] & T^{(0)} = T' \arrow[r, "\phi_X"] \arrow[d, "\pi_0"] \arrow[dr, "\theta"] & X \arrow[d, "\pi"] \\
& & & Z \arrow[r, "\phi"'] & Y
\end{tikzcd}$$

3.a. Localisation at primes of height one.

The aim of this section is to formulate a slight generalisation of a well known result of commutative algebra, which motivates the definition of the schemes $T^{(h)}$ (see Lemma 3.2). As before, for a ring A we let $Fr(A)$ denote the total ring of fractions of A.

We begin with the following elementary observation.

Lemma 3.1. *Let R be an integral domain and let $f : R \to A$ be a ring homomorphism. Suppose that f is flat and that A is finite over R and reduced. Then we can view R as contained in A and $Fr(A) \cong A \otimes_R Fr(R)$.*

Proof. We show that there is a ring homomorphism

$$g : Fr(A) \to A \otimes_R Fr(R)$$

given by sending x/a to $x \otimes (1/a)$, which is an isomorphism. Note that f is injective. Indeed, if r is a non-zero element of R, then the map on R given by multiplication by r is injective on R, hence, because of flatness, the map on A given by multiplication by $f(r)$ is injective too. It also follows that every non-zero element of R is regular in A, that is it is not a zero-divisor. We write $R \subset A$. Let a be a regular element of A. Since A is finite over R it is integral over it and so there exist $r_0, r_1, \ldots, r_{n-1}$ in R such that

$$a^n + r_{n-1}a^{n-1} + \cdots + r_1 a + r_0 = 0 \ .$$

Because a is regular we can assume that $r_0 \neq 0$ and we see that a is invertible in $A \otimes_R Fr(R)$, indeed

$$a \cdot \left(a^{n-1} \otimes \frac{1}{r_0} + a^{n-2} \otimes \frac{r_{n-1}}{r_0} + \cdots + 1 \otimes \frac{r_1}{r_0} \right) = -1 \ .$$

So the map g is well defined and is clearly an isomorphism.

Next we note a generalization of the well-known fact that a normal noetherian ring equals the intersections of its localizations at the primes of height one.

Lemma 3.2. *Let R be a noetherian, normal ring which is a subring of a ring A. We assume that A is a finite, flat R-module. Then*

$$A = \bigcap_{p,\ ht(p)=1} A_p \ ,$$

where p runs over the prime ideals in R, which are of height 1.

Proof. We show that there is an exact sequence

$$0 \to Fr(A)/A \to \bigoplus_{p, \mathrm{ht}(p)=1} Fr(A)/A_p \ .$$

Let q be a prime ideal of R and let B be an R-algebra. We write B_q for the ring $S^{-1}B$ obtained by localising B with respect to the multiplicative set $S = R \setminus q$. So $B_q = B \otimes_R R_q$. Since, by assumption, A is flat over R we have, as above, that every non-zero element r of R does not divide zero in A. In particular the canonical map $A \to Fr(A)$ induces an injection of A_q into $Fr(A)$ and the intersection is to be viewed in $Fr(A)$. The fact that $R = \bigcap_{p,\ \mathrm{ht}(p)=1} R_p$, can be interpreted as giving the exactness of the sequence

$$0 \to Fr(R)/R \to \bigoplus_{p,\mathrm{ht}(p)=1} Fr(R)/R_p \ .$$

By the flatness of A over R we deduce the exactness of the sequence

$$0 \to (Fr(R)/R) \otimes_R A \to \bigoplus_{p,\mathrm{ht}(p)=1} (Fr(R)/R_p) \otimes_R A \ .$$

Again by flatness we can consider $A = R \otimes_R A$ and $A_p = R_p \otimes_R A$ as subrings of $Fr(R) \otimes_R A$, which by Lemma 3.1 we can identify with $Fr(A)$. So the last exact sequence can be rewritten as

$$0 \to Fr(A)/A \to \bigoplus_{p,\mathrm{ht}(p)=1} Fr(A)/A_p \ ,$$

which concludes the proof.

3.b. Partial normalisation; definition of $N^{(h)}$ and $T^{(h)}$.
Let U be an open affine in Y, we want to define a ring $N^{(h)}(U)$ by partial normalisation of the ring $O_{T'}(\theta^{-1}(U))$, where $\theta := \phi \circ \pi_0$ (see diagram before Sect. 3.a). For ease of notation let us write $W := \theta^{-1}(U)$. The scheme W is affine and open in T', and $O_{T'}(W)$ is a finite and flat $O_Y(U)$-algebra. Let $K_Y(U)$ denote the ring of fractions of $O_Y(U)$. By Lemma 3.1 we can write

$$O_Y(U) \subset O_{T'}(W) \quad \text{and} \quad Fr(O_{T'}(W)) = O_{T'}(W) \otimes_{O_Y(U)} K_Y(U) \ .$$

Note that

$$W = \pi^{-1}(U) \times_Y \phi^{-1}(U) \text{ and } O_{T'}(W) = O_X(\pi^{-1}(U)) \otimes_{O_Y(U)} O_Z(\phi^{-1}(U)) \ ,$$

and, with the obvious notation,

$$O_{T'}(W) \otimes_{O_Y(U)} K_Y(U) = K_X(\pi^{-1}(U)) \otimes_{K_Y(U)} K_Z(\phi^{-1}(U)) \ ,$$

the tensor product of the fraction fields. The extensions K_X/K_Y and K_Z/K_Y are finite and separable, thus the tensor product is a separable K_Y-algebra, and hence semi-simple. So we see that $O_{T'}(W)$ is contained in the reduced ring $Fr(O_{T'}(W))$ and T' is reduced.

Notation. For a subring B of $Fr(O_{T'}(W))$ we write $\widetilde{B}$ for its integral closure.

We are going to define a finite sub-algebra of $\widetilde{O_{T'}(W)}$. Let us apply Lemma 3.1 to $O_Y(U) \to \widetilde{O_{T'}(W)}$. We obtain

$$O_Y(U) \subset O_{T'}(W) \subset \widetilde{O_{T'}(W)}$$

and clearly $Fr(O_{T'}(W)) = Fr(\widetilde{O_{T'}(W)})$.

Let now S denote a subring of $Fr(O_{T'}(W))$ which contains $O_Y(U)$ and which is such that every non-zero element of $O_Y(U)$ is regular in S. For any codimension one point ξ of U we put $S_\xi = S_p$, where $p = p_\xi$ is the ideal of height one in $O_Y(U)$ corresponding to ξ. Recall that we denote by ξ_h the generic point of the component b_h of the branch locus $b(X/Y)$. Fix an integer h with $1 \leq h \leq m$. For any codimension one point ξ of U let

$$S_U^{(h)}(\xi) = \begin{cases} \widetilde{O_{T'}(W)}_\xi & \text{if } \xi \in \{\xi_1, \ldots, \xi_h\} \\ O_{T'}(W)_\xi & \text{if not .} \end{cases} \tag{3.2}$$

So for any ξ we have the inclusions

$$O_{T'}(W) \subset S_U^{(h)}(\xi) \subset Fr(O_{T'}(W)) \ .$$

Definition. $N^{(h)}(U) := \bigcap_{\xi \in U,\ \mathrm{codim}(\xi)=1} S_U^{(h)}(\xi)$.

Remark. (a) It follows from Lemma 3.2 that $N^{(h)}(U)$ is a finite subalgebra of $\widetilde{O_{T'}(W)}$ and for all ξ in U we have $(\widetilde{O_{T'}(W)})_\xi = (\widetilde{O_{T'}(W)_\xi})$. (More generally for any prime q of R we know that $R \backslash q$ consists of non-zero divisors of $\widetilde{A}$, so normalisation and localisation commute, namely $(\widetilde{A_q}) = (\widetilde{A})_q$.)

(b) Since for any h the divisor b_h is the closure of ξ_h, we see that ξ_h belongs to U if and only if U meets b_h.

(c) If ξ is in U but differs from all of the ξ_h, then $O_{T'}(W)_\xi = \widetilde{O_{T'}(W)}_\xi$ and so for $h = m$ we get

$$N^{(m)}(U) := \bigcap_{\xi \in U,\ \mathrm{codim}(\xi)=1} \widetilde{O_{T'}(W)}_\xi = \widetilde{O_{T'}(W)} \ ,$$

where the second equality follows from Lemma 3.2.

The next proposition summarizes some further basic properties of $N^{(h)}(U)$, which allow us to define the coherent O_Y-algebra $N^{(h)}$ used to construct $T^{(h)}$.

Proposition 3.3. *Let U be an affine open in Y and let h be any integer with $1 \leq h \leq m$.*

a) *Let ξ be a point of codimension one in U, then*

$$N^{(h)}(U)_\xi = S_U^{(h)}(\xi) \ .$$

b) *Let $D(f)$ be a standard open subset of the affine open subset U in Y. Then*

$$N^{(h)}(D(f)) = N^{(h)}(U)_f \ .$$

c) *The assignment* $U \mapsto N^{(h)}(U)$ *defines a coherent* O_Y*-algebra* $N^{(h)}$ *such that for any affine open subset of* Y *the restriction of* $N^{(h)}$ *to* U *is isomorphic to the quasi-coherent module associated to* $N^{(h)}(U)$.

Definition. For $1 \leq h \leq m$, let $T^{(h)} := \mathbf{Spec}(N^{(h)})$ be the affine Y-scheme which is defined by the coherent O_Y-algebra $N^{(h)}$ of the previous proposition (see [EGAI], I.9.1.4) or [Ha] II, Ex. 5.17).

Part (a) shows that we can rewrite the definition of the sub-$O_Y(U)$-algebra $N^{(h)}(U)$ in $\widetilde{O_{T'}(W)}$ as

$$N^{(h)}(U) = \bigcap_{\xi \in U,\ \mathrm{codim}(\xi)=1} N^{(h)}(U)_\xi \ . \tag{3.3}$$

Compare with Lemma 3.2.

Proof. (a) Let $R = O_Y(U)$ and let $A = O_{T'}(W)$. Let $p' = p_\xi$ be the prime of height one in R corresponding to ξ and, if ξ_h lies in U, let p_h denote the prime corresponding to ξ_h. We shall also put $S^{(h)}(p') = S_U^{(h)}(\xi)$. From the sequence of inclusions $A \subset S^{(h)}(p') \subset \widetilde{A_{p'}}$ we deduce that $A \subset N^{(h)}(U) \subset \widetilde{A}$. Also, if p is a prime of height one in R which is different from all of the p_h, then $A \subset N^{(h)}(U) \subset A_p$. So we deduce the desired equality in this case:

$$N^{(h)}(U)_{p'} = A_{p'} = S^{(h)}(p') \ .$$

Suppose now that $p' = p_h$ for some h, say $h = 1$ after reordering if necessary. Since $N^{(h)}(U) \subset S^{(h)}(p_1)$ we deduce $N^{(h)}(U)_{p_1} \subset \widetilde{A_{p_1}}$. To show the reverse inclusion we start by checking

$$\left(\bigcap_p S^{(h)}(p)\right)_{p_1} = \bigcap_p S^{(h)}(p)_{p_1} \ .$$

The left hand side is clearly contained in the right hand side. The other inclusion can easily be checked for finite intersections. Let x be an element of the right hand side. Since x is in $Fr(A)$, we can write $x = a/s$ with a in A and s in $R \setminus \{0\}$. If s is not in p_1, then x belongs to the left hand side. If s belongs to p_1, let $q_2, \dots , q_n$ be the primes of height one different from p_1 which contain s. (They are finite in number because R is noetherian.) Thus x lies in $\bigcap_{p \notin \{p_1, q_2, \dots, q_n\}} S^{(h)}(p)$. Now, x also lies in $\bigcap_{p \in \{p_1, q_2, \dots, q_n\}} S^{(h)}(p)_{p_1}$ and because we are intersecting over a finite set we see that x belongs to $\left(\bigcap_{p \in \{p_1, q_2, \dots, q_n\}} S^{(h)}(p)\right)_{p_1}$. Write $x = b/t$ with t in $R \setminus p_1$ and b in $\bigcap_{p \in \{p_1, q_2, \dots, q_n\}} S^{(h)}(p)$. Then $x = tx/t$ with tx in $\bigcap_p S^{(h)}(p)$.

It remains to verify the inclusion $\widetilde{A_{p_1}} \subset S^{(h)}(p)_{p_1}$. This is clear for p one of the p_i, because then $\widetilde{A} \subset \widetilde{A_p} = S^{(h)}(p)$. So let us suppose p is not any of the p_i. What has to be checked is $\widetilde{A_{p_1}} \subset (A_p)_{p_1}$. Let g be an element of $\widetilde{A_{p_1}}$, say $g = \widetilde{a}/s_1$ with $\widetilde{a}$ in $\widetilde{A}$ and s_1 in $R \setminus p_1$. By Lemma 3.1 we can write $\widetilde{a} = a/t$, with t in $R \setminus \{0\}$ and so $g = a/(s_1 t)$. If t does not belong to p, then g is in $(A_p)_{p_1}$. Analogously, if t does not belong to p_1, then g is in $A_{p_1} \subset (A_p)_{p_1}$. So assume t lies in $p \cap p_1$, then $s_1 t$ does too and we are presented with two cases:

Case 1: $V(p) \cap V(p_1) \neq \emptyset$.
Let M be a maximal ideal containing both p and p_1. Since Y is regular R_M is a regular local ring and hence a factorial ring. The ideals p and p_1 are of height one in R_M and so they are principal. Let r and r_1 denote generators of p and p_1 respectively. The fact that p and p_1 are distinct implies that $r \notin p_1$ and $r_1 \notin p$. The factorization of $s_1 t$ in R_M is $s_1 t = r^n r_1^{n_1} u/v$, with u in $R \setminus p \cup p_1$ and v not in M. From this we obtain the equality $vst_1 = ur^n r_1^{n_1}$, which holds in R. So we can write g as

$$g = \frac{a}{s_1 t} = \frac{va}{vs_1 t} = \frac{va}{ur^n r_1^{n_1}} = \frac{1}{ur^n}\left(\frac{va}{r_1^{n_1}}\right) .$$

Now, $va/r_1^{n_1}$ belongs to A_p and so we see that g lies in $(A_p)_{p_1}$.

Case 2: $V(p) \cap V(p_1) = \emptyset$.
Let r_1 be a uniformizing parameter for the discrete valuation ring R_{p_1}. Here $R = p + p_1$ and also $R = p + p_1^2$. So we can find r_1' in R such that

$$r_1' \equiv r_1 \bmod p_1^2 \quad \text{and} \quad r_1' \equiv 1 \bmod p ,$$

so r_1' is a generator of $p_1 R_{p_1}$ not belonging to p. In R_{p_1} we write $s_1 t = (r_1')^{n_1} u/v$ with u and v not in p_1 and we deduce that $vs_1 t = (r_1')^{n_1} u$ with $(r_1')^{n_1}$ not in p and u not in p_1. Then

$$g = \frac{a}{s_1 t} = \frac{va}{vs_1 t} = \frac{1}{u}\left(\frac{va}{(r_1')^{n_1}}\right)$$

and we again conclude that g lies in $(A_p)_{p_1}$.

We now proceed to prove part (b) of the proposition. We begin with a lemma.

Lemma 3.4.

$$N^{(h)}(U)_f = \bigcap_{\xi} S_U^{(h)}(\xi)_f .$$

Proof. We prove that the right hand side is contained in the left. The other inclusion is clear. Let x be an element of $\bigcap_\xi S_U^{(h)}(\xi)_f$. Since x is in $Fr(A)$ we can write $x = a/s$ with s in $R \setminus \{0\}$. Let $\{q_1, \ldots, q_n\}$ be the primes of height one in R which contain s. By definition x lies in $S_U^{(h)}(p)$ if p is not one of the q_i. Since for all $1 \leq i \leq n$, x belongs to $S_U^{(h)}(q_i)_f$, for all i we can find n_i such that $f^{n_i} x$ is in $S_U(q_i)$. Thus there is an $\ell \geq 1$ such that $f^\ell x$ belongs to $\bigcap_p S_U^{(h)}(p) = N^{(h)}(U)$ and we deduce that x is in $N^{(h)}(U)_f$.

Let P_f (resp. P_U) denote the set of primes of height one in $O_Y(D(f))$ (resp. $O_Y(U)$). On the one hand, the previous lemma implies that

$$N^{(h)}(U)_f = \bigcap_{p \in P_U} S_U^{(h)}(p)_f \ .$$

On the other hand, it follows from the definition that

$$N^{(h)}(D(f)) = \bigcap_{p \in P_f} S_{D(f)}^{(h)}(p) \ .$$

Indeed, since θ is affine we have that $W = \theta^{-1}(U) \to U$ is induced by the inclusion $O_Y(U) \subset O_{T'}(W)$, hence $\theta^{-1}(D(f)) = D_W(f)$. It thus follows that for any p in P_f

$$S_{D(f)}^{(h)}(p) = S_U^{(h)}(p)_f \ .$$

To conclude the proof of part (b) of the proposition it suffices to show that if p is in $P_U \setminus P_f$, then

$$S_U^{(h)}(p)_f = Fr(O_{T'}(W)) \ .$$

Let $A = O_{T'}(W)$. And write x in $Fr(A)$ as $x = a/s$ with a in A and s in $R \setminus \{0\}$. If s is not in p, then x belongs to $S_U^{(h)}(p)$ and thus to $S_U^{(h)}(p)_f$. If s lies in p, then we work inside the discrete valuation ring R_p. Since s and f both are both in p, there exist integers n and k such that s^n and f^k have the same valuation in R_p. So in R_p we can write

$$s^n = f^k \frac{u}{v} \ ,$$

with u and v in $R \setminus p$. We rewrite $x = (as^{n-1}v)/s^n v$ as $(as^{n-1}v)/f^k u$ to get x in $S_U^{(h)}(p)_f$. This concludes the proof of part (b) of the proposition.

Let us prove part (c) of the proposition. Write $F_U^{(h)}$ for the coherent sheaf on U associated to the $O_Y(U)$-module $N^{(h)}(U)$. Note that $F_U^{(h)}(D(f)) = F_U^{(h)}(U)_f = N^{(h)}(U)_f$, so part (b) implies that

$$F_U^{(h)}(D(f)) = N^{(h)}(D(f)) \ .$$

Lemma 3.5. *Let U and V be open affine subsets of Y, with $V \subset U$, then*

$$F_U^{(h)}|_V = F_V^{(h)} \ .$$

Proof. To show this identity of sheaves on V it is sufficient to show they have the same sections over the elements in a basis $\mathcal{B}$ for the topology of V. We choose for $\mathcal{B}$ the set of principal open sets of U which are contained in V. Let $D(f)$ be an element of $\mathcal{B}$. From the previous displayed formula we deduce that

$$F_U^{(h)}|_V(D(f)) = F_U^{(h)}(D(f)) = N^{(h)}(D(f)) \ .$$

Note that $D(f) = D(f|_V)$. Since $D(f|_V)$ is a principal open subset of the affine V, we also deduce that

$$F_V^{(h)}(D(f)) = F_V^{(h)}(D(f|_V)) = N^{(h)}(D(f)) \ .$$

This implies that $F_U^{(h)}|_V$ and $F_V^{(h)}$ coincide on $\mathcal{B}$.

We now glue the sheaves $F_U^{(h)}$ together into a sheaf $N^{(h)}$. Let U and V be open affine subsets of Y. The intersection $U \cap V$ is again affine, because Y is proper (and hence separated) and so the previous lemma implies

$$F_U^{(h)}|_{U\cap V} = F_V^{(h)}|_{U\cap V} = F_{U\cap V}^{(h)} \ .$$

This is sufficient to show the existence of $N^{(h)}$ having the properties announced in the proposition.

3.c. Local structure of $T^{(h)}$.
Let $q_Z : \mathcal{U} = Z_S \to Z$ denote the étale neighbourhood constructed in Sect. 2 using a sufficiently small étale neighbourhood $q : S = \mathrm{Spec}(A_y) \to Y$ of $y = \phi(z)$. The aim of this section is to describe the structure of $T^{(h)} \times_Y S$ and relate it to the partial normalisation of $T_S^{'}$ (see Prop. 3.6).

Write U for the image of q. Let us assume that y belongs to the intersection of the divisors $b_1, \ldots, b_n$, with $1 \leq n \leq m$ and that U does not intersect the remaining divisors of the branch locus. We have that on U the divisors $b_1, \ldots, b_n$ are defined by sections $a_1, \ldots, a_n$, which means that, if ξ_ℓ denotes the generic point of b_ℓ, then for any point ξ of codimension one in U

$$v_\xi(a_\ell) = \begin{cases} 0 & \text{if } \xi \neq \xi_\ell \\ 1 & \text{if } \xi = \xi_\ell \end{cases} \ .$$

Note that by the construction of A_y, the sequence $a_1, \ldots, a_n$ is part of a regular system of parameters at y, in particular $n \leq \dim Y$. In what follows we shall denote by the same letter points of codimension one and their associated primes.

For $1 \leq \ell \leq n$ consider the divisor $q^*(b_\ell)$ of S. It can be written

$$q^*(b_\ell) = \sum_{\eta \in S^{(1)}} v_\eta(a_\ell)\overline{\{\eta\}} \ ,$$

where η runs over the set $S^{(1)}$ of codimension one points of S. We have that $q(\eta) = \xi_i$ if and only if a_i belongs to η, so $q^*(b_\ell)$ can also be written

$$q^*(b_\ell) = \sum_{q(\eta)=\xi_\ell} \overline{\{\eta\}} \ .$$

Recall that for $\widetilde{x}$ in $\widetilde{X}$ above y, in (2.2) we have defined $J = J(\widetilde{x})$ to be the set (of indices) of irreducible components of the branch locus b covered by a component of the ramification divisor, that go through $\widetilde{x}$. We again identify J with the set $\{1, \ldots, n\}$. For any h, let us partition J into

$$J'_h = J'_h(\widetilde{x}) = \{\ell \in J : 1 \leq \ell \leq h\} \tag{3.5}$$

and

$$J''_h = J''_h(\widetilde{x}) = \{\ell \in J : h+1 \leq \ell \leq m\} \ .$$

So for $h \geq n$, the set J''_h is empty. Recall that $C_{j,x,i}$ is the integral closure of $B_{j,x} \otimes_{A_y} B_{x,i}$ inside $Fr(B_{j,x}) \otimes_{K_y} Fr(B_{x,i})$. This algebra appears in the next proposition as $C^{(h)}_{j,x,i}$ for $h = m$. The proposition generalizes the remark after Prop. 2.6.

Proposition 3.6. *For any integer h with $1 \leq h \leq m$, we have:*
a) $(T'_S)^{(h)} = (T^{(h)})_S =: T^{(h)}_S$
b) $T^{(h)}_S = Spec(O_{T^{(h)}}(S))$ *, where* $O_{T^{(h)}}(S) = \prod_{1\leq i \leq r, 1 \leq j \leq s} C^{(h)}_{j,x,i}$ *with*

$$C^{(h)}_{j,x,i} := \bigoplus_{\alpha \in A(i)} B_{j,x} \theta^{(h)}_i(\alpha)$$

and for $\alpha = (\alpha_{i,\ell})$

$$\theta^{(h)}_i(\alpha) = \prod_{\ell \in J'_h} t_{i,\ell}^{-\alpha_{i,\ell}} \otimes \prod_{\ell \in J} t_{i,\ell}^{\alpha_{i,\ell}} \ .$$

Proof. Let us fix some notation. Let

$$R := O_Y(U) \quad , \quad K := Fr(R) \quad , \quad L := Fr(A_y)$$

and let the total ring of fractions of $O_{T'}(\theta^{-1}(U))$ be denoted by F. So F is also the total ring of fractions of $N^{(h)}(U)$ and of the integral closure $\widetilde{O_{T'}(\theta^{-1}(U))}$.

To prove part (a) we write the two sides of the first equality as intersections over the set $S^{(1)}$ of codimension one points η of S. Consider the right hand side. By definition $T^{(h)} = \mathbf{Spec}(N^{(h)})$, so $T^{(h)} \otimes_Y S = \mathbf{Spec}(q^*(N^{(h)}))$ and because S is affine

$$T^{(h)} \otimes_Y S = \operatorname{Spec}(q^*(N^{(h)})(S)) = \operatorname{Spec}(N^{(h)}(U) \otimes_R A_y) \ ,$$

see [EGAI] Cor. 9.1.9. Below we will show that

$$N^{(h)}(U) \otimes_R A_y = \bigcap_{\eta \in S^{(1)}} (N^{(h)}(U) \otimes_R A_y)_\eta \tag{3.6}$$

(see after (3.6)). Now we consider the left hand term in Prop. 3.6 (a). The scheme $(T'_S)^{(h)}$ is the normalisation of the scheme T'_S along the divisor $q^*(b_1) \cup \ldots \cup q^*(b_h)$. Also $T'_S = \operatorname{Spec}(O_{T'}(\theta^{-1}(U)) \otimes_R A_y)$. Let us denote by P_h the set of codimension one points η of S such that there is h' with $1 \le h' \le h$ and $v_\eta(q^*(b_{h'})) \ge 1$. Then

$$(T'_S)^{(h)} = \mathrm{Spec}(N_S^{(h)}(S))$$

with $N_S^{(h)}(S) = \bigcap_{\eta \in S^{(1)}} N_S^{(h)}(S)(\eta)$ where

$$N_S^{(h)}(S)(\eta) = \begin{cases} (O_{T'}(\theta^{-1}(U)) \otimes_R A_y)^{\sim}_\eta & \text{if } \eta \in P_h \\ \\ (O_{T'}(\theta^{-1}(U)) \otimes_R A_y)_\eta & \text{if } \eta \notin P_h \end{cases}$$

Claim. We claim that since S is étale over U, taking the integral closure commutes with étale base change and localisation, namely

$$(O_{T'}(\theta^{-1}(U)) \otimes_R A_y)^{\sim}_\eta = (O_{T'}(\theta^{-1}(U))^{\sim} \otimes_R A_y)_\eta \ .$$

Indeed, $O_{T'}(\theta^{-1}(U))^{\sim}$ is finite over R, so $B := O_{T'}(\theta^{-1}(U))^{\sim} \otimes_R A_y$ is integral over A_y and hence

$$O_{T'}(\theta^{-1}(U)) \otimes_R A_y \subset B \subset (O_{T'}(\theta^{-1}(U)) \otimes_R A_y)^{\sim} \ .$$

We show that B is normal. Note that being the base change by an étale morphism $\mathrm{Spec}(B)$ is open in the regular scheme $\mathrm{Spec}(O_{T'}(\theta^{-1}(U))^{\sim})$ and so is regular. This implies that B_p is normal for every p, which proves the claim.

Thus

$$N_S^{(h)}(S) = \Big(\bigcap_{\eta \in P_h} O_{T'}(\theta^{-1}(U))^{\sim} \otimes_R A_{y,\eta} \Big) \cap \Big(\bigcap_{\eta \notin P_h} O_{T'}(\theta^{-1}(U)) \otimes_R A_{y,\eta} \Big) \ .$$

From this and the definition of $N^{(h)}(U)$ (see also (3.2)) we deduce that

$$N_S^{(h)}(S) = N^{(h)}(U) \otimes_R A_y \ ,$$

which by (3.6) shows part (a) of the proposition.

We now look more closely at $N^{(h)}(U) \otimes_R A_y$. This will give us $T^{(h)} \otimes_Y S$ and part (b) of the proposition. We have the inclusions

$$R \subset O_{T'}(\theta^{-1}(U)) \subset N^{(h)}(U) \subset O_{T'}\widetilde{(\theta^{-1}(U))} \ .$$

Since A_y is flat over R we also have the inclusions

$$A_y = R \otimes_R A_y \subset O_{T'}(\theta^{-1}(U)) \otimes_R A_y \subset N^{(h)}(U) \otimes_R A_y \\ \subset O_{T'}\widetilde{(\theta^{-1}(U))} \otimes_R A_y \ .$$

The A_y-algebras so obtained are finite over A_y and every non zero element of A_y defines a regular element in them by flatness, so by Lemma 3.1 the total ring of fractions E of the last three algebras equals $O_{T'}\widetilde{(\theta^{-1}(U))} \otimes_R L$.

We have that $L = A_y \otimes_R K$ and we claim that this implies $E = A_y \otimes_R F$. For this we need to see that L/K is algebraic which can be seen as in [Mi]

I Prop. 3.19. Indeed, a non-zero element a of A_y is invertible in $A_y \otimes_R K$ and A_y is a domain so that, as we saw above,

$$(O_{T'}(\theta^{-1}(U)) \otimes_R A_y)^{\sim} = O_{T'}(\theta^{-1}(U))^{\sim} \otimes_R A_y \ .$$

Let ξ run over the set $U^{(1)}$ of codimension one points in U. From the inclusion

$$\frac{F}{N^{(h)}(U)} \hookrightarrow \prod_{\xi} \frac{F}{N^{(h)}(U)_\xi}$$

we obtain the inclusion

$$\frac{F \otimes_R A_y}{N^{(h)}(U) \otimes_R A_y} \hookrightarrow \prod_{\xi} \frac{F \otimes_R A_y}{N^{(h)}(U)_\xi \otimes_R A_y} \ ,$$

from which we deduce the equality

$$N^{(h)}(U) \otimes_R A_y = \bigcap_{\xi \in U^{(1)}} N^{(h)}(U)_\xi \otimes_R A_y \ .$$

Claim. We claim that

$$N^{(h)}(U) \otimes_R A_y = \bigcap_{\eta \in S^{(1)}} (N^{(h)}(U) \otimes_R A_y)_\eta \ .$$

Indeed

$$N^{(h)}(U)_\xi \otimes_R A_y = \begin{cases} O_{T'}\widetilde{(\theta^{-1}(U))} \otimes_R A_{y,\xi} & \text{if } \xi \in U \cap \{\xi_1, \dots, \xi_h\} \\ O_{T'}(\theta^{-1}(U)) \otimes_R A_{y,\xi} & \text{if } \xi \notin U \cap \{\xi_1, \dots, \xi_h\} \end{cases}$$

In any case $N^{(h)}(U)_\xi \otimes_R A_y$ is finite and flat over $A_{y,\xi}$, which is a domain and is normal. We deduce from Lemma 3.2, that

$$N^{(h)}(U)_\xi \otimes_R A_y = \bigcap_{\eta \to \xi} (N^{(h)}(U)_\xi \otimes_R A_y)_\eta \ , \tag{3.6}$$

where η runs over the points in $S^{(1)}$ such that $q(\eta) = \xi$. The claim follows from the fact that $(N^{(h)}(U)_\xi \otimes_R A_y)_\eta = (N^{(h)}(U) \otimes_R A_y)_\eta$. Note that we have also obtained equation (3.6).

So we have to consider the $(N^{(h)}(U)_\xi \otimes_R A_y)_\eta$. From the work in Sect. 2 and the definition of $N^{(h)}$ we have

$$(N^{(h)}(U) \otimes_R A_y)_\eta = \begin{cases} \prod_{\substack{1 \le i \le r \\ 1 \le j \le s}} (C_{j,x,i})_\eta & \text{if } q(\eta) \in \{\xi_1, \dots, \xi_h\} \\ \prod_{\substack{1 \le i \le r \\ 1 \le j \le s}} (B_{j,x} \otimes_{A_y} B_{x,i})_\eta & \text{if } q(\eta) \notin \{\xi_1, \dots, \xi_h\} \end{cases}$$

and we are reduced to describing $C^{(h)}_{j,x,i} := \bigcap_\eta E^{(h)}_{j,x,i}(\eta)$, for a fixed pair (i,j) in the given range, where

$$E^{(h)}_{j,x,i}(\eta) = \begin{cases} (C_{j,x,i})_\eta & \text{if } q(\eta) \in \{\xi_1, \dots, \xi_h\} \\ (B_{j,x} \otimes_{A_y} B_{x,i})_\eta & \text{if } q(\eta) \notin \{\xi_1, \dots, \xi_h\} \end{cases} .$$

We show

$$C^{(h)}_{j,x,i} := \bigcap_{\eta \in S^{(1)}} E^{(h)}_{j,x,i}(\eta) = \bigoplus_{\alpha \in A(i)} B_{j,x}\theta^{(h)}_i(\alpha) .$$

Note the equivalence of conditions on η:

$$q(\eta) \in \{\xi_1, \dots, \xi_h\} \Longleftrightarrow \exists\, \ell \in J'_h : a_\ell \in \eta ,$$

and let Q_h denote the set of η which satisfy one of the conditions in the equivalence. Write $B_{j,x,\eta} := (B_{j,x})_\eta$.

Case 1: $\eta \in Q_h$. In this case, by Prop.2.6,

$$E^{(h)}_{j,x,i}(\eta) = (C_{j,x,i})_\eta = B_{j,x,\eta}[\{t^{-1}_{i,\ell} \otimes t_{i,\ell} \ : \ \ell \in J(\widetilde{x})\}] .$$

However, if a_ℓ does not belong to η, then $t^{-1}_{i,\ell} \otimes t_{i,\ell}$ lies in $B_{j,x,\eta}[1 \otimes t_{i,\ell}]$. Indeed, then $t^{e_{i,\ell}}_{i,\ell} = a_\ell$ is a unit in $B_{j,x,\eta}$ and hence $t^{-1}_{i,\ell} = t^{e_{i,\ell}-1}_{i,\ell}/a_\ell$ belongs to $B_{j,x,\eta}$. Thus, since we are assuming that η is in Q_h, to show that ℓ does not belong to J'_h, it suffices to show that a_ℓ does not belong to η. Hence in this case

$$\begin{aligned} (C_{j,x,i})_\eta &= B_{j,x,\eta}[\{1 \otimes t_{i,\ell} \ : \ \ell \in J''_h\} \cup \{t^{-1}_{i,\ell} \otimes t_{i,\ell} \ : \ \ell \in J'_h\}] \\ &= \bigoplus_{\alpha \in A(i)} B_{j,x,\eta}\theta^{(h)}_i(\alpha) . \end{aligned} \tag{3.6}$$

Case 2: $\eta \notin Q_h$. In this case

$$E^{(h)}_{j,x,i}(\eta) = (B_{j,x} \otimes_{A_y} B_{x,i})_\eta = B_{j,x,\eta}[\{1 \otimes t_{i,\ell} \ : \ \ell \in J(\widetilde{x})\}] .$$

We have thus shown that for every η of codimension one in S, $E^{(h)}_{j,x,i}(\eta)$ is contained in the right hand side of equation (3.6). Taking the intersection over all η and applying Lemma 3.2 to deduce $\bigcap_\eta B_{j,x,\eta} = B_{j,x}$ we obtain

$$C^{(h)}_{j,x,i} = \bigcap_\eta E^{(h)}_{j,x,i}(\eta) \subset \bigoplus_{\alpha \in A(i)} B_{j,x}\theta^{(h)}_i(\alpha) .$$

To show the reverse inclusion let z be an element of the right hand side. Since for any η of codimension one $B_{j,x}$ is contained in $B_{j,x,\eta}$, we see that z certainly belongs to $B_{j,x,\eta}[\{1 \otimes t_{i,\ell} \ : \ \ell \in J''_h\} \cup \{t^{-1}_{i,\ell} \otimes t_{i,\ell} \ : \ \ell \in J'_h\}]$, and so it belongs to $E^{(h)}_{j,x,i}(\eta)$ in case η lies in Q_h. If η does not lie in Q_h, then for ℓ in J'_h we have $B_{j,x,\eta}[1 \otimes t_{i,\ell}] = B_{j,x,\eta}[t^{-1}_{i,\ell} \otimes t_{i,\ell}]$ and so again z belongs to $E^{(h)}_{j,x,i}(\eta)$. This concludes the proof of the proposition.

3.d. The inverse different and the interpolating forms $\Lambda^{(h)}$.
Consider the cartesian diagram of schemes

$$\begin{tikzcd} X' = Y' \times_Y X \arrow[r, "\phi'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\phi"] & Y \end{tikzcd}$$

and let $\mathcal{D}_{X/Y}$ be the different of X/Y, which we recall is the O_X-module defined as the annihilator of the module of relative differentials.

Lemma 3.7. *Suppose that π and ϕ are finite and flat, then*

$$\phi'^*(\mathcal{D}_{X/Y}) = \mathcal{D}_{X'/Y'} \ .$$

Proof. We show that the sections of the two sheaves coincide over the basis for the topology of Y' given by the $\phi^{-1}(U)$, where U runs over the affine opens of Y. Let U be one such and put $V = \pi^{-1}(U)$, $V' = (\pi \circ \phi')^{-1}(U)$, $U' = \phi^{-1}(U)$, $A = O_Y(U)$, $B = O_X(V)$, $A' = O_{X'}(U')$, and $B' = O_{X'}(V')$. By definition $\mathcal{D}_{X/Y}(V) = \mathrm{Ann}_B(\Omega^1_{B/A})$. The module $\Omega^1_{B/A}$ is finite over B. Let $\{d_1, \dots, d_r\}$ be a system of generators for it. Then $x \mapsto (xd_1, \dots, xd_r)$ induces the injection

$$B/\mathcal{D}_{X/Y}(V) \hookrightarrow \prod_{1 \le i \le r} Bd_i \ .$$

Since ϕ' is flat, B' is flat over B, so we have the injection

$$B'/\mathcal{D}_{X/Y}(V) \cdot B' \hookrightarrow \prod_{1 \le i \le r} B'd_i \ ,$$

and hence $\mathcal{D}_{X/Y}(V) \cdot B' = \mathrm{Ann}_{B'}(\sum_{1 \le i \le r} B'd_i)$. Now, $B' = B \otimes_A A'$ and $\Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes_{A'} B'$. We deduce from this that $\sum_{1 \le i \le r} B'd_i = \Omega^1_{B'/A'}$ and hence that

$$\mathcal{D}_{X/Y}(V) \cdot B' = \mathrm{Ann}_{B'}(\Omega^1_{B'/A'}) = \mathcal{D}_{X'/Y'}(V') \ .$$

This concludes the proof of the lemma.

Using the fact that flat base change commutes with higher direct images (see *e.g.* [Ha] III 9.3), we deduce from the lemma that, in the previous notation,

$$\phi^*(\pi_*(\mathcal{D}_{X/Y})) = (\pi_0)_*(\mathcal{D}_{T'/Z}) \ ,$$

and also that

$$q^*_{T^{(h)}}(\mathcal{D}_{T^{(h)}/Z}) = \mathcal{D}_{T^{(h)}_S/Z_S} \ .$$

Lemma 3.8. *The square root of $\mathcal{D}_{T_S^{(h)}/Z_S}$ exists and so that of $\mathcal{D}_{T^{(h)}/Z}$ also exists.*

Proof. This follows from the local description given in the previous section. Indeed, by Prop. 3.6 (b) we have that $O_{T^{(h)}}(S) = \prod_{1\le i\le r, 1\le j\le s} C_{j,x,i}^{(h)}$ with

$$C_{j,x,i}^{(h)} = B_{j,x}[\{1\otimes t_{i,\ell} \ : \ \ell \in J_h''\} \cup \{t_{i,\ell}^{-1}\otimes t_{i,\ell} \ : \ \ell \in J_h'\}] \ .$$

Since for any ℓ in J_h', $t_{i,\ell}^{-1}\otimes t_{i,\ell}$ is a unit in $C_{j,x,i}^{(h)}$ we obtain that

$$\mathcal{D}_{T_S^{(h)}/Z_S} = \prod_{i,j} C_{j,x,i}^{(h)} \cdot d_{j,i}^{(h)} \tag{3.8}$$

with $d_{j,i}^{(h)} = \prod_{\ell\in J_h''}(1\otimes t_{i,\ell})^{(e_{i,\ell}-1)}$ (see for instance [Mi] I Sect. 3 Ex. 3.9). So the square roots exist because the $e_{i,\ell}$ are odd.

For future reference we also note the equality

$$q_Z^*((\pi_0)_*(\mathcal{D}_{T^{(h)}/Z}^{-1/2})) = (\pi_{0,S})_*(\mathcal{D}_{T_S^{(h)}/Z_S}^{-1/2}) \ . \tag{3.8}$$

Definition 3.9. For any $1 \le h \le m$ we let

$$\Lambda^{(h)} := (\pi_h)_*(\mathcal{D}_{T^{(h)}/Z}^{-1/2}) \ .$$

This is a locally free sheaf on Z, which gives rise to a form on Z when endowed with the trace form. In particular

$$\Lambda^{(0)} = (\pi_0)_*(\mathcal{D}_{T'/Z}^{-1/2}) = \phi^*(\pi_*(\mathcal{D}_{X/Y}))$$

and

$$\Lambda^{(m)} = (\pi_Z)_*(\mathcal{D}_{T^{(m)}/Z}^{-1/2}) = (\pi_Z)_*(O_T) \ .$$

Lemma 3.10. *For $1 \le h, h' \le m$ the forms $(\Lambda^{(h)}, Tr)$ and $(\Lambda^{(h')}, Tr)$ agree on the generic fiber of Z.*

In fact the forms are isomorphic outside the inverse image in Z of the branch locus b.

Remark. As noted in [S2] and [L-W], in case Y is a complex algebraic curve the existence of the square root of the inverse different amounts to the existence of a canonical choice of a theta characteristic on Y (or also to a canonical spin structure on Y).

3.e. Local structure of the modules $\Lambda^{(h)}$.
Let $q_Z : \mathcal{U} = Z_S \to Z$ denote the neighbourhood constructed in Sect. 2, using a neighbourhood S of $y = \phi(z)$, and write

$$\Lambda^{(h)}(S) := (q_Z^*(\Lambda^{(h)}))(Z_S) \ .$$

Recall that for $\widetilde{x}$ in $\widetilde{X}$ above y, and for any h we had defined in (3.5) a partition of the set $J = J(\widetilde{x})$ into the union of $J'_h = J'_h(\widetilde{x})$ and $J''_h = J''_h(\widetilde{x})$. Also recall the notation introduced for Lemma 2.5, where we had parametrised the characters of the group H_i in terms of sequences $(\alpha_{i,\ell})$. For $\alpha = (\alpha_{i,\ell})$, write

$$\partial_i^{(h)}(\alpha) := \left(\prod_{\ell \in J'_h} t_{i,\ell}^{-\alpha_{i,\ell}} \prod_{\ell \in J''_h} t_{i,\ell}^{-\epsilon_{i,\ell}}\right) \otimes \prod_{\ell \in J} t_{i,\ell}^{\alpha_{i,\ell}} \ ,$$

where $\epsilon_{i,\ell} = 0$ or $e_{i,\ell}$ depending on whether $\alpha_{i,\ell}$, which by definition satisfies $0 \le \alpha_{i,\ell} < e_{i,\ell}$, is strictly smaller or strictly larger than $e_{i,\ell}/2$. Put

$$D_{j,i}^{(h)}(\alpha) := B_{j,x}\partial_i^{(h)}(\alpha) \quad \text{and} \quad D_{j,i}^{(h)} := \bigoplus_{\alpha \in A(i)} D_{j,i}^{(h)}(\alpha) \ .$$

Note that $D_{j,i}^{(h)}(\alpha)$ is a rank 1 module, so that the above should be viewed as a decomposition into eigenspaces according to the characters of H_i.

Lemma 3.11.

$$\Lambda^{(h)}(S) = \prod_{1 \le j \le s, 1 \le i \le r} D_{j,i}^{(h)} \ .$$

Proof. This follows from equation (3.8) and Prop. 3.6. Indeed by (3.8) we can write

$$\Lambda^{(h)}(S) = \prod_{i,j} C_{j,x,i}^{(h)} \cdot (d_{j,i}^{(h)})^{-1/2}$$

and by (3.6)(b)

$$C_{j,x,i}^{(h)} := \bigoplus_{\alpha \in A(i)} B_{j,x}\theta_i^{(h)}(\alpha) \ .$$

Therefore we obtain a basis for the (j,i)-th component of $\Lambda^{(h)}(S)$ by considering the products

$$\prod_{\ell \in J'_h} t_{i,\ell}^{-\alpha_{i,\ell}} \otimes \prod_{\ell \in J'_h} t_{i,\ell}^{\alpha_{i,\ell}} \prod_{\ell \in J''_h} t_{i,\ell}^{\alpha_{i,\ell}-(e_{i,\ell}-1)/2}$$

with α running through $A(i)$. We rewrite this and obtain the desired result after a permutation of the α's. For ℓ in J''_h consider $1 \otimes t_{i,\ell}^{\alpha_{i,\ell}-(e_{i,\ell}-1)/2}$. When $(e_{i,\ell}-1)/2 \le \alpha_{i,\ell} \le e_{i,\ell}$, we put $\alpha'_{i,\ell} := \alpha_{i,\ell} - (e_{i,\ell}-1)/2$, so that $\alpha'_{i,\ell}$ lies between 0 and $e_{i,\ell}/2$. When $0 \le \alpha_{i,\ell} < (e_{i,\ell}-1)/2$, we put $\alpha'_{i,\ell} :=$

$\alpha_{i,\ell}+(e_{i,\ell}+1)/2$, so that in this case $\alpha'_{i,\ell}$ lies between $e_{i,\ell}/2$ and $e_{i,\ell}$ and we write

$$1\otimes t_{i,\ell}^{\alpha_{i,\ell}-(e_{i,\ell}-1)/2}=(t_{i,\ell}^{-e_{i,\ell}}\otimes t_{i,\ell}^{e_{i,\ell}})(1\otimes t_{i,\ell}^{\alpha_{i,\ell}-(e_{i,\ell}-1)/2})\ ,$$

which gives $1\otimes t_{i,\ell}^{\alpha_{i,\ell}-(e_{i,\ell}-1)/2}=t_{i,\ell}^{-e_{i,\ell}}\otimes t_{i,\ell}^{\alpha'_{i,\ell}}$.

We are now in the position to prove that the $\Lambda^{(h)}$ have the expected properties.

Proposition 3.12. *a) For any $1\le h\le m-1$, there are short exact sequences of* locally free *O_Z-modules*

$$0\longrightarrow I^{(h)}\stackrel{\text{diag}}{\longrightarrow}\Lambda^{(h)}\oplus\Lambda^{(h+1)}\longrightarrow\mathcal{G}^{(h)}\longrightarrow 0\ .$$

Here $I^{(h)}$ is given by the intersection of $\Lambda^{(h)}$ and $\Lambda^{(h+1)}$ viewed inside the generic fiber, $\mathcal{G}^{(h)}$ is the O_Z-module generated by $\Lambda^{(h)}$ and $\Lambda^{(h+1)}$ and the map into $\mathcal{G}^{(h)}$ is the difference map.

b) For any $1\le h\le m$, the form $(\Lambda^{(h)},(-1)^hTr)\oplus(\Lambda^{(h+1)},(-1)^{h+1}Tr)$ is metabolic with lagrangian $I^{(h)}=\mathcal{G}^{(h)\vee}$.

Proof. Part (b) is clear (see the example at the end of Sect. 1.c), so we only need to check that the modules are locally free. Because $\mathcal{G}^{(h-1)}(S)=\Lambda^{(h-1)}(S)+\Lambda^{(h)}(S)$ and $I^{(h-1)}(S)=\Lambda^{(h-1)}(S)\cap\Lambda^{(h)}(S)$, from the previous lemma we obtain the decompositions

$$\mathcal{G}^{(h-1)}(S)=\prod_{\alpha\in A(i)}\prod_{1\le j\le s,1\le i\le r}\mathcal{G}_{j,i}^{(h-1)}(\alpha)$$

and

$$I^{(h-1)}(S)=\prod_{\alpha\in A(i)}\prod_{1\le j\le s,1\le i\le r}I_{j,i}^{(h-1)}(\alpha)\ ,$$

where $\mathcal{G}_{j,i}^{(h-1)}(\alpha):=D_{j,i}^{(h-1)}(\alpha)+D_{j,i}^{(h)}(\alpha)$ and $I_{j,i}^{(h-1)}(\alpha):=D_{j,i}^{(h-1)}(\alpha)\cap D_{j,i}^{(h)}(\alpha)$.

Remark 3.13. Our aim is to compare the forms on $\Lambda^{(0)}$ and on $\Lambda^{(m)}$. The fact that for each h the forms $\Lambda^{(h)}\oplus\Lambda^{(h+1)}$ are metabolic should be interpreted in the framework of forms in triangulated categories (here the derived category $D^b(Z)$ of bounded complexes of locally free sheaves on Z). Then our construction should say that $\Lambda^{(0)}\oplus\Lambda^{(m)}$ is metabolic in this more general context. Note that, as shown in [Ba2] the usual Witt group of Z equals the Witt group of $D^b(Z)$.

Proposition 3.14.

a) *If $h \notin J(\widetilde{x})$, then for all i and j and all α in $A(i)$,*

$$I_{j,i}^{(h-1)}(\alpha) = \mathcal{G}_{j,i}^{(h-1)}(\alpha) = D_{j,i}^{(h-1)}(\alpha) = D_{j,i}^{(h)}(\alpha) \ .$$

b) *Denote by $A_h(i)$ the set of sequences α in $A(i)$ such that $e_{i,h}/2 < \alpha_{i,h}$. If $h \in J(\widetilde{x})$, then*

$$\mathcal{G}_{j,i}^{(h-1)}(\alpha) = B_{j,x}\delta_i^{(h-1)}(\alpha) \quad and \quad I_{j,i}^{(h-1)}(\alpha) = B_{j,x}\gamma_i^{(h-1)}(\alpha)$$

where

$$\delta_i^{(h-1)}(\alpha) = \begin{cases} \partial_i^{(h)}(\alpha) & \alpha \notin A_h(i) \\ \partial_i^{(h-1)}(\alpha) & \alpha \in A_h(i) \end{cases}$$

$$\gamma_i^{(h-1)}(\alpha) = \begin{cases} \partial_i^{(h-1)}(\alpha) & \alpha \notin A_h(i) \\ \partial_i^{(h)}(\alpha) & \alpha \in A_h(i) \end{cases}$$

c) *The eigenspaces $\mathcal{G}_{j,i}^{(-)}(\alpha)$ and $D_{j,i}^{(-)}(\alpha)$ are related as follows. For any couple (i,j) and any h in J*

$$D_{j,i}^{(h)}(\alpha) = \begin{cases} \mathcal{G}_{j,i}^{(h-1)}(\alpha) & \alpha \notin A_h(i) \\ \\ \mathcal{G}_{j,i}^{(h-1)}(\alpha)({t_{i,h}}^{e_{i,h}-\alpha_{i,h}} \otimes 1) & \alpha \in A_h(i) \end{cases}$$

The proposition follows from the equalities

$$\partial_i^{(h-1)}(\alpha) = (t_{i,h}^{\alpha_{i,h}} \otimes 1)\partial_i^{(h)}(\alpha) \qquad \text{if } 0 \le \alpha_{i,h} < e_{i,h}/2$$

$$\partial_i^{(h)}(\alpha) = (t_{i,h}^{e_{i,h}-\alpha_{i,h}} \otimes 1)\partial_i^{(h-1)}(\alpha) \quad \text{if } e_{i,h}/2 < \alpha_{i,h} < e_{i,h}$$

4. Proofs of the main theorems

4.a. Proof of Theorem 0.1.

We showed in Prop. 3.12 that the form

$$(\Lambda^{(h)}, (-1)^h Tr) \oplus (\Lambda^{(h+1)}, (-1)^{h+1} Tr)$$

is metabolic with lagrangian $\mathcal{G}^{(h)\vee}$, so the Main Lemma (see Cor. 1.15 (b)) implies that for $0 \le h \le m-1$ in $H^*(Z_{et}, \mathbf{Z}/2\mathbf{Z})$

$$w_t(\Lambda^{(h)}, (-1)^h Tr) \cdot w_t(\Lambda^{(h+1)}, (-1)^{h+1} Tr) = d_t(\mathcal{G}^{(h)}) \ , \tag{4.0}$$

where, we recall, $d_t(V) = \sum_{i=0}^n (1+(-1)t)^{n-i} c_i(V) t^{2i}$. This implies that

$$w_t(\Lambda^{(0)}, Tr) \cdot w_t(\Lambda^{(m)}, (-1)^m Tr)^{(-1)^{m+1}} = \prod_{1 \le h \le m} d_t(\mathcal{G}^{(h-1)})^{(-1)^{h-1}} \ .$$

To obtain the claim of Thm. 0.1 from this we simply have to remember the equalities just prior to Lemma 3.10.

Remark. As noted after the statement of Thm. 0.1, if $d = b_{m+1}$ denotes an irreducible divisor on Y not contained in the branch locus b and $b' = b \cup b_{m+1}$, then for the statement of the theorem we might as well work with b' instead of b. This can be seen as follows, let $F = \Lambda^{(m+1)} = (\pi_Z)_*(O_T)$, then the sequence of Prop. 3.12 corresponding to $h = m$ is

$$0 \to F \to F \oplus F \to \mathcal{G}^{(m)} = F \to 0$$

and so by the Main Lemma 1.15

$$w_t((\pi_Z)_*(O_T), Tr_{T/Z}) w_t((\pi_Z)_*(O_T), -Tr_{T/Z}) = d_t(\mathcal{G}^{(m)}) = d_t(F) \ .$$

4.b. Proof of Theorem 0.2.

We begin by reducing the theorem to the étale case by using the base change $\phi : Z \to Y$. For this we use (1) functoriality: for both kind of classes $\phi^* w_i(-) = w_i(\phi^*(-))$, and (2) the fact that, as we saw in Thm. 2.2 the pull-back map $\phi^* : H^*(Y) \to H^*(Z)$ is injective.

To see the equality of the first Hasse-Witt classes, recall from Sect. 1.e.1 that $w_1(F, B_F) = w_1(\det(F), \det(B_F))$, which shows that we have the equality

$$w_1((\pi_Z)_*(O_T), Tr) = w_1(\phi^*(\pi_*(\mathcal{D}_{X/Y}^{-1/2})), Tr)$$

because the forms coincide on the generic fiber and thus have isomorphic determinants. Moreover these determinants equal the function field discriminant $(d_{T/Z}) = (d_{T'/Z})$, so the first part of Thm. 0.2 follows (see Sect. 1.h). The same reasoning also shows that

$$\begin{cases} w_1(\Lambda^{(0)}, Tr) = w_1(\Lambda^{(h)}, Tr) & 0 \le h \le m \\ \\ c_1(\Lambda^{(0)}) = c_1(\Lambda^{(h)}) & 0 \le h \le m \end{cases} .$$

We now consider the second invariants. From the Main Lemma 1.15 (c) we deduce that

$$w_2(\Lambda^{(h-1)}, Tr) + w_2(\Lambda^{(h)}, Tr) = c_1(\Lambda^{(h)}) + c_1(\mathcal{G}^{(h-1)}) \ .$$

Thus we obtain

$$w_2((\pi_Z)_*(O_T), Tr) + w_2(\phi^*(\pi_*(\mathcal{D}_{X/Y}^{-1/2})), Tr) = \sum_{h=1}^{m} c_1(\Lambda^{(h)}) + c_1(\mathcal{G}^{(h-1)}) \ .$$

Remark 4.1. Note that when m is even the right hand side of the last expression equals $\det(\mathcal{G})$, where

$$\det(\mathcal{G}) = \sum_{h=1}^{m} \det(\mathcal{G}^{(h)})^{(-1)^h} \ .$$

By the result in the étale case, presented in Sect. 1.h, we see that

$$w_2((\pi_Z)_*(O_T), Tr) = w_2(\pi_Z) + (2) \cup (d_{T/Z}) \ .$$

Thus by the result for w_1 and the functoriality of the Galois theoretic w_2, we are reduced to showing the next theorem.

Theorem 4.2. *The following equality holds in $H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z})$:*

$$\sum_{1\leq h\leq m} \left(c_1(\Lambda^{(h)}) + c_1(\mathcal{G}^{(h-1)})\right) = \phi^*(\rho(X/Y)) \ ,$$

where $\rho(X/Y)$ has been defined in terms of ramification data in (0.1).

The proof is given in the next section and makes heavy use of the local calculations performed above.

4.c. Determinantal calculation.

For the proof of Thm. 4.2 we start by examining both sides of the desired equality. The right hand side is defined as a *divisor* of Z and we use the same notation for its class in $\mathrm{Pic}(Z)$, as well as for the image of this class in $H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z})$. Now, from the very definition of Chern classes, we first observe that the left hand side can also be considered as the image in $H^2(Z_{et}, \mathbf{Z}/2\mathbf{Z})$ of an element of $\mathrm{Pic}(Z)$. Namely it is the image of the divisor defined by $\sum_{1\leq h\leq m} \det(\Lambda^{(h)}) + \det(\mathcal{G}^{(h-1)})$. In fact, as in [E-K-V], p.176, for any h, the inclusion $\alpha_h : \Lambda^{(h)} \longrightarrow \mathcal{G}^{(h-1)}$, induces an exact sequence

$$(4.2) \qquad 0 \longrightarrow \det(\Lambda^{(h)}) \longrightarrow \det(\mathcal{G}^{(h-1)}) \longrightarrow \mathrm{coker}(\det(\alpha_h)) \longrightarrow 0$$

Hence there exists a divisor $\Delta^{(h)}$ in $Div(Z)$ such that in $\mathrm{Pic}(Z)$

$$-\det(\Lambda^{(h)}) + \det(\mathcal{G}^{(h-1)}) = [\Delta^{(h)}] \ ,$$

where we denote by $[D]$ the class of the divisor D. We conclude that in $\mathrm{Pic}(Z) \otimes \mathbf{F}_2$

$$(4.2) \qquad \sum_{1\leq h\leq m} \det(\Lambda^{(h)}) + \det(\mathcal{G}^{(h-1)}) = \sum_{1\leq h\leq m} [\Delta^{(h)}] \ .$$

Therefore to show the theorem we are reduced to showing the following congruence in $Div(Z)$

$$(4.2) \qquad \Delta \equiv \phi^*(\rho(X/Y)) \mod 2 \ ,$$

where we denote by Δ the divisor $\sum_{1\leq h\leq m} \Delta^{(h)}$.

To show (4.2) it suffices to work in $Div(Z_S)$, where as before S is a suitable étale neighbourhood of a point y in Y and where X_S and Z_S are as described in Sect. 2. For further notation refer to the diagram there. So we have to show in $Div(Z_S)$

$$(4.2) \qquad q_Z^*(\Delta) \equiv q_Z^*(\phi^*(\rho(X/Y))) \mod 2 \ .$$

We start by examining the right hand side of the congruence. By diagram commutativity we deduce that $q_Z^*(\phi^*(\rho(X/Y))) = \phi_S^*(q^*(\rho(X/Y)))$. By definition $\rho(X/Y) = \pi_*(\Gamma)$, where Γ is a divisor on X (see (0.1)). We then deduce that $q_Z^*(\phi^*(\rho(X/Y))) = \phi_S^*(q^*(\pi_*(\Gamma)))$. By [EGAII] II 1.5.2, under our hypotheses we have that $q^*(\pi_*(\Gamma)) = \pi_{S*}(q_X^*(\Gamma))$. Since X_S and S are étale over X and Y respectively, the divisor $q_X^*(\Gamma)$ of X_S corresponds for $X_S \to S$ to what Γ is for $X \to Y$. Let us denote this divisor by Γ_S. Hence we have proved that the right hand side of the congruence (4.2) is equal to ${\phi_S}^*(\pi_{S*}(\Gamma_S))$, namely ${\phi_S}^*(\rho(X_S/S))$. We record this as

$$q_Z^*(\phi^*(\rho(X/Y))) = {\phi_S}^*(\rho(X_S/S)) \ . \tag{4.2}$$

We now describe $\rho(X_S/S)$. The points of codimension 1 which are ramified in $X_S \to S$ considered as points of codimension 1 of S are those above the generic points ξ_ℓ of the branch divisors b_ℓ, with ℓ in $J(\widetilde{x})$, which as usual we identify with the set $\{1, \ldots, n\}$. We let $\{\eta_{k,\ell},\ 1 \le k \le r_l,\ 1 \le l \le n\}$ be those primes. By definition we have

$$\rho(X_S/S) = \sum_{k,\ell} \Big(\sum_{\nu \to \eta_{k,\ell}} \frac{({e_\nu}^2 - 1)}{8} f(\nu) \Big) \eta_{k,\ell} \ . \tag{4.2}$$

It now follows from the description of X_S given in (2.4), that for each ℓ such that $1 \le \ell \le n$ and each $\nu \to \eta_{k,\ell}$, the e_ν run through the set $\{e_{i,\ell}, 1 \le i \le r\}$. So we have to determine, for each $e_{i,\ell}$, how many ν have this ramification index and for each such ν, the index $f(\nu)$ (or at least its parity). From the decomposition of $O_X(S)$ given in *loc. cit.* we observe that a prime of $O_X(S)$ will be the product of a prime in one component and the whole ring in the others. This observation leads us to study what happens in each factor $A_y[t_{i,1}, ..., t_{i,n}]$ of the decomposition. If follows from the definition of $A_y[t_{i,1}, ..., t_{i,n}]$, that for each $\eta_{k,\ell}$, the primes ν of this algebra above $\eta_{k,\ell}$ will have the same inertia index. We are thus led to the following claim.

Claim. *For each pair (i, ℓ) with $1 \le i \le r$ and $1 \le \ell \le n$ the sum*

$$\Sigma(i,\ell) := \sum_{\substack{\nu \to \eta_{k,\ell} \\ e_\nu = e_{i,\ell}}} f(\nu)$$

is odd. More precisely

$$\Sigma(i,\ell) = e'_{i,\ell} := \prod_{\substack{1 \le h \le n \\ h \ne \ell}} e_{i,h} \ .$$

Note in passing, that the index $f(\nu)$ is odd as it is the residue class degree associated to the extension $B_{\widetilde{x}}/B_{\widetilde{x}}^{I(x,i)}$, which is of odd degree.

To prove the claim let us simplify notation a little and write $A = A_y$, so that $S = \mathrm{Spec}(A)$. Recall that $X_S = \mathrm{Spec}(B)$, where

$$B = A[t_{1,1}, \dots , t_{1,n}] \times \cdots \times A[t_{r,1}, \dots , t_{r,n}] \ .$$

Let us treat the case $n = 2$, so that B equals

$$A[T_{1,1}, T_{1,2}]/(T_{1,1}^{e_{1,1}} - a_1, T_{1,2}^{e_{1,2}} - a_2) \times \cdots \times A[T_{r,1}, T_{r,2}]/(T_{r,1}^{e_{r,1}} - a_1, T_{r,2}^{e_{r,2}} - a_2) \ .$$

Let us consider $\Sigma(i, \ell)$ for $\ell = 1$. Thus $\eta_{k,1}$ is a prime of height one in A, which contains a_1 and which defines an element z in S. We are looking for the number of elements in the fiber $X_{S,z}$ and for ν in this fiber, we want to determine $f(\nu)$. By definition

$$X_{S,z} = \mathrm{Spec}(B \otimes_A k(z))$$

and, since $\eta_{k,1}$ contains a_1, we see that $B \otimes_A k(z)$ equals

$$k(z)[T_{1,1}, T_{1,2}]/(T_{1,1}^{e_{1,1}}, T_{1,2}^{e_{1,2}} - a_2) \times \cdots \times k(z)[T_{r,1}, T_{r,2}]/(T_{r,1}^{e_{r,1}}, T_{r,2}^{e_{r,2}} - a_2) \ .$$

A prime in this algebra is the product of a prime in one of the components with the other (full) components. Thus, for instance, the primes with ramification index $e_{1,1}$ correspond to the primes in the first component, that is to the primes of $k(z)[T_{1,1}, T_{1,2}]$ which contain $(T_{1,1}^{e_{1,1}}) + (T_{1,2}^{e_{1,2}} - a_2)$. In turn these are determined by the primes in the algebra $k(z)[T_{1,2}]/(T_{1,2}^{e_{1,2}} - a_2)$, which we find by decomposing the polynomial $P = T_{1,2}^{e_{1,2}} - a_2$ into irreducible factors. Now $a_2 \neq 0$ in $k(z)$, the index $e_{1,2}$ is prime to the residue characteristic of $k(z)$ (by tameness) and the field $k(z)$ contains the $e_{1,2}$-th roots of unity, so it follows that P decomposes into a product of q_1 polynomials, which are irreducible over $k(z)$ and of same degree f_1. This gives q_1 elements $\nu_{1,1}, \dots \nu_{1,q_1}$ in $X_{S,z}$ with $f_1 = f(\nu_{1,1}) = \ldots = f(\nu_{1,q_1})$. Hence the sum $\Sigma(i, \ell)$ equals

$$f(\nu_{1,1}) + \cdots f(\nu_{1,q_1}) = q_1 f_1 = e_{1,2} \ ,$$

which is $e'_{1,1}$ in this special case. This completes the proof of the claim.

Thus finally we deduce from (4.2) and the claim

$$\rho(X_S/S) \equiv \sum_{k,\ell} \Big(\sum_{1 \le i \le r} \frac{e_{i,\ell}^2 - 1}{8} \Big) \eta_{k,\ell} \bmod 2 \ ,$$

and therefore

$$\phi_S{}^*(\rho(X_S/S)) \equiv \phi_S{}^*\Big(\sum_{k,\ell} \Big(\sum_{1 \le i \le r} \frac{e_{i,\ell}^2 - 1}{8} \Big) \eta_{k,\ell}\Big) \bmod 2 \ . \tag{4.2}$$

We now consider the left hand side of the congruence (4.2). Using the fact that taking determinants commutes with base change, we obtain that

$$q_Z^*(\Delta) = \Delta(S) := \sum_{1\le h\le m} \Delta^{(h)}(S) \ ,$$

where the $\Delta^{(h)}(S)$ are obtained as in (4.2), via an exact sequence involving $\det(\Lambda^{(h)}(S))$ and $\det(\mathcal{G}^{(h-1)}(S))$, (see Sect. 3.d). The following is clear.

Lemma 4.3. *Let e be an odd integer. Then*

$$\sum_{e/2<\alpha<e} (e-\alpha) = \sum_{n=1}^{(e-1)/2} n = \frac{e^2-1}{8} \ .$$

From Prop. 3.14 (c) we deduce that if h is not in $J(\tilde{x})$, then $\Delta^{(h)}(S)$ is trivial. If instead ℓ is in $J(\tilde{x})$, then using the lemma, Prop. 3.14 (c) and taking the discriminant and the product over the α, we obtain

$$\det(\mathcal{G}_{j,i}^{(\ell-1)}) = (t_{i,\ell}^{-\frac{e_{i,\ell}^2-1}{8}})\det(D_{j,i}^{(\ell)}) \ .$$

Thus, for ℓ in $J(\tilde{x})$,

$$\det(\mathcal{G}^{(\ell-1)}(S)) = \prod_{1\le j\le s}\prod_{1\le i\le r}(t_{i,\ell}^{-\frac{e_{i,\ell}^2-1}{8}})\det(\Lambda^{(\ell)}(S)) \ . \tag{4.3}$$

and $\Delta^{(\ell)}(S)$ is defined as a Cartier divisor of Z_S, by the function

$$\beta^{(\ell)} = \prod_{1\le j\le s}\prod_{1\le i\le r}(t_{i,\ell}^{-\frac{e_{i,\ell}^2-1}{8}}) \ .$$

We then conclude that $\Delta(S)$ is defined by

$$\beta = \prod_{\ell\in J(\tilde{x})}\prod_{1\le j\le s}\prod_{1\le i\le r}(t_{i,\ell}^{-\frac{e_{i,\ell}^2-1}{8}}) \ .$$

For any ℓ in $J(\tilde{x})$ we put $N_\ell = \prod_{1\le i\le r} e_{i,\ell}$ and we write $N = \prod_\ell N_\ell$. Using the fact that for all i we have $t_{i,\ell}{}^{e_{i,\ell}} = a_\ell$, we then obtain for the N-th power

$$\beta^N = \prod_{\ell\in J(\tilde{x})} a_\ell{}^{s\frac{N}{N_\ell}\sum_{1\le i\le r} -e'_{i,\ell}(\frac{e_{i,\ell}^2-1}{8})} \ .$$

Since the integers N, N_l, s and the $e'_{i,\ell}$ are odd we obtain that

$$\beta \equiv \beta^N \equiv \prod_{l\in J(\tilde{x})} a_l^{\sum_{1\le i\le r}(\frac{e_{i,\ell}^2-1}{8})} \mod 2 \ .$$

Therefore we conclude that

$$\Delta(S) \equiv \phi_S{}^*(\sum_{k,\ell}(\sum_{1\leq i\leq r}\frac{e_{i,\ell}^2-1}{8})\eta_{k,\ell}) \bmod 2 \ .$$

Hence the result follows from this last equation together with (4.2), (4.2) and (4.2).

5. Appendix. Simplicial techniques

5.a. Simplicial objects.

We recall the definition of a simplicial object in a category $\mathcal{C}$. Let Δ denote the category whose objects are the ordered sets $[n]=\{0,1,2,\ldots,n\}$, whose morphisms are all non-decreasing monotone maps. This category can be shown to be equivalent to the category, denoted by the same letter, generated by the objects $[n]$ and for $0\leq i\leq n$ by the maps

$$\partial_n^i:[n-1]\to[n] \quad \sigma_n^i:[n+1]\to[n] \ ,$$

which are respectively the increasing injection which does not take the value i in $[n]$, and the non-decreasing surjection which takes the value i twice in $[n]$. (To complete the identification one should also add a list of identities which describe the commutation rules satisfied by these maps, see [G-Z] II Lemma 2.2, [Mac] pp.172-173, [Mac-Mo] VIII 7, [Ma1] Sect. 2 or [Go-Ja] I (1.2).) The maps ∂_n^i are called (co)faces and the maps σ_n^i (co)degeneracies.

A *simplicial (resp. cosimplicial) object* in a category $\mathcal{C}$ is a functor X : $\Delta^{opp}\to\mathcal{C}$ (resp. $Y:\Delta\to\mathcal{C}$). We write $X_n=X([n])$ and $\partial_i^n=X(\partial_n^i)$ for the faces and $\sigma_i^n=X(\sigma_n^i)$ for the degeneracies on X. So a simplicial object in $\mathcal{C}$ is determined by the X_n and the identities satisfied by the faces and degeneracies. Simplicial objects in a category $\mathcal{C}$ form a category $S\mathcal{C}$.

If $\mathcal{C}=$ Set is the category of sets, one speaks of a simplicial set. We let $\mathbf{S}=S$Set. Starting with a simplicial set X one obtains (in a functorial way) a topological space $|X|$, called the *realization* of X. Conversely, given a topological space T one defines a simplicial set $S(T)$, called the *singular set*. This is the origin of simplicial techniques. In fact the realization functor is left adjoint to the singular functor (see *e.g.* [Go-Ja] I Prop. 2.2).

Example 5.1. Given a (small) category $\mathcal{B}$ one obtains a simplicial set $N(\mathcal{B})$ called the *nerve* of the category, by letting

$$N(\mathcal{B})_n = Hom(E_n,\mathcal{B}) \ ,$$

where E_n is $[n]$ viewed as a category, and Hom denotes the set of functors, see [Mac-Mo] VIII 7 or [Seg]. (Note that any ordered set $(E,\leq)$ defines a category whose objects are the elements of the set and where there is a morphism from object x to object y precisely if $x\leq y$ in E.) The set $N(\mathcal{B})_n$ can be identified with composable strings of n arrows in $\mathcal{B}$. Faces are then given by suitable compositions and degeneracies by inserting identity arrows.

Example 5.2. As a particular case we consider that of a groupoid $\mathcal{B}$, that is a category in which every morphism is invertible. The trivial groupoid $T(U)$ on a set U is the category whose objects are the elements of U and the morphisms are given by $U \times U$. Composition is given by $(x,y) \circ (y,z) = (x,z)$. (This is equivalent to the category with one object and one arrow.) Then the nerve $NT(U) = N(T(U))$ is given by

$$NT(U): \quad U \quad \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} \quad U \times U \quad \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} \quad U \times U \times U \quad \cdots$$

where we leave it to the reader to make the degeneracies and faces explicit.

Example 5.3. If we consider a group G as a category with one object e, then the nerve construction gives the *classifying simplicial set* of G, denoted $\mathbf{B}G$. One can check that this is the simplicial set (with only one vertex) given by

$$\mathbf{B}G: \quad e \quad \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} \quad G \quad \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} \quad G \times G \quad \cdots$$

Here the two maps $G \to e$ are the same, $e \to G$ is given by the inclusion of the identity, and the maps $G \times G \to G$ are $\partial_0 = pr_2$ (second projection), $\partial_1 = \mu$ (multiplication), and $\partial_2 = pr_1$ (first projection). The realization $|\mathbf{B}G|$ of $\mathbf{B}G$ is an Eilenberg-MacLane space of the form $K(G,1)$, that is it has G as fundamental group and all other homotopy groups reduced to the identity.

Example 5.4. The nerve of the trivial groupoid on G gives the *"universal bundle"* :

$$\mathbf{E}G := NT(G) \ ,$$

which maps to $\mathbf{B}G$.

Further examples of simplicial objects are given by actions of groups (X,G). A G-sheaf on a scheme X, can then be identified with a simplicial sheaf on the simplicial scheme associated to the action (X,G).

5.b. The Amitsur complex.

The Amitsur complex is a cosimplicial group defined as follows (see [D-G] III, Sect. 4 n. 6). Let Y be a scheme or more generally a sheaf and as before Thm. 1.6 let Y' be a covering of Y, that is a sheaf epimorphism. Let F be a contravariant functor on sheaves on Y, with values in a category $\mathcal{C}$. Let $K_{n-1} = (Y'/Y)^n$ denote the n-fold fiber product over Y of Y' with itself, with coordinates numbered from 0 to $n-1$. Consider the simplicial sheaf

$$\cdots \quad K_2 = (Y'/Y)^3 \quad \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \; K_1 = (Y'/Y)^2 \quad \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \quad K_0 = Y' \ ,$$

where for $0 \le i \le n$ the face maps $\partial_i^n : K_n \to K_{n-1}$ are defined by omission of the i-th coordinate and the degeneracies $\sigma_i^n : K_n \to K_{n+1}$ are obtained by "duplication" of i-th coordinate. Next apply the contravariant functor F to this, to get the cosimplicial object in the target category $\mathcal{C}$,

$$F(K_0) \quad \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \quad F(K_1) \quad \begin{smallmatrix}\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\end{smallmatrix} \quad F(K_2) \quad \cdots \ ,$$

with induced maps: cofaces $\partial_n^i = F(\partial_i^n) : F(K_{n-1}) \to F(K_n)$ and codegeneracies $\sigma_n^i = F(\sigma_i^n) : F(K_{n+1}) \to F(K_n)$. We will be interested in the case where

$$F(-) = Mor_{Sh}(-, G) \ ,$$

which, for G a sheaf of groups, takes values in the category of groups, and we will then write

$$C^n(Y'/Y, G) = Mor_{Sh}((Y'/Y)^{n+1}, G) \ .$$

The resulting cosimplicial group is the *Amitsur complex* defined by Y'/Y and G. Define further a subset $H^0(Y'/Y, G)$ of C^0 as the equalizer of the maps $\partial_1^1,\ \partial_1^0 : C^0 \to C^1$, that is

$$H^0(Y'/Y, G) := \{x \in C^0(Y'/Y, G) \mid \partial_1^1(x) = \partial_1^0(x)\} \ .$$

This is in fact a subgroup of C^0. The set of 1-*cocycles* is

$$(5.4) \qquad Z^1(Y'/Y, G) := \{f \in C^1(Y'/Y, G) \mid \partial_2^1(f) = \partial_2^2(f) \cdot \partial_2^0(f)\} \ .$$

Now, x in C^0 acts on the right on f in C^1 by the rule

$$f \cdot x = (\partial_1^1(x))^{-1} \cdot f \cdot \partial_1^0(x) \ ,$$

and Z^1 is stable under this action (use the equalities between maps from C^0 to C^2 : $\partial_2^2\partial_1^1 = \partial_2^1\partial_1^1$, $\partial_2^0\partial_1^1 = \partial_2^2\partial_1^0$, and $\partial_2^0\partial_1^0 = \partial_2^1\partial_1^0$).

Consider the orbit space

$$(5.4) \qquad H^1(Y'/Y, G) := Z^1(Y'/Y, G)/C^0(Y'/Y, G) \ .$$

This is the cohomology set we need in Sect.1.d. It has as distinguished point the unit in C^1.

Remark 5.5. When the group G is abelian, then the Amitsur complex is a cosimplicial complex of abelian groups, and so its cohomology equals that of the associated normal complex with differential given by the alternating sum $\partial^n = \sum_i (-1)^i \partial_i$ (see [Ma1] Thm. 22.1, [Cu] Sect. 5). This is the complex considered in say [K-O1] Chapt. V or [Knu] II.2, where one can see how it gives rise to higher cohomology groups and how it is related to Galois and Čech cohomology. See also [Mi] Chapt. I Prop. 4.6.

5.c. The homotopy category of simplicial sheaves.
As was seen in the main text homotopy classes of maps between simplicial objects can play a clarifying role in our subject. We just say a few words to introduce some relevant notations and refer to the work of Jardine–say– for more details. The starting point for what we are going to describe is the remarkable (and by now classical) fact that one can study spaces up to homotopy by using simplicial sets. Namely, by inverting certain morphisms called weak equivalences in the category $\mathbf{S}$ of simplicial sets, one obtains a category denoted $\mathrm{Ho}(\mathbf{S})$ equivalent to the category of CW-complexes with morphisms homotopy classes of maps. The equivalence is given by the realization and singular functors, see [Go-Ja] Thm. I.11.4. By abstracting what was necessary for the definition and study of the homotopy category $\mathrm{Ho}(\mathbf{S})$, Quillen arrived at the notion of a closed model category. Given a closed model category $\mathcal{C}$ one can define the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ to be the category with same objects as $\mathcal{C}$ and with morphisms between X and Y given by the set

$$[X, Y] := \pi(W_X, W_Y)$$

of homotopy classes between certain objects W_X and W_Y, which are weakly equivalent to X and Y (cofibrant and fibrant..., see [Go-Ja] p. 75).

In [Br], Brown has considered a closed model structure on the category of sheaves of simplicial sets on topological spaces. Then Jardine has defined a homotopy category for any Grothendieck site in [J5]. It is this last work that lays the foundations in the right generality for our needs. For instance it is there that one finds the definition of cohomology of a (fibrant) simplicial sheaf X, with coefficients in an abelian sheaf F as

$$H^i(X, F) := [X, K(F, i)] \ ,$$

where $K(F, i)$ is the simplicial abelian sheaf obtained by iterated application of the construction giving $\mathbf{B}F$ (see [J5] Sect. 2 and [J3] Sect. 3; the motivation for this again comes from topology, see *e.g.* [Go-Ja] III.2.19). Recall that this entered the definition of the Galois theoretic classes in Sect. 1.f. A further application of these ideas is given in the next section.

5.d. Torsors and homotopy.
Here we sketch a proof of Prop. 1.7, which states that for any sheaf of groups G on Y, there is a bijection

$$H^1(Y, G) \leftrightarrow [Y, \mathbf{B}G] \ ,$$

where the left hand side is defined in terms of the cohomology of the Amitsur complex. We follow Jardine [J1].

As a first step in obtaining the description of torsors under G, in terms of homotopy classes of maps into $\mathbf{B}G$, we give an alternative interpretation of Amitsur 1-cocycles. So let

$$f : Y' \times_Y Y' \to G$$

be an element of $Z^1(Y'/Y, G)$ (see (5.4)). Using f we want to define a map from the simplicial sheaf $NT(Y'/Y)$ to the classifying sheaf $(\mathbf{B}G/Y)$. Both of these simplicial objects are defined in the Appendix (5.a). That is, we want a simplicial map

$$\begin{array}{lcccccc} NT(Y'/Y): & Y' & \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} & Y' \times_Y Y' & \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} & Y' \times_Y Y' \times_Y Y' & \cdots \\ (f_i)\downarrow & \downarrow & & f\downarrow & & \downarrow & \\ (\mathbf{B}G/Y): & Y & \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} & G & \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} & G \times_Y G & \cdots \end{array}$$

By unwinding the definitions it can be seen that putting $f_1 = f$ and

$$f_2(x, y, z) = (f_1(x, y), f_1(y, z))$$

makes the second square commutative, precisely because f is supposed to be a 1-cocycle. Proceeding in this way and after having checked that two cocycles defining the same element of H^1 give homotopic maps, one obtains a bijection

$$H^1(Y'/Y, G) \leftrightarrow \pi(NT(Y'/Y), \mathbf{B}G) \ ,$$

where $\pi(-,-)$ denotes homotopy classes of simplicial maps. It is the content of [J1] Prop. 1.1, that for a locally trivial fibration $V \to Y$, the canonical map from V into the classifying space $NT(V_0)$ of the fundamental groupoid induces a bijection

$$\pi(NT(V_0), \mathbf{B}G) \leftrightarrow \pi(V, \mathbf{B}G) \ .$$

This can be paraphrased by saying that the fundamental groupoid preserves weak equivalences and that a hypercover has the fundamental groupoid as Čech resolution. (Also $\mathbf{B}G$ is right adjoint to fundamental groupoid.) An important result of [J5], inspired by Brown's [Br] Thm. 1.1 and Sect. 3 (5), shows that the set $[Y, \mathbf{B}G]$ is given by

$$[Y, \mathbf{B}G] = \lim_{\rightarrow} \pi(Z, \mathbf{B}G) \ ,$$

where the limit is taken over simplicial homotopy classes represented by locally trivial fibrations $Z \to Y$. This a generalized Verdier hypercovering theorem. Thus putting everything together we obtain the expected result.

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