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# The strongly perfect lattices of dimension 10

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*Dedicated to Prof. Jacques Martinet*

Résumé. Cet article donne une classification des réseaux fortement parfaits en dimension 10. A similitude près il y a deux tels réseaux, $K_{10}'$ et son réseau dual.

Abstract. This paper classifies the strongly perfect lattices in dimension 10. There are up to similarity two such lattices, $K_{10}'$ and its dual lattice.

## 1. Introduction

This paper is a continuation of a series of papers on lattices and spherical designs ([Ven], [BaV], [Mar1]). For the basic definitions we refer to these papers and to the books [Mar] and [CoS].

We study lattices, in Euclidean space, whose minimal vectors form a spherical 4-design. Such lattices are called strongly perfect lattices. They form an interesting class of lattices. In particular strongly perfect lattices are perfect lattices in the sense of Voronoi (cf. [Mar]). In small dimensions there are only few such lattices and a complete classification was known up to dimension 9 and for dimension 11. Up to similarity they are root lattices of types $A_1, A_2, D_4, E_6, E_7, E_8$ and their duals ([Ven]). There are no such lattices in dimensions 3, 5, 9, and 11. In this paper we classify the strongly perfect lattices in dimension 10. This is the first dimension where less trivial lattices do appear. Up to similarity there are exactly two such lattices $K_{10}'$ and its dual lattice (cf. Theorem 3.1). This was conjectured in [Ven]. For a description of these lattices we refer to [Mar, Chap.VIII, paragraphe 5].

## 2. Some general equations

**2.1. General notation.** For a lattice $\Lambda$ in $n$-dimensional Euclidean space we denote by $\Lambda^*$ its dual lattice. Important invariants of the lattice $\Lambda$ are its

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determinant $\det(\Lambda)$, its minimum $\min(\Lambda)$ and its Hermite invariant, which is defined as

$$\gamma(\Lambda) := \frac{\min(\Lambda)}{\det(\Lambda)^{1/n}}.$$

The Hermite invariant is an invariant of the similarity class of $\Lambda$. As a function of $\Lambda$, $\gamma$ is bounded by a constant $\gamma_n := \max\{\gamma(\Lambda) \mid \Lambda \subset \mathbb{R}^n$ is a lattice $\}$ that depends only on the dimension $n$. The exact values for $\gamma_n$ are known for $n \leq 8$. For higher dimensions, only upper bounds for $\gamma_n$ are known. We use Rogers' bound, which gives that $\gamma_{10} \leq 2.2752$ ([CoS, Table 1.2]). Closely related to the Hermite invariant is the Bergé-Martinet invariant

$$(\gamma')^2(\Lambda) := \gamma(\Lambda)\gamma(\Lambda^*) = \min(\Lambda)\min(\Lambda^*)$$

which takes into account also the dual lattice.

There are also general bounds on the number of shortest vectors, the Kissing number, of an $n$-dimensional lattice. For $n = 10$ the bound is $2 \cdot 297$ (see [CoS, Table 1.5]).

For $a \in \mathbb{R}$, $a > 0$ we let

$$\Lambda_a := \{x \in \Lambda \mid (x, x) = a\}.$$

For $\alpha \in \Lambda^*$, $m := \min(\Lambda)$ and $i \in \mathbb{N}$ let

$$N_i(\alpha) := \{x \in \Lambda_m \mid (\alpha, x) = i\}$$

and $n_i(\alpha) := |N_i(\alpha)|$.

If $\Lambda$ is an integral lattice, that is $\Lambda \subset \Lambda^*$, we let for $p \in \mathbb{N}$

$$\Lambda_{(p)} := \{v \in \Lambda \mid (v, v) \equiv 0 \pmod{p}\}.$$

In general $\Lambda_{(p)}$ is only a subset of $\Lambda$, but in many cases that we consider, it turns out to be a sublattice of $\Lambda$.

**2.2. Designs and strongly perfect lattices.** Let $(\mathbb{R}^n, (,))$ be the Euclidean space of dimension $n$. In this section we assume that $m \in \mathbb{R}$, $m > 0$ and $X \subseteq S^{n-1}(m) = \{y \in \mathbb{R}^n \mid (y, y) = m\}$ is such that $X \cap -X = \emptyset$ and $X \cup -X$ is a spherical 4-design. Let $s := |X|$.

Then by the definition of designs in [Ven] one has for all $\alpha \in \mathbb{R}^n$:

$$(D2) \qquad \sum_{x \in X} (x, \alpha)^2 = \frac{sm}{n}(\alpha, \alpha)$$

and

$$(D4) \qquad \sum_{x \in X} (x, \alpha)^4 = \frac{3sm^2}{n(n+2)}(\alpha, \alpha)^2.$$

Substituting $\alpha := \xi_1\alpha_1 + \xi_2\alpha_2$ in $(D2)$ and comparing coefficients, one finds

$$(D11) \quad \sum_{x\in X}(x,\alpha_1)(x,\alpha_2) = \frac{sm}{n}(\alpha_1,\alpha_2) \text{ for all } \alpha_1,\alpha_2 \in \mathbb{R}^n.$$

Writing $\alpha$ as a linear combination of 4 vectors, $(D4)$ implies that for all $\alpha_1,\dots,\alpha_4 \in \mathbb{R}^n$

$$(D1111) \qquad \sum_{x\in X}(x,\alpha_1)(x,\alpha_2)(x,\alpha_3)(x,\alpha_4) =$$

$$= \frac{sm^2}{n(n+2)}((\alpha_1,\alpha_2)(\alpha_3,\alpha_4) + (\alpha_1,\alpha_3)(\alpha_2,\alpha_4) + (\alpha_1,\alpha_4)(\alpha_2,\alpha_3))$$

In particular

$$(D13) \qquad \sum_{x\in X}(x,\alpha_1)(x,\alpha_2)^3 = \frac{3sm^2}{n(n+2)}(\alpha_1,\alpha_2)(\alpha_2,\alpha_2)$$

$$(D22) \quad \sum_{x\in X}(x,\alpha_1)^2(x,\alpha_2)^2 = \frac{sm^2}{n(n+2)}(2(\alpha_1,\alpha_2)^2 + (\alpha_1,\alpha_1)(\alpha_2,\alpha_2))$$

**Lemma 2.1.** *Let $\alpha \in \mathbb{R}^n$ be such that $(x,\alpha) \in \{0,\pm1,\pm2\}$ for all $x \in X$. Let $N_2(\alpha) := \{x \in X \cup -X \mid (x,\alpha) = 2\}$ and put*

$$c := \frac{sm}{6n}(\frac{3m}{n+2}(\alpha,\alpha) - 1).$$

*Then $|N_2(\alpha)| = c(\alpha,\alpha)/2$ and*

$$\sum_{x\in N_2(\alpha)} x = c\alpha.$$

*Proof.* Put $\alpha_2 = \alpha$ in $(D11)$ and $(D13)$. Then $(D13) - (D11)$ reads as

$$\sum_{x\in X}(x,\gamma)(x,\alpha)((x,\alpha)^2 - 1) = (\alpha,\gamma)\frac{sm}{n}(\frac{3m}{n+2}(\alpha,\alpha) - 1) \text{ for all } \gamma \in \mathbb{R}^n.$$

The left hand side is

$$6\sum_{x\in N_2(\alpha)}(x,\gamma)$$

and hence

$$(\sum_{x\in N_2(\alpha)} x - c\alpha, \gamma) = 0$$

for all $\gamma \in \mathbb{R}^n$, where $c$ is the constant of the lemma. Taking the scalar product with $\alpha$, one sees that $2|N_2(\alpha)| = c(\alpha,\alpha)$. □

Recall that a lattice $L \subset \mathbb{R}^n$ is called strongly perfect, if its minimal vectors form a 4-design.

**Lemma 2.2** ([Ven, Théorème 10.4]). *Let $L$ be a strongly perfect lattice of dimension $n$. Then the Bergé-Martinet invariant*

$$(\gamma')^2(L) = \min(L)\min(L^*) \geq \frac{n+2}{3}.$$

*Proof.* Let $m := \min(L)$, $2s := |L_m|$, $m' := \min(L^*)$ and and $r := mm'$. For $\alpha \in (L^*)_{m'}$ let $n_i := |\{x \in L_m \mid (x,\alpha) = \pm i\}|$. Then by $(D2)$ and $(D4)$ $\sum_{i=1}^r i^2 n_i = \frac{sr}{n}$ and $\sum_{i=1}^r i^4 n_i = \frac{3sr^2}{n(n+2)}$. The difference is

$$\sum_{i=2}^{r}(i^4 - i^2)n_i = \frac{rs}{n}(\frac{3r}{n+2} - 1).$$

Since the left hand side is a sum of non negative numbers, the right hand side is non negative and therefore $r \geq \frac{n+2}{3}$. One also sees that equality implies that $n_2 = n_3 = \ldots = 0$. □

**2.3. Gauss sums and the Milgram-Braun formula.** For the classification of (strongly perfect) lattices it is helpful to know restrictions on the possible genera of even lattices. Let $L$ be an even lattice in the Euclidean space $(\mathbb{R}^n, (,))$ and $L^*$ be its dual lattice. Then the bilinear form $(,)$ induces a quadratic form $q : L^*/L \to \mathbb{R}/\mathbb{Z}$ on the finite abelian group $D := L^*/L$ by $q(x+L) := \frac{1}{2}(x,x) + \mathbb{Z}$. Then the Gauss sum $G(L) = G(D,q)$ is defined as

$$G(D,q) := \frac{1}{\sqrt{|D|}} \sum_{d \in D} \exp(2\pi i q(d)).$$

The following is known as the Milgram-Braun formula.

**Lemma 2.3** (see [Scha, Corollary 5.8.2], [MiH, Appendix 4]).

$$G(L) = \exp(\frac{2\pi i}{8})^n.$$

Moreover if $(D,q) = (D_1,q_1) \perp (D_2,q_2)$ then $G(D,q) = G(D_1,q_1) \cdot G(D_2,q_2)$. Since $D$ is the orthogonal sum of its Sylow $p$-subgroups, and $G(L)$ is independent of the even lattice $L$, it suffices to calculate $G(D,q)$ for anisotropic orthogonally indecomposable $p$-groups.

**Lemma 2.4** ([Scha, Corollary 5.8.3]). *If $|D| = 1$, then $G(D) = 1$. If $|D| = p$ for some odd prime $p$, $D = \langle x \rangle$, then*

$$G(D) = \begin{cases} \left(\frac{q(x)}{p}\right) & \text{if } p \equiv 1 \pmod 4 \\ i\left(\frac{q(x)}{p}\right) & \text{if } p \equiv 3 \pmod 4 \end{cases}.$$

For $p = 2$ one has 8 nonisometric anisotropic orthogonally indecomposable quadratic 2-groups:

**Lemma 2.5.** *Let $p = 2$ and $(D, q)$ be a nontrivial anisotropic orthogonally indecomposable 2-group. Then one of the following three possibilities occurs:*

(1) $D = \langle x, y\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *with* $q(ax + by) = \frac{1}{2}(a^2 + ab + b^2) + \mathbb{Z}$. *In this case* $G(D, q) = -1$.

(2) $D = \langle x\rangle \cong \mathbb{Z}/2\mathbb{Z}$ *with* $q(x) = \frac{\epsilon}{4} + \mathbb{Z}$ *where* $\epsilon = \pm 1$. *Here* $G(D, q) = \exp(\frac{2\pi i}{8})^{\epsilon}$.

(3) $D = \langle x\rangle \cong \mathbb{Z}/4\mathbb{Z}$ *with* $q(x) = \frac{\epsilon}{8} + \mathbb{Z}$ *where* $\epsilon = \pm 1, \pm 3$ *and* $G(D, q) = \exp(\frac{2\pi i}{8})^{\epsilon}$.

*Proof.* By [Cas, Lemma 8.4.1], any regular quadratic 2-group $(D, q)$ is isometric to the orthogonal sum copies of

$$(\mathbb{Z}/(2^{e-1}\mathbb{Z}) = \langle x\rangle, q(x) = \frac{a}{2^e} + \mathbb{Z}) \quad \text{with } a = \pm 1, \pm 3 \text{ and } e \in \mathbb{N}$$

and

$$(\mathbb{Z}/(2^{e-1}\mathbb{Z})^2 = \langle x, y\rangle \text{ with } q(ax+by) = \frac{1}{2^e}\begin{cases} ab \\ \text{or} \\ a^2 + ab + b^2 \end{cases} + \mathbb{Z}) \text{ and } e \in \mathbb{N}.$$

One now easily finds the anisotropic quadratic spaces among these. □

This explicit calculation of the Gauss sums has the following important consequence.

**Lemma 2.6.** *Let $L$ be an even lattice of dimension $n$ with* $n \equiv 2 \pmod 8$. *Then exactly one of the following holds:*

(a) *The number of primes* $p \equiv 3 \pmod 4$ *such that $p$ divides* $\det(L)$ *to an odd power is odd.*

(b) $2^2$ *divides* $\det(L)$ *and the number of orthogonal components of type (2) or (3) in Lemma 2.5 in the discriminant group of any maximal even overlattice of $L$ is* $\equiv 2 \pmod 4$.

## 3. The strongly perfect lattices of dimension 10

In this section we prove the following theorem:

**Theorem 3.1.** *There are exactly two similarity classes of strongly perfect lattices in dimension 10. They are represented by $K'_{10}$ and its dual lattice $(K'_{10})^*$.*

The lattices $K'_{10}$ and $(K'_{10})^*$ are described in [Mar, Chap. VIII, Tableau 5.9', 5.11]. $K'_{10}$ has minimum 4, determinant $2^2 \cdot 3^5$ and kissing number $2 \cdot 135$. The rescaled dual lattice $\sqrt{6}(K'_{10})^*$ has minimum 6, determinant $2^8 \cdot 3^5$ and kissing number $2 \cdot 120$.

**Lemma 3.2.** *The proper overlattices of $K'_{10}$ or $(K'_{10})^*$ have a strictly smaller minimum.*

*Proof.* Let $\Lambda$ be such an overlattice of $K'_{10}$ and $p := [\Lambda : K'_{10}]$ its index. Then by Rogers' bound ([CoS, Table 1.2])

$$\frac{\min(\Lambda)^{10}}{p^2 \det(K'_{10})} \leq \gamma_{10}^{10} \leq 3.5 \frac{\min(K'_{10})^{10}}{\det(K'_{10})}.$$

If $\min(\Lambda) = \min(K'_{10})$, then $p = 1$. Analogously one shows the lemma for $(K'_{10})^*$. □

Because of this lemma, we only have to show Theorem 3.1 under the assumption that the strongly perfect lattice is generated by its minimal vectors.

Now let $\Lambda$ be a strongly perfect lattice. We assume that $\Lambda$ is rescaled such that $\min(\Lambda) = 1$ and choose $X \subseteq \Lambda_1$ such that $X \cap (-X) = \emptyset$ and $X \cup (-X) = \Lambda_1$.

**3.1. Some invariants.** In this subsection we use the equalities in Section 2, to determine $s := |X|$ and $r := \min(\Lambda^*) = (\gamma')^2(\Lambda)$.

By Lemma 2.2 one has $r \geq 4$. But in fact equality holds here, which means that $\Lambda$ is of minimal type in the sense of [Ven].

**Lemma 3.3.** *$r = 4$ and $n_2(\alpha) = 0$ for all $\alpha \in (\Lambda^*)_4$.*

*Proof.* Let $\alpha \in (\Lambda^*)_r$. By the bounds on $s$ and $r$ given in Subsection 2.1, one has

$$C := \sum_{i=2}^{r} \frac{i^4 - i^2}{12} n_i(\alpha) = \frac{sr}{120}(\frac{r}{4} - 1) \leq 3.$$

Since $(i^4 - i^2) = i^2(i-1)(i+1)$ is divisible by 12 for all $i \in \mathbb{Z}$, $C$ is integral, $C = n_2(\alpha) + 6n_3(\alpha) + \ldots$. Therefore $n_3(\alpha) = n_4(\alpha) = \ldots = 0$ and $n_2(\alpha) \leq 3$. Write $r =: p/q$ with $p, q \in \mathbb{N}$ and $\gcd(p,q) = 1$. Assume that $p \neq 4q$. Then $s = 480 n_2(\alpha) q^2/(p(p-4q))$. Since $\gcd(p,q) = 1$ one has that $p$ divides $480 n_2$ and $q^2$ divides $s \leq 297$. One finds the following possibilities:

| $n_2$ | $s$ | $p$ | $q$ | $c$ |
|---|---|---|---|---|
| 1 | 96 | 5 | 1 | 2/5 |
| 1 | 125 | 24 | 5 | 5/12 |
| 1 | 243 | 40 | 9 | 9/20 |
| 1 | 169 | 60 | 13 | 13/30 |
| 2 | 192 | 5 | 1 | 4/5 |
| 2 | 250 | 24 | 5 | 5/6 |
| 2 | 289 | 80 | 17 | 17/20 |
| 3 | 288 | 5 | 1 | 6/5 |
| 3 | 245 | 36 | 7 | 7/6 |

This table lists the possible values $n_2(\alpha)$, $s$, $p$, $q$ that satisfy the conditions above. The last column gives the constant $c = \frac{s}{60}(\frac{r}{4}-1)$ of Lemma 2.1.

Since $n_i(\alpha) = 0$ for all $i \leq 3$, the conditions of Lemma 2.1 are satisfied and

$$\sum_{x \in N_2(\alpha)} x = c\alpha$$

where $c$ is the constant in the last column of the table above. If $n_2(\alpha) = 1$, then $N_2(\alpha) = \{x\}$ and therefore

$$c^2(\alpha,\alpha)\frac{p}{q} = 1.$$

This is a contradiction in the four cases where $n_2(\alpha) = 1$.

Since $N_2(\alpha)$ consists of minimal vectors (of square length 1) in a lattice, the scalar products $(x_1, x_2) \leq 1/2$ for all $x_1, x_2 \in N_2(\alpha)$. Hence if $n_2(\alpha) = 2$, then

$$3 \geq (\sum_{x \in N_2(\alpha)} x, \sum_{x \in N_2(\alpha)} x) = c^2(\alpha,\alpha) = c^2 p/q$$

which leads to a contradiction in the cases where $n_2 = 2$.

In the remaining two cases $n_2 = 3$. Let $N_2(\alpha) := \{x, y, z\}$. Then $c(x,\alpha) = 2c = (x,x) + (x,y) + (x,z)$. Analogous equations for $y$ and $z$ show that

$$2c - 1 = (x,y) + (x,z) = (x,y) + (y,z) = (x,z) + (y,z)$$

whence $(x,y) = (2c-1)/2 > 1/2$ which is a contradiction.

Therefore $n_2(\alpha) = 0$ and $r = 4$. □

We now determine the possible values for $s$:

**Lemma 3.4.** $s = 5s_0$ *where* $s_0 \in \{24, 32, 27, 25\}$.

*Proof.* From equation $(D2)$, with $\alpha \in (\Lambda^*)_4$, one gets that $2s/5 = n_1(\alpha) \in \mathbb{Z}$. Therefore $s$ is divisible by 5,

$$s = 5s_0.$$

For any $\alpha \in \Lambda^*$ write $(\alpha,\alpha) = p/q$ with $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$. Since $\frac{1}{12}((D4) - (D2)) \in \mathbb{Z}$ one gets that

$$(\star) \qquad \frac{s_0 p(p-4q)}{2^5 \cdot 3q^2} \in \mathbb{Z}.$$

In particular $q^2 \mid s_0 \leq 45$, hence $q \in \{1, 2, 3, 4, 5, 6\}$.

If $q$ is even, then $128 = 2^7$ divides $s_0$ which is a contradiction. Hence $q$ is odd.

If $q = 5$ then $5^2 \mid s_0$ and therefore $s_0 = 25$.

If $q = 3$ then $3^3 \mid s_0$ and therefore $s_0 = 27$.

So assume that $q = 1$, i.e. $(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Lambda^*$. If $4 \mid (\alpha, \alpha)$ for all $\alpha \in \Lambda^*$, then $1/\sqrt{2}\Lambda^* =: \Gamma$ is an even lattice with $\min(\Gamma) = 2$ and $\min(\Gamma^*) = 2$. Hence $\Gamma_2 \subseteq (\Gamma^*)_2$. But then for all $\alpha \in (\Lambda^*)_4$ one has $\frac{1}{2}\alpha \in \Lambda_1$ and $(\alpha, \frac{1}{2}\alpha) = 2$ contradicts the statement $n_2(\alpha) = 0$ of Lemma 3.3.

Therefore there is $\alpha \in \Lambda^*$ with $4 \not| (\alpha, \alpha)$ and hence $8 \mid s_0$, $s_0 \in \{8, 16, 24, 32, 40\}$.

If $2^5$ does not divide $s_0$ then $\Lambda^*$ is an even lattice. Assume also that $3 \not| s_0$, hence $s_0 \neq 24, 32$. By $(D22)$ for all $\alpha, \beta \in \Lambda^*$, one has

$$(\star\star) \qquad \sum_{x \in X} (x, \alpha)^2 (x, \beta)^2 = (s/120)(2(\alpha, \beta)^2 + (\alpha, \alpha)(\beta, \beta)).$$

Since $3 \not| s$, this implies that 3 divides $(2(\alpha, \beta)^2 + (\alpha, \alpha)(\beta, \beta))$ for all $\alpha, \beta \in \Lambda^*$, and therefore

$$\Gamma_{(3)} = \{v \in \Lambda^* \mid (v, v) \equiv 0 \pmod 3\} = 3\Lambda \cap \Lambda^*$$

is a sublattice of $\Lambda^*$ of index 3. Since $\sqrt{3}^{-1}\Gamma_{(3)}$ is an even lattice, the determinant $d := \det(\Lambda^*)$ is divisible by $3^{10-2} = 3^8$. But $\min(\Lambda) = 1$ and $\det(\Lambda) = 1/d$ implies that

$$\gamma(\Lambda)^{10} = d \leq (\gamma_{10})^{10} < 3719 < 3^8$$

which is a contradiction. $\square$

The main result of this subsection is the next lemma, which is stated without proof in [Ven, Théorème 13.2].

**Lemma 3.5.** *$s = 120$ or $s = 135$.*

*Proof.* By Lemma 3.4, we have to exclude the cases $s = 160$ and $s = 125$. Assume first that $s = 160$. Then by $(\star)$ the norms of all vectors in $\Lambda^*$ are integral and either 0 or 1 modulo 3. We also have $\min(\Lambda^*) = 4$ and $\min(\Lambda) = 1$. Assume that there is $\alpha \in \Lambda^*$ with $(\alpha, \alpha) = 6$. Then $n_i(\alpha) = 0$ for all $i \geq 3$ and $n_2(\alpha) = 4$. Let $N_2(\alpha) = \{x_1, x_2, x_3, x_4\}$. Then by Lemma 2.1 $x := x_1 + x_2 + x_3 + x_4 = 4/3\alpha$. But $16/9 \cdot 6 = (x, x) > 10$ contradicts the fact that $(x_i, x_j) \leq 1/2$ for all $1 \leq i \neq j \leq 4$. Therefore $(\Lambda^*)_6 = \emptyset$. Let $\Gamma := \sqrt{2}\Lambda^*$. Then $\Gamma$ is an even (and hence integral) lattice. Since 3 does not divide $s$ the equation $(\star\star)$ of the proof of Lemma 3.4 shows that $\Gamma_{(3)}$ is a sublattice of $\Gamma$ of index 3, and $3^8$ divides $\det(\Gamma) =: d$. Since $\min(\Gamma_{(3)}) \geq 18$, one gets that

$$d \geq \frac{18^{10}}{9\gamma_{10}^{10}} > 10^8.$$

On the other hand $\min(\Gamma^*) = 1/2$ and therefore

$$d \leq (2\gamma_{10})^{10} < 4 \cdot 10^7$$

which is a contradiction.

Now assume that $s = 125$. By equation ($\star$) of the proof of Lemma 3.4 one has that $\Gamma := \sqrt{5}/\sqrt{2}\Lambda^*$ is an even lattice. By ($\star\star$), since $3 \not| s$, the set $\Gamma_{(3)}$ is a sublattice of $\Gamma$ of index 3.

Assume that there is a vector $\alpha \in (\Lambda^*)_{24/5}$. Then $n_2(\alpha) = 7$ and $v := \sum_{x \in N_2(\alpha)} x = \frac{35}{12}\alpha$ by Lemma 2.1. Since the $x \in N_2(\alpha)$ are minimal vectors of $\Lambda$, one has $(v,v) \leq 7 + 7 \cdot 6 \cdot \frac{1}{2} = 28$. But $(v,v) = (\frac{35}{12})^2 \frac{24}{5} > 40$ which is a contradiction.

Therefore $\min(\Gamma_{(3)}) \geq 18$ and the determinant $d := \det(\Gamma)$ satisfies

$$10^8 < \frac{18^{10}}{9 \cdot \gamma_{10}^{10}} \leq d \leq \frac{\gamma_{10}^{10} \cdot 5^{10}}{2^{10}} < 4 \cdot 10^7$$

which is a contradiction. □

**3.2. The case** $s = 120$**.** In this subsection we assume that $\Lambda$ is a strongly perfect lattice of minimum 1 and dimension 10, such that $|\Lambda_1| = 2 \cdot 120$. We also assume that $\Lambda$ is generated by its minimal vectors.

Let $\Gamma := \Lambda^*$. Then equation ($\star$) of the proof of Lemma 3.4 shows that $\Gamma$ is an even lattice of minimum 4.

We show the following theorem:

**Theorem 3.6.** *Let $\Gamma$ be an even lattice of dimension 10 and of minimum $\geq 4$ such that the minimum of $\Gamma^*$ is $\geq 1$ and $\Gamma^*$ is generated by its minimal vectors. Then $\Gamma$ is isometric to either $K'_{10}$ or $Q_{10}$.*

The lattice $Q_{10}$ ([CoS]), (denoted by $F_5$ in [Sou]) has minimum 4, determinant $2^2 4^4$ and $2 \cdot 130$ vectors of norm 4. The dual lattice of $Q_{10}$ is similar to $Q_{10}$ and not strongly perfect. Therefore all strongly perfect lattices with kissing number $2 \cdot 120$ are similar to $(K'_{10})^*$, which is strongly perfect.

Let $\Gamma$ be such a lattice as in the theorem and $d := \det(\Gamma)$. Since $\min(\Gamma^*) = 1$, one has $d \leq \gamma_{10}^{10} < 3719$. Let $D := \Gamma^*/\Gamma$. Then $q : D \to \mathbb{Q}/\mathbb{Z}; x + \Gamma \mapsto \frac{1}{2}(x,x) + \mathbb{Z}$ is a non degenerate quadratic form on the finite abelian group $D$. Since $\Gamma^*$ is generated by its minimal vectors, the group $D$ is generated by vectors $x$ with $q(x) = \frac{1}{2} + \mathbb{Z}$. Therefore one has:

**Lemma 3.7.** *Let $p \neq 2$ be a prime and $D_p$ be the $p$-primary component of $D$. Then $D_p$ is not cyclic. If $|D_p| = p^2$ then $D_p$ is hyperbolic.*

*Proof.* Let $|D| = lp^\alpha$ with $p \not| l$. Then $D_p = lD$. Therefore $D_p$ is generated by the isotropic vectors $lx$ with $x \in X$. This implies that $D_p$ is not cyclic, and $D_p$ is hyperbolic, if $\alpha = 2$. □

We classify the lattices $\Gamma$ according to the 2 possibilities of Lemma 2.6:

**Lemma 3.8.** *If $\Gamma$ satisfies case (b) of Lemma 2.6 then $\Gamma \cong Q_{10}$.*

*Proof.* Let $d = \det(\Gamma)$ be the determinant of $\Gamma$.

•We first claim that $2^4$ divides $d$, and $2^6$ divides $d$ if the maximal even overlattice of $\Gamma$ has determinant divisible by 8. Let $D_2$ be the Sylow-2-subgroup of the quadratic module $(\Gamma^*/\Gamma, q)$. Then $D_2$ is generated by elements $x$ with $q(x) = 1/2 + \mathbb{Z}$. The condition (b) of Lemma 2.6 implies that $D_2$ contains an element $y$ with $q(y) \in \frac{1}{8}\mathbb{Z} - \frac{1}{2}\mathbb{Z}$. Write $y = \sum_{i=1}^t x_i$ as sum of vectors $x_i \in D_2$ with $q(x_i) = 1/2 + \mathbb{Z}$. Then

$$q(y) = \sum_{i=1}^{t} q(x_i) + \sum_{1\leq i<j\leq t} b_q(x_i, x_j).$$

Therefore, there are vectors $x, x' \in D_2$ with $b_q(x, x') = \pm 1/4 + \mathbb{Z}$. Hence $\langle x, x'\rangle \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ is a submodule of $D_2$ and therefore $2^4$ divides $d$.

If 8 divides the determinant of a maximal even overlattice of $\Gamma$, then $D_2$ contains an element $y$ with $q(y) = \epsilon/8 + \mathbb{Z}$, with $\epsilon = \pm 1, \pm 3$. One concludes that there is a pair of elements $x, x' \in D_2$, $q(x) = q(x') = 1/2 + \mathbb{Z}$ and $b_q(x, x') = 1/8 + \mathbb{Z}$. The order of $\langle x, x'\rangle$ is at least $2^6$.

Now $d/2^4 < 3719/2^4 < 233$ and hence all the primes $p$ that divide $d$ are $\leq 13$. Moreover if a prime $p > 3$ divides $d$ then $p^2$ is the largest $p$-power dividing $d$ and $D_p$ is hyperbolic. If $3^3$ divides $d$ then also $3^4$ divides $d$ and $d = 2^4 \cdot 3^4$.

•Let $L$ be a maximal even overlattice of $\Gamma$. Then $\det(L) = 2^2$ and $L$ is in the genus of $D_{10}$, $\det(L) = 4 \cdot 2$ and $L$ is in the genus of $E_7 \perp D_3$ or $\det(L) = 2^2 \cdot 3^2$ and $L$ is in the genus of $A_5^2$. Calculating the respective genera with [MAG] one sees that $\min(L^*) \geq 1$ implies that $L \cong D_{10}$.

•We claim that no prime $p > 2$ divides $d$. Otherwise $\Gamma^*$ contains a lattice $M^* = \langle D_{10}^*, v\rangle$ with $M^*/D_{10}^* = p$. In the coordinates with respect to an orthonormal basis of the sublattice $\mathbb{Z}^{10}$ of $D_{10}$ write $v = 1/p(a_1, \ldots, a_{10})$ with $a_i \in \mathbb{Z}$ and $|a_i| \leq (p-1)/2$. Let $n_i := |\{1 \leq j \leq 10 \mid |a_j| = i\}$ for $i = 0, \ldots, \frac{p-1}{2}$. Then

$$\sum_{i=1}^{10} a_i^2 = \sum_{i=1} (p-1)/2 i^2 n_i \geq p^2. \tag{1}$$

Multiplying by the integers $2, \ldots, p-1$ and reducing the coefficient $a_i/p$ modulo $\mathbb{Z}$, one finds in total $(p-1)/2$ inequalities (1), where the $n_i$ are permuted cyclicly. Summing them up, one finds

$$(\sum_{i=1}^{(p-1)/2} i^2)(n_1 + \ldots + n_{(p-1)/2}) \geq \frac{p-1}{2} p^2$$

If $p \geq 7$ this implies that $n_1 + \ldots + n_{(p-1)/2} \geq 11$ which is a contradiction. For $p = 3, 5$ one checks by hand, that there are no such lattices, using the fact that

$$||v - (\pm\frac{1}{2})^{10}||^2 = \sum_{i=0}^{10}(2a_i - p)^2 \geq 4 \cdot p^2.$$

• One finds 336 overlattices of $D_{10}^*$ of index 2 with minimum 1, which fall into 2 isometry classes, represented by, say, $L$ and $L'$. The overlattices of $L$ and $L'$ with minimum 1 and index 2 fall into 3 isometry classes. These three lattices have up to isometry a unique overlattice, say $M^*$, of index 2, minimum 1. The lattice $M$ is isometric to $Q_{10}$. As in the proof of Lemma 3.2 one sees that $M^*$ has no proper overlattice of minimum 1. □

**Lemma 3.9.** *If $\Gamma$ satisfies case (b) of Lemma 2.6 then $\Gamma \cong K'_{10}$.*

*Proof.* Let $\Gamma$ be such a lattice.

• We first show that $2^2$ divides $d$: Let $\Gamma' := \langle \Gamma, x\rangle$ for any $x \in X$. Then $\Gamma'$ is an integral lattice with $2 \mid [\Gamma' : \Gamma]$. Therefore $2^2 \mid [\Gamma^*/(\Gamma')^*][\Gamma' : \Gamma] \mid d$.

Since $d < 3719$ and $p^3$ divides $d$ for some prime $p \equiv 3 \pmod 4$, one has that either $7^3 \mid d$ or $3^3 \mid d$. In the first case $d = 7^3 \cdot 2^2$ or $d = 7^3 \cdot 2^3$. In the second case $d = 2^2 \cdot 3^3 d_0$ with $d_0 < 34$ and all odd primes divide $d_0$ to an even power. Therefore one finds that

• $d = \det(\Gamma)$ is one of the following:

$$2^2 7^3, 2^3 7^3, 2^2 3^5, 2^3 3^5, 2^a 3^3 \ (a = 2, \dots, 7), \text{ or } 2^2 3^3 5^2.$$

• The maximal even overlattices of $\Gamma$ are all isometric to $D_4 \perp E_6$: Let $L$ be a maximal even overlattice of $\Gamma$. Then $\det(L)$ is one of $7, 2^2 \cdot 7, 2^3 \cdot 7, 3, 2^2 \cdot 3$, or $2^3 \cdot 3$ and for each determinant there is only one genus of maximal even lattices. If $\det(L) = 2^3 \cdot 7$, then the Sylow 2-subgroup of $\Gamma^*/\Gamma$ is isometric to the one of $L^*/L$. But the latter does not contain an isotropic vector, which contradicts the fact that $2x$ is isotropic for all $x \in (\Gamma^*)_1$. Therefore $\det(L) \neq 2^3 \cdot 7$. For the other determinants one can list all the lattices in the genus (e.g. with [MAG]). The property that $\min(L^*) \geq 1$ excludes all possibilities except for $L \cong D_4 \perp E_6$.

• There is a unique orbit of overlattices of index 3 of $(D_4 \perp E_6)^*$ under the automorphism group that consists of lattices of minimum $\geq 1$. Let $M^*$ be such a lattice. All the overlattices of $M^*$ one index 2 or 5 contain vectors of length $< 1$. Therefore $\Gamma^*$ is an overlattice of $M^*$ of index 3, and there are only 48 such lattices of minimum $\geq 1$. All these lattices are isometric to $(K'_{10})^*$. □

### 3.3. The case $s = 135$.

In this section we prove the following

**Theorem 3.10.** *Let $\Lambda$ be a strongly perfect lattice of dimension 10 with $2 \cdot 135$ minimal vectors. Then $\Lambda$ is similar to $(K'_{10})^*$.*

Let $\Lambda$ be a strongly perfect lattice of minimum 1 and dimension 10, such that $|\Lambda_1| = 2 \cdot 135$. We assume again that $\Lambda$ is generated by its minimal vectors. Equation $(\star)$ of the proof of Lemma 3.4 shows that

$$\Gamma := \frac{\sqrt{3}}{\sqrt{2}}\Lambda^*$$

is an even lattice of minimum 6. Let $d := \det(\Gamma)$. Since $\min(\Gamma^*) = 2/3$, one has $d \leq (\frac{3}{2})^{10}\gamma_{10}^{10} < 214439$. Let $D := \Gamma^*/\Gamma$.

**Lemma 3.11.** $\Gamma_{(4)} \subseteq \Gamma \cap 2\Gamma^*$ *is a sublattice of* $\Gamma$ *of index 2 or 4.*

*Proof.* Since $(D13)$ and $(D11)$ hold for all $\alpha_2 \in \mathbb{R}^{10}$, one finds that for all $\alpha \in \mathbb{R}^{10}$

$$\sum_{x\in X}(x,\alpha)^3 x = \frac{3}{2}(\alpha,\alpha)\alpha \text{ and } \sum_{x\in X}(x,\alpha)x = 9\alpha.$$

In particular if $\alpha \in \Gamma$, then $(x,\alpha)^3 - (x,\alpha)$ is divisible by 6 and hence $(\frac{3}{2}(\alpha,\alpha) - 9)\alpha \in 6\Gamma^*$. If 4 divides $(\alpha,\alpha)$ then this shows that $\alpha \in 2\Gamma^*$, and hence $\Gamma_{(4)} \subseteq \Gamma \cap 2\Gamma^*$. In particular $(\alpha,\beta)$ is even for all $\alpha,\beta \in \Gamma_{(4)}$ and $\Gamma_{(4)}$ is a lattice.

Let $\alpha_1,\alpha_2 \in \Gamma - \Gamma_{(4)}$. Then $\alpha_1 - \alpha_2 \in \Gamma_{(4)}$ if and only if $(\alpha_1,\alpha_2)$ is even. Since $\min(\Gamma^*)^2\frac{s}{n(n+2)} = \frac{4\cdot 135}{120} = \frac{9}{2}$ the equality $(D1111)$ shows that for all $\alpha_1,\ldots,\alpha_4 \in \Gamma$

$$(\alpha_1,\alpha_2)(\alpha_3,\alpha_4) + (\alpha_1,\alpha_3)(\alpha_2,\alpha_4) + (\alpha_1,\alpha_4)(\alpha_2,\alpha_3) \in 2\mathbb{Z}.$$

Hence $|\Gamma/\Gamma_{(4)}| \leq 4$. Since $\Gamma \neq \Gamma_{(4)}$ and the index is a power of 2, one has $|\Gamma/\Gamma_{(4)}| = 2$ or 4. $\square$

**Corollary 3.12.** $2^8 \mid d = \det(\Gamma)$.

*Proof.* Let $\Gamma_{(4)}$ be as in Lemma 3.11 and $\alpha \in \Gamma - \Gamma_{(4)}$. Since $\Gamma_{(4)} \subseteq 2\Gamma^*$, all the scalar products in the lattice $\Gamma' := \langle\Gamma_{(4)},\alpha\rangle$ are divisible by 2. Therefore $2^{10}$ divides $\det(\Gamma')$. Since $[\Gamma : \Gamma'] \leq 2$, $2^8$ divides $d$. $\square$

Since the norms of the minimal vectors in $\Gamma^*$ are $2/3$, it is clear that 3 divides $d$. Moreover, if $2^9$ divides $d$, then also $2^{10}$ divides $d$. If we are in case (b) of Lemma 2.6, then the argument of Lemma 3.8 shows that also $2^{10}$ divides $d$. Hence we have the following possibilities for $d$:

**Lemma 3.13.** *If* $\Gamma$ *satisfies case (a) of Lemma 2.6, then* $\det(\Gamma)$ *is one of*

$$2^8\cdot3\cdot13^2,\ 2^8\cdot3\cdot11^2,\ 2^8\cdot3\cdot7^2,\ 2^{10}\cdot3\cdot7^2,\ 2^8\cdot3\cdot5^3,\ 2^8\cdot3\cdot5^2,\ 2^{10}\cdot3\cdot5^2,$$
$$2^{11}\cdot3\cdot5^2,\ 2^8\cdot3^3\cdot5^2,\ 2^8\cdot3^5,\ 2^{8+a}\cdot3^3,\ 2^{8+b}\cdot3,\ \textit{where } 0\leq a\leq 4 \textit{ and } 0\leq b\leq 8.$$

*If* $\Gamma$ *satisfies case (b) of Lemma 2.6, then* $\det(\Gamma)$ *is one of*

$$2^{10}\cdot3^4,\ 2^{11}\cdot3^4,\ 2^{10+a}\cdot3^2,\quad \textit{where } 0\leq a\leq 4.$$

If one could prove the existence of a norm 10 vector in $\Gamma$, then this would exclude most of the possible determinants. But unfortunately, we did not succeed in proving this directly.

**Remark 3.14.** If $\Gamma$ contains a vector of square length 10, then $3^4$ divides $\det(\Gamma)$.

*Proof.* Let $\alpha \in \Gamma$ with $(\alpha, \alpha) = 10$. Since $10\frac{2}{3} < 9$, the scalar products $(\alpha, x)$ with $x \in X$ are $0, \pm 1, \pm 2$. By Lemma 2.1 $|N_2(\alpha)| = 5$ and $\sum_{x \in N_2(\alpha)} x = \alpha$. Calculating the norm, one finds that

$$10 = (\alpha, \alpha) = (\sum_{x \in N_2(\alpha)} x, \sum_{x \in N_2(\alpha)} x) \leq 5 \cdot \frac{2}{3} + 5 \cdot 4 \cdot \frac{1}{3} = 10$$

since the vectors in $N_2(\alpha)$ are minimal vectors in a lattice. Therefore $(x, y) = \frac{1}{3} + \delta_{xy}\frac{1}{3}$ for all $x, y \in N_2(\alpha)$ and $N_2(\alpha)$ generates a lattice isomorphic to $\sqrt{3}^{-1} A_5$ of determinant $2 \cdot 3^{-4}$. In particular one has an epimorphism $\mathbb{Z}_3 \otimes_{\mathbb{Z}} \langle \Gamma, N_2(\alpha) \rangle \to (\mathbb{Z}_3 \otimes_{\mathbb{Z}} \sqrt{3}^{-1} A_5)/(\mathbb{Z}_3 \otimes_{\mathbb{Z}} A_5)^*$ which contains $\mathbb{Z}_3 \otimes \Gamma$ in its kernel. Therefore the order of the discriminant group $\Gamma^*/\Gamma$ is divisible by $3^4$. $\square$

**Lemma 3.15.** *There is an even overlattice* $\Gamma' := \langle 3x_1, 3x_2, \Gamma \rangle$ *isometric to an even orthogonally decomposable lattice* $\Gamma' \cong A_2 \perp L$ *with* $\Gamma \subseteq \Gamma' \subseteq \Gamma + 3\Gamma^*$.

*Proof.* Assume first that there are no vectors of norm 8 in $\Gamma$. Then the minimum of the lattice $\Gamma_{(4)}$ of Lemma 3.11 is 12. Since the index of $\Gamma_{(4)}$ in $\Gamma$ is at most 4, one finds $12/(4d)^{1/10} \leq \gamma_{10}$, hence $d \geq 12^{10}/(4\gamma_{10})^{10} > 10^6$. This contradicts the fact that $\det(\Gamma) \leq (\frac{3}{2})^{10}\gamma_{10}^{10} < 214439$. Therefore there is a vector $\alpha \in \Gamma$ with $(\alpha, \alpha) = 8$.

One calculates that $n_2(\alpha) = 2$ and $2(x_1 + x_2) = \alpha$ where $N_2(\alpha) = \{x_1, x_2\}$. Since $(x_i, x_i) = 2/3$ and the $x_i$ are minimal vectors in $\Gamma^*$, one gets $(x_i, x_j) = 1/3$. Let $\Gamma' := \langle 3x_1, 3x_2, \Gamma \rangle$. Then $\Gamma'$ is an even lattice and $L' := \langle 3x_1 - \alpha, 3x_2 - \alpha \rangle \leq \Gamma'$ is a sublattice of $\Gamma'$ isometric to $A_2$. Moreover $(x_i, 3x_j - \alpha) = \delta_{ij} - 1$. So $\langle x_1, x_2 \rangle \leq (\Gamma')^*$ generate the dual lattice of $L'$ and hence $\Gamma' \cong A_2 \perp L$ for some 8-dimensional even lattice $L$. $\square$

**Lemma 3.16.** *The maximal even overlattice of the form* $A_2 \perp L$ *of* $\Gamma$ *is* $A_2 \perp L_0$*, with* $L_0$ *isometric to one of*

$$E_8,\ D_4 \perp A_4,\ D_4 \perp A_2 \perp A_2,\ \textit{or}\ D_6 \perp A_2.$$

*Proof.* Using the fact that the rank of the Sylow-2-subgroup of the discriminant group of an even lattice is congruent to the dimension modulo 2, one finds with Lemma 3.13 that the possible determinants of $L_0$ are $1, 2^2, 5, 2^2 \cdot 5, 2 \cdot 4, 3^2, 2^2 \cdot 3^2, 2 \cdot 4 \cdot 3^2, 2^2 \cdot 3$, and $2 \cdot 4 \cdot 3$. If $\det(L_0) = 2^2$, then the discriminant group is isometric to the anisotropic group of Lemma 2.5 (1) and $\gamma(L_0) = -1 \neq (-1)^8$ contradicting Lemma 2.3. Similarly, there is no maximal even 8-dimensional lattice of determinant $3^2$. For all the other determinants there is a unique genus of maximal even lattices $L_0$. If $\det(L_0) = 1$, then clearly $L_0 \cong E_8$. There is a unique even 8-dimensional lattice of determinant 5, but its dual has minimum $2/5 < 2/3$, therefore $\det(L_0) \neq 5$. The lattices $L_0$ of determinant $2^2 \cdot 5$ ly in the genus of $D_4 \perp A_4$. This genus consists of 2 lattices, but only for $D_4 \perp A_4$, the dual lattice has minimum $\geq 2/3$. There are 2 lattices $L_0$ of determinant $4 \cdot 2$, namely $A_1 \oplus D_7$ and $\sqrt{2}A_1 \oplus E_7$, for both of which the dual lattice has minimum $< 2/3$. If $\det(L_0) = 2^2 \cdot 3^2$, then $L_0$ lies in the genus of $A_2 \perp A_2 \perp D_4$. This genus consists of 3 lattices, but only for $A_2 \perp A_2 \perp D_4$, the dual lattice has minimum $\geq 2/3$. There is a unique genus of maximal even lattices $L_0$ with $\det(L_0) = 2 \cdot 4 \cdot 3^2$, namely the one of $A_3 \perp A_2 \perp A_2 \perp A_1$. It consists of 8 lattices, of which all the dual lattices contain vectors of norm $< 2/3$. In the last 2 cases $\det(L_0) = 2^2 \cdot 3$ or $2 \cdot 4 \cdot 3$, $\Gamma$ satisfies the alternative (b) of Lemma 2.6. There are in total 5 lattices in the two genera, only one of which, $D_6 \perp A_2$, has a dual lattice of minimum $\geq 2/3$. $\square$

The most complicated case is $L_0 \cong E_8$. Since there is an even overlattice $A_2 \perp L$ of $\Gamma$ generated by 2 vectors and the 2-rank of $\Gamma^*/\Gamma$ is at least 8, the 2-rank of the discriminant group of $L$ is at least 4. By computer calculation one checks:

**Lemma 3.17.** *The sublattices $L$ of $E_8$ such that $\min(L^*) \geq 2/3$ and the 2-rank of $L^*/L$ is at least 4 are contained in $D_4 \perp D_4$.*

**Lemma 3.18.** *No prime $p \geq 7$ divides $\det(\Gamma)$.*

*Proof.* Assume that a prime $p \geq 7$ divides $\det(\Gamma)$. Then $\det(\Gamma)$ is one of $2^8 \cdot 3 \cdot 13^2$, $2^8 \cdot 3 \cdot 11^2$, $2^8 \cdot 3 \cdot 7^2$, or $2^{10} \cdot 3 \cdot 7^2$. In particular the maximal even overlattice of Lemma 3.16 of $\Gamma$ is $A_2 \perp E_8$. By Lemma 3.17 $\Gamma^*$ is an overlattice of $L^* := A_2^* \perp D_4^* \perp D_4^*$. We choose coordinates for this lattice, such that $v := (a_1, \ldots, a_{10})$ has norm $(v,v) := \frac{2}{3}(a_1^2 + a_1a_2 + a_2^2) + \sum_{i=3}^{10} a_i^2$. Note that the vectors with integral coordinates ly in $L^*$. Assume that $p = 7, 11$, or 13 divides $\det(\Gamma)$. Then $\Gamma^*$ contains a lattice $\langle L^*, \frac{1}{p}v \rangle$, where the coordinates of $v$ are $(a_1, \ldots, a_{10}) \in \mathbb{Z}^{10}$ with $-\frac{p-1}{2} \leq a_i \leq \frac{p-1}{2}$. Since the Sylow-$p$-subgroup of $\Gamma^*/\Gamma$ is a hyperbolic plane, we may also assume that $\frac{1}{p}v$ is isotropic, i.e. $p^2 \mid (v,v)$. Since $\min(\Gamma^*) = 2/3$ we have that $(v,v) \geq p^2$. Now $\frac{2}{3}(a_1^2 + a_1a_2 + a_2^2) \leq a_1^2 + a_2^2$, because the difference is $\frac{1}{3}(a_1^2 - 2a_1a_2 + a_2^2) = \frac{1}{3}(a_1 - a_2)^2 \geq 0$. Therefore $(v,v) \leq \sum_{i=1}^{10} a_i^2 =: q^+(v)$. Working with this bigger quadratic form $q^+$ we can argue as in the proof of Lemma 3.8: Let $n_i := |\{1 \leq j \leq 10 \mid |a_j| = i\}$ for $i = 0, \ldots \frac{p-1}{2}$. Then

$(\sum_{i=1}^{(p-1)/2} i^2)(n_1+\ldots+n_{(p-1)/2}) \geq \frac{p-1}{2}p^2$ implies that $n_1+\ldots+n_{(p-1)/2} \geq 11$ which is a contradiction. □

*Proof of Theorem 3.10.* It remains to consider the sublattices of $A_2 \perp L_0$ for the lattices $L_0$ in Lemma 3.16 that have one of the determinants not divisible by 7, 11, or 13 that are listed in Lemma 3.13.

- We first consider the case that $L_0 \cong D_6 \perp A_2$. Then we are in case (b) of Lemma 2.6 and hence $2^{10}$ divides the determinant of $\Gamma$. Since $2^4 \not| \det(L_0)$, $\Gamma$ is contained in a lattice $A_2 \perp L$ with $[L_0 : L] = 2$. One finds 3 isometry classes of such lattices $L$, such that $\min(L^*) \geq 2/3$. Calculating sublattices of index 2 and testing isometry, one finds 22 sublattice of index 4 of $A_2 \perp L_0$, 32 of index 8 and 8 of index 16, such that the minimum of the dual lattices is $\geq 2/3$. The latter 8 lattices have determinant $3^2 \cdot 2^{10}$ and minimum $\leq 4$. Therefore $\Gamma$ is of index 2 or 3 of one of those 8 lattices, but one finds no such sublattices such that the minimum of the dual lattice is $\geq 2/3$. Therefore $L_0 \not\cong A_2 \perp D_6$.

In all the other cases we are in case (a) of Lemma 2.6.

- Now assume that $L_0 \cong D_4 \perp A_2 \perp A_2$. One finds 2 sublattices $L$ of index 2 of $L_0$ where the dual lattice has minimum $\geq 2/3$. $A_2 \perp L$ has 13 sublattices $L'$ of index 2 and 4 sublattices $L'$ of index 4 with $\min((L')^*) \geq 2/3$. The latter 4 lattices have determinant $2^8 3^3$ but the Sylow-2-subgroup of the discriminant group has only rank 6. So $\Gamma$ is contained in one of these lattices of index divisible by 2. One finds no sublattices $M$ of these 4 lattices of index 2 with $\min(M^*) \geq 2/3$. So also this case is impossible.
- If $L_0 \cong D_4 \perp A_4$ then $\det(\Gamma) = 2^8 \cdot 3 \cdot 5^3$ and $\Gamma$ is a sublattice $M$ of index 5 of one of the 9 sublattices $L$ of $A_2 \perp D_4 \perp A_4$ with $\min(L^*) = 2/3$ and $L^*/L \cong \mathbb{F}_2^8 \times \mathbb{F}_3 \times \mathbb{F}_5$. With the computer one finds no such sublattices $M$ such that $M^*/M$ has an elementary abelian Sylow 5-subgroup and $\min(M^*) = 2/3$. Therefore $5^3$ does not divide $\det(\Gamma)$.
- The hardest case is that $L_0 \cong E_8$. By Lemma 3.17, $\Gamma$ is contained in $A_2 \perp D_4 \perp D_4$ of index divisible by 4. One finds 36 isometry classes of sublattices $L$ of index 4 in $A_2 \perp D_4 \perp D_4$ such that $\min(L^*) \geq 2/3$, 5 of which satisfy $L^*/L \cong \mathbb{F}_2^8$. If $2^8$ is the highest 2-power that divides $\det(\Gamma)$, i.e. $\det(\Gamma)$ is one of $2^8 \cdot 3 \cdot 5^2$, $2^8 \cdot 3^3 \cdot 5^2$, $2^8 \cdot 3^5$, $2^8 \cdot 3^3$, $2^8 \cdot 3$ then $\Gamma$ is contained in one of these five lattices $M$ of index $5, 3 \cdot 5, 3^2, 3$, or 1. Since all the lattices $M$ have vectors of length 4, the latter case is impossible. One finds no sublattices $N$ of index 5 of the lattices $M$ such that $N^*$ has minimum $\geq 2/3$. So the first two cases are also impossible. The lattices $M$ have 2 sublattices $N$ of index 3, such that the minimum of the dual lattice is $\geq 2/3$. These two lattices still contain vectors of length 4. Therefore $\Gamma$ is a sublattice of index 3 of one of these two lattices. Up to isometry there is a unique such lattice $\Gamma$ such that $\min(\Gamma^*) \geq 2/3$. This lattice $\Gamma$ is similar to $(K'_{10})^*$.

If $2^{10}$ divides $\det(\Gamma)$, then $\Gamma$ is contained in one of the (up to isometry) 60 sublattices $M$ of index 2 in the 36 lattices $L$ above that have $\min(M^*) \geq 2/3$. Only for 8 of the 60 lattices $M$ the Sylow 2-subgroup of the discriminant group is of rank $\geq 8$.

If $2^{10}$ is an exact divisor of $\det(\Gamma)$, i.e. $\det(\Gamma)$ is one of $2^{10} \cdot 3 \cdot 5^2$, $2^{10} \cdot 3^3$, $2^{10} \cdot 3$, then $\Gamma$ is a sublattice of one of these 8 lattices $N$ of index $5, 3$, or 1. Since all the lattices $N$ contain vectors of length 4, the last case is impossible. One finds no sublattices $O$ of index 3 or 5 of $N$ such that $\min(O^*) \geq 2/3$. Therefore this case is impossible.

The 60 lattices $M$ above contain up to isometry 14 sublattices $O$ of index 2 such that $\min(O^*) \geq 2/3$. All these lattices have vectors of length 4, and one finds no sublattices of indes 2 or 3 of these lattices such that the minimum of the dual lattice is $\geq 2/3$. So we finally proved Theorem 3.10. $\square$

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