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Congruences modulo ℓ between ε factors for cuspidal representations of $GL(2)$

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Pour Jacques Martinet

Résumé. Titre français : Congruences modulo ℓ entre facteurs ϵ des représentations cuspidales de $GL(2)$
Soient $\ell \neq p$ deux nombres premiers distincts, F un corps local non archimedien de caractéristique résiduelle p, $\overline{\mathbf{Q}}_\ell$ une clôture algébrique du corps des nombres ℓ-adiques, et $\overline{\mathbf{F}}_\ell$ le corps résiduel de $\overline{\mathbf{Q}}_\ell$. On conjecture que la correspondance locale de Langlands pour $GL(n, F)$ sur $\overline{\mathbf{Q}}_\ell$ respecte les congruences modulo ℓ entre les facteurs L et ϵ de paires, et que la correspondance locale de Langlands sur $\overline{\mathbf{F}}_\ell$ est caractérisée par des identités entre de nouveaux facteurs L et ϵ. Nous allons le démontrer lorsque $n = 2$.

Abstract. Let $\ell \neq p$ be two different prime numbers, let F be a local non archimedean field of residual characteristic p, and let $\overline{\mathbf{Q}}_\ell, \overline{\mathbf{Z}}_\ell, \overline{\mathbf{F}}_\ell$ be an algebraic closure of the field of ℓ-adic numbers $\mathbf{Q}_\ell$, the ring of integers of $\overline{\mathbf{Q}}_\ell$, the residual field of $\overline{\mathbf{Z}}_\ell$. We proved the existence and the unicity of a Langlands local correspondence over $\overline{\mathbf{F}}_\ell$ for all $n \geq 2$, compatible with the reduction modulo ℓ in [V5], without using L and ε factors of pairs.

We conjecture that the Langlands local correspondence over $\overline{\mathbf{Q}}_\ell$ respects congruences modulo ℓ between L and ε factors of pairs, and that the Langlands local correspondence over $\overline{\mathbf{F}}_\ell$ is characterized by identities between new L and ε factors. The aim of this short paper is prove this when $n = 2$.

Introduction

The Langlands local correspondence is the unique bijection between all irreductible $\overline{\mathbf{Q}}_\ell$-representations of $GL(n, F)$ and certain ℓ-adic representations of an absolute Weil group W_F of dimension n, for all integers $n \geq 1$,

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which is induced by the reciprocity law of local class field theory

$$W_F^{ab} \simeq F^*$$

when $n = 1$ (W_F^{ab} is the biggest abelian Hausdorff quotient of W_F), and which respects L and ε factors of pairs [LRS], [HT], [H2].

Let $\psi : F \to \overline{\mathbf{Z}}_\ell^*$ be a non trivial character. We denote by $\mathrm{Cusp}_R\, GL(n,F)$ the set of isomorphism classes of irreducible cuspidal R-representations of $GL(n,F)$. When $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(n,F)$, Henniart [H1] showed that π is characterized by the epsilon factors of pairs $\varepsilon(\pi,\sigma)$ for all $\sigma \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(m,F)$ and for all $m \leq n-1$ (note that $L(\pi,\sigma) = 1$), using the theory of Jacquet, Piatestski-Shapiro, and Shalika [JPS1].

Does this remain true for cuspidal irreductible $\overline{\mathbf{F}}_\ell$-representations of $GL(n,F)$? We need first to define the epsilon factors of pairs.

Let $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(n,F)$. It is known that the constants of the epsilon factors of pairs $\varepsilon(\pi,\sigma)$ belong to $\overline{\mathbf{Z}}_\ell$ for all $\sigma \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(m,F)$ and for all $m \leq n-1$, and that the conductor does not change by reduction modulo ℓ (this is proved by Deligne [D] for the irreducible representations of the Weil group, and by the local Langlands correspondence over $\overline{\mathbf{Q}}_\ell$ is true for cuspidal representations).

Now let $\pi \in \mathrm{Cusp}_{\overline{\mathbf{F}}_\ell}\, GL(n,F)$. Then π lifts to $\mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(n,F)$ [V1, III.5.10]. By reduction modulo ℓ, one can define epsilon factors of pairs $\varepsilon(\pi,\sigma)$ for all $\sigma \in \mathrm{Cusp}_{\overline{\mathbf{F}}_\ell}\, GL(m,F)$ and for all $m \leq n-1$. Let q be the order of the residual field of F. We expect that π is characterized by the epsilon factors $\varepsilon(\pi,\sigma)$ for all σ, when the multiplicative order of q modulo ℓ is $> n-1$; otherwise, π should be characterized by less naive but natural epsilon factors. The same should be true when π is replaced by an $\overline{\mathbf{F}}_\ell$-irreducible representation of the Weil group W_F.

The existence [V4] of an integral Kirillov model for $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(n,F)$ seems to be an adequate tool to solve the problem. The description of the representation π on the Kirillov model is given by the central character ω_π and by the action of the symmetric group S_n (the Weyl group of $GL(n,F)$). The action of S_n is related with the $\varepsilon(\pi,\sigma)$ for all σ as above [GK, see the end of paragraph 7]. When $n = 2$ Jacquet and Langlands [JL] described the action of S_2 on the Kirillov model in terms of $\varepsilon(\pi,\chi) = \varepsilon(\pi \otimes \chi)$ for all $\overline{\mathbf{Q}}_\ell$-characters χ of F^*, using the Fourier transform on F^*.

In the case $n = 2$ and only in this case, we will prove that two integral $\pi, \pi' \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, GL(2,F)$ have the same reduction modulo ℓ if and only if their central characters have the same reduction modulo ℓ and the factors $\varepsilon(\pi \otimes \chi)$, $\varepsilon(\pi' \otimes \chi)$ have the same reduction modulo ℓ for integral $\overline{\mathbf{Q}}_\ell$-characters χ of F^* when ℓ does not divide $q-1$. When ℓ divides $q-1$ this remains true with new epsilon factors taking into account the natural congruences modulo ℓ satisfied by the $\varepsilon(\pi \otimes \chi)$ for all χ. By reduction modulo ℓ, we get that the local Langlands $\overline{\mathbf{F}}_\ell$-correspondence for $n = 2$ is characterized by the equality on L and new ε factors of pairs. The field $\overline{\mathbf{F}}_\ell$ can be replaced by any algebraically closed field R of characteristic ℓ.

The case $n = 3$ could be treated probably, but the general case $n \geq 4$ remains an open and interesting question.

1. Integral Kirillov model

The definition of the L and ϵ factors of pairs [JPS1] uses the Whittaker model, or what is equivalent the Kirillov model. We showed [V4] that these models are compatible with the reduction modulo ℓ.

We denote by O_F the ring of integers of F. Let R be an algebraically closed field of characteristic $\neq p$, and let $\psi : F \to R^*$ be a character such that O_F is the biggest ideal on which ψ is trivial. We extend ψ to a R-character of the group N of strictly upper triangular matrices of $G = GL(n, F)$ by $\psi(n) = \psi(\sum n_{i,i+1})$ for $n = (n_{i,j}) \in N$. The mirabolic subgroup P of G is the semi-direct product of the group $GL(n-1, F)$ embedded in $GL(n, F)$ by

$$g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

and of the group F^{n-1} embedded in $GL(n, F)$ by

$$x \to \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

The representation $\tau_R := \operatorname{ind}_{P,N} \psi$ of the mirabolic subgroup P (compact induction) is called mirabolic. It is irreducible (this is a corollary of [V4 prop.1]), but it is not admissible when $n \geq 2$.

Lemma. $\operatorname{End}_{RP} \tau_R \simeq R$.

Proof. This is a general fact: the representation τ_R is absolutely irreducible [V1, I.6.10], hence $\operatorname{End}_{RP} \tau_R \simeq R$. From the Schur's lemma [V1, I.6.9] $\operatorname{End}_{RP} \tau_R \simeq R$ when the cardinal of R is strictly bigger than $\dim_R \tau_R$ (countable dimension). There exists an algebraically closed field R' which contains R and of uncountable cardinal. Two RP-endomorphisms of τ_R which are proportional over R' are proportional over R. $\square$

Theorem. *An irreducible R-representation π of G is cuspidal if and only if extends the mirabolic representation τ_R.*

Proof. This results from [BZ] and [V1]. Suppose that π is cuspidal. Then $\pi|_P$ is the mirabolic representation: when $R = \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ see [BZ, 5.13 & 5.20], when $R = \overline{\mathbf{F}}_\ell$, π lifts to $\overline{\mathbf{Q}}_\ell$ [V1, III.5.10] where it is true then reduce. Conversely, suppose $\pi|_P = \tau_R$ and $R = \overline{\mathbf{Q}}_\ell$ or $\overline{\mathbf{F}}_\ell$. Then π is cuspidal [V1, III.1.8]. The case of a general R is deduced from this two cases by the next lemma. $\square$

Let G be the group of rational points of a reductive connected group over F. We denote by $\mathrm{Irr}_R\, G$ the set of isomorphism classes of irreducible R-representations of G.

Lemma. (1) *A non zero homomorphism of algebraically closed fields $f : R \to R'$ gives a natural injective map $\pi \to f_*(\pi) : \mathrm{Irr}_R\, G \to \mathrm{Irr}_{R'}\, G$ which respects cuspidality.*

(2) *Let $\pi' \in \mathrm{Cusp}_{R'}\, G$. Then there exists an unramified character χ of G such that $\pi' \otimes \chi = f_*(\pi)$ with $\pi \in \mathrm{Cusp}_R\, G$.*

Proof. This results from [V1].

(1) f_* respects irreducibility [V1, II.4.5], and commutes with the parabolic restriction. Hence it respects cuspidality. The linear independence of characters [V1, I.6.13] shows that if $\pi, \pi' \in \mathrm{Irr}_R\, G$ are not isomorphic then $f_*\pi, f_*\pi'$ are not isomorphic.

(2) Let Z be the center of G. The group of rational characters $X(Z)$ is a subgroup of finite index in the group $X(G)$. This implies that there exists an unramified character χ of G such that the quotient Z/Z_o of Z by the kernel Z_o of the central character ω of $\pi' \otimes \chi$ is profinite. Hence the values of ω are roots of unity. We deduce that $\pi' \otimes \chi$ has a model on R [V1, II.4.9]. $\square$

Let $\pi \in \mathrm{Cusp}_R\, GL(n, F)$ of central character ω. The realisation of π on the mirabolic representation τ_R is called the Kirillov model $K(\pi)$ of π. It is sometimes useful to use the Whittaker model instead of the Kirillov model. By adjonction and the theorem $\mathrm{Hom}_{RG}(\pi, \mathrm{Ind}_{G,N}\, \psi) \simeq R$ (the unicity of the Whittaker model); the Whittaker model $W(\pi)$ is the unique realisation of π in $\mathrm{Ind}_{G,N}\, \psi$. By definition

$$W(g) = (\pi(g)W)(1)$$

for all $g \in G$ and for all Whittaker functions $W \in W(\pi)$. We denote by $\Gamma(j)$ the subgroup of matrices $k \in GL(n, O_F)$ of the form

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \in GL(n-1, O_F), d \in O_F^*, c \in p_F^j O_F$$

for any integer $j > 0$. The smallest $j > 0$ such that π contains a non-zero vector transforming under $\Gamma(j)$ according to the one dimensional character

$$\omega_j(k) = \omega(d)$$

for $k \in \Gamma(j)$ as above, is called the conductor of π and denoted f.

Theorem. *Let $\pi \in \mathrm{Cusp}_R\, GL(n,F)$ of central character $\omega = \omega$ and conductor f.*

(1) *The restriction from G to P induces a G-equivariant isomorphism*

$$W \to W|_P : W(\pi) \simeq K(\pi)$$

from the Whittaker model to the Kirillov model.

(2) *Let $\pi' \in \mathrm{Cusp}_R\, GL(n,F)$. There is a natural isomorphism $W \to W'$: $W(\pi) \to W(\pi')$ of R-vector spaces defined by the condition $W|_P = W'|_P$.*

(3) *There is unique function $W_\pi \in W(\pi)$ such that*

$$W_\pi|_{GL(n-1,F)} = 1_{GL(n-1,O_F)}.$$

The function W_π is called the new vector of π and generates the space of vectors of π transforming under $\Gamma(f)$ according to ω_f.

(4) *$W(\pi)$ is contained in the compactly induced representation $\mathrm{ind}_{G,NZ}\, \psi \otimes \omega_\pi$.*

Proof. (1) There exists $W \in W(\pi)$ with $W(1) \neq 0$, and $f : W \to W_P$ is a non zero P-equivariant map from π to $\mathrm{Ind}_N^P\, \psi$. The map f is injective of image $\mathrm{ind}_N^P\, \psi$, because $\mathrm{End}_R\, \tau_R \simeq R$. We get also (2).

(3) The space of τ_R is isomorphic by restriction to $G' = GL(n-1,F)$, to the space of $\mathrm{ind}_{N',G'}\, \psi$ where $N' = N \cap G'$. As ψ is trivial on O_F, the characteristic function of $GL(n-1,O_F)$ belongs to $\mathrm{ind}_{N'}^{G'}\, \psi$. For the conductor [JPS2].

(4) Let $W \in W(\pi)$. The function $x \to W(xg)$ on the parabolic standard subgroup PZ is locally constant of compact support modulo NZ for all $g \in G$. As $G = PZGL(n,O_F)$, the function W is of compact support modulo NZ. $\square$

Let $\pi \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell}\, G$. Let $E/\mathbf{Q}_\ell$ be an extension contained in a finite extension of the maximal unramified extension of $\mathbf{Q}_\ell$. Example: the extension E/Q_ℓ generated by the values of ψ. The ring of integers O_E is principal. An O_E-free module L with an action of G such that L is a finite type $O_E G$-module and such that $\overline{\mathbf{Q}}_\ell \otimes_{O_E} L \simeq \pi$ is called an O_E-integral structure of π. If such an L is exists, π is called integral, the representation $r_\ell L = L \otimes_{O_E} \overline{\mathbf{F}}_\ell$ is of finite length. One calls $\overline{\mathbf{Z}}_\ell \otimes_{O_E} L$ an integral structure of π. When L, L' are two integral structures of π, then the semi-simplifications of $r_\ell L, r_\ell L'$ are isomorphic (see [V1, II.5.11.b] when E/Q_ℓ is finite, and [Vig4, proof of theorem 2, page 416] in general). When $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell}\, G$ is integral, $r_\ell L = L \otimes_{O_E} \overline{\mathbf{F}}_\ell$ is irreducible; the isomorphism class $r_\ell \pi$ of $r_\ell L$ is called the reduction of π; any irreducible cuspidal $\overline{\mathbf{F}}_\ell$-representation of G is the reduction of an integral irreducible cuspidal $\overline{\mathbf{Q}}_\ell$-representation of G. For all these facts see [V1, III.5.10].

A function with values in $\overline{\mathbf{Q}}_\ell$ is called integral, when its values belong to $\overline{\mathbf{Z}}_\ell$. We denote by $K(\pi, \overline{\mathbf{Z}}_\ell)$, resp. $W(\pi, \overline{\mathbf{Z}}_\ell)$, the set of integral functions in the Kirillov model, resp. Whittaker model, of $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$. Let Λ be the maximal ideal of $\overline{\mathbf{Z}}_\ell$. The reduction modulo ℓ of an integral function f is the fonction $r_\ell f$ with values in $\overline{\mathbf{Z}}_\ell/\Lambda \simeq \overline{\mathbf{F}}_\ell$ deduced from f.

Theorem. (A) *Let $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ with central character ω_π. Then the following properties are equivalent:*

(A.1) *ω_π is integral.*

(A.2) *π is integral.*

(A.3) *$K(\pi, \overline{\mathbf{Z}}_\ell)$ is a $\overline{\mathbf{Z}}_\ell$-structure of π, called the integral Kirillov model.*

(A.4) *$W(\pi, \overline{\mathbf{Z}}_\ell)$ is a $\overline{\mathbf{Z}}_\ell$-structure of π, called the integral Whittaker model.*

(B) *When π is integral, we have*

(B.1) *The restriction to P from $W(\pi, \overline{\mathbf{Z}}_\ell)$ to $K(\pi, \overline{\mathbf{Z}}_\ell)$ is an isomorphism.*

(B.2) *The integral Kirillov model is $\overline{\mathbf{Z}}_\ell P$- generated by any function f with $f(1) = 1$. The integral Whittaker model $W(\pi, \overline{\mathbf{Z}}_\ell)$ is $\overline{\mathbf{Z}}_\ell G$ generated by the new vector.*

(B.3) $\overline{\mathbf{F}}_\ell \otimes_{\overline{\mathbf{Z}}_\ell} K(\pi, \overline{\mathbf{Z}}_\ell) = K(r_\ell\pi, \overline{\mathbf{F}}_\ell)$ *is the Kirillov model, and* $\overline{\mathbf{F}}_\ell \otimes_{\overline{\mathbf{Z}}_\ell} W(\pi, \overline{\mathbf{Z}}_\ell) = W(r_\ell\pi, \overline{\mathbf{F}}_\ell)$ *is the Whittaker model of $r_\ell\pi$.*

Proof. The equivalence of (A1) (A2) [V1, II.4.12]; for the rest [V4 th.2] and the last theorem. □

Corollary. *Let $\pi, \pi' \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral, with central character $\omega_\pi, \omega_{\pi'}$. Then $r_\ell\pi = r_\ell\pi'$ if and only if*

$$(*) \qquad r_\ell\omega_\pi = r_\ell\omega_{\pi'}, \quad r_\ell\pi(w)(f) = r_\ell\pi'(w)(f)$$

for all $w \in S_n$, and for all f in the integral Kirillov model.

Proof. Use (B.3) and $\mathrm{End}_{\overline{\mathbf{F}}_\ell} \tau_{\overline{\mathbf{F}}_\ell} \simeq \overline{\mathbf{F}}_\ell$. □

Questions. Can one define an integral Kirillov or Whittaker model for $\pi \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell} G$ integral and not cuspidal ? What is the action of S_n in the Kirillov model ?

2. The case $n = 2$

We can go further in the case $n = 2$. Let $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ where $G = GL(2, F)$. The restriction of $GL(2, F)$ to $GL(1, F) = F^*$ gives an isomorphism from $K(\pi)$ to the space $C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$ of locally constant functions $F^* \to \overline{\mathbf{Q}}_\ell$ with compact support, which respects the natural $\overline{\mathbf{Z}}_\ell$-structures $K(\pi, \overline{\mathbf{Z}}_\ell) \simeq C_c^\infty(F^*, \overline{\mathbf{Z}}_\ell)$. The unique non trivial element of S_2 is represented by

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The action of $\pi(w)$ on the Kirillov model was described by Jacquet and Langlands [JL, Prop. 2.10 p. 46], using Fourier transform for complex representations.

We choose a $\mathbf{Q}_\ell$-Haar measure dx on F^*. The Fourier transform of $f \in C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$ with respect to dx is

$$\hat{f}(\chi) := \int_{F^*} f(x)\chi(x)dx$$

for any character $\chi : F^* \to \overline{\mathbf{Q}}_\ell^*$.

We choose a uniformizing parameter p_F of F. A function $f \in C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$ is determined by the set of functions $f_n \in C_c^\infty(O_F^*, \overline{\mathbf{Q}}_\ell)$ defined by $f_n(x) := f(p_F^{-n}x)$ for all $n \in \mathbf{Z}$. The functions f_n depend on the choice of p_F. Extension by zero allows to consider $C_c^\infty(O_F^*, \overline{\mathbf{Q}}_\ell)$ as a subspace of $C_c^\infty(F^*, \overline{\mathbf{Q}}_\ell)$, because O_F^* is open in F^*. We have

$$\hat{f}(\chi) = \sum_n \hat{f}_n(\chi)\chi(p_F^{-n}).$$

For a given character χ, the sum is finite. The functions $\hat{f}_n(\chi)$ depend only on the restriction of χ to O_F^*. Set $\hat{O}_F^* := \mathrm{Hom}(O_F^*, \overline{\mathbf{Q}}_\ell)$. One introduces the formal series

$$f(x, X) := \sum_{n\in\mathbf{Z}} f_n(x)X^n, \quad \hat{f}(\chi, X) := \sum_{n\in\mathbf{Z}} \hat{f}_n(\chi)X^n$$

for all $x \in O_F^*$ and for all $\chi \in \hat{O}_F^*$.

Jacquet and Langlands [JL Prop. 2.10 page 46] proved that the action of $\pi(w)$ on the Kirillov model is given by:

$$(\pi(w)f)_n\hat{}\,(\chi) \;=\; c(\pi \otimes \chi^{-1})\; \hat{f}_m(\chi^{-1}\omega_\pi^{-1})$$

for all $\chi \in \hat{O}_F^*$, all integers $n \in \mathbf{Z}$, where $m = -n - f(\pi \otimes \chi^{-1})$, for some constant $c(?) \in \overline{\mathbf{Q}}_\ell^*$ and some integer $f(?) \in \mathbf{Z}$. The formula and $c(\pi \otimes \chi^{-1})$ are independent of the choice of dx. The formula is equivalent to

$$(\pi(w)f)\hat{}\,(\chi, X) \;=\; \varepsilon(\pi \otimes \chi^{-1})\hat{f}(\chi^{-1}\omega_\pi^{-1}, X^{-1})$$

for all $\overline{\mathbf{Q}}_\ell$-characters χ of O_F^*, where the epsilon factor is

$$\varepsilon(\pi \otimes \chi^{-1}) = c(\pi \otimes \chi^{-1})X^{f(\pi\otimes\chi^{-1})}.$$

On calls $c(\pi)$ the constant and $f(\pi)$ the conductor of the epsilon factor $\varepsilon(\pi)$. They both depend on the choice of the non trivial character $\psi : F \to \overline{\mathbf{Z}}_\ell^*$ which was fixed, but not on the choice of dx or on p_F. Jacquet and Langlands used complex representations but their method is valid when the field of complex numbers is replaced by $\overline{\mathbf{Q}}_\ell$, because one uses only integrals of locally constant functions on compact sets. There is no problem of vanishing because we work on $\overline{\mathbf{Q}}_\ell$.

We suppose that dx is a $\mathbf{Z}_\ell$-Haar measure on F^* wich is not divisible by ℓ. Let

$$\mathcal{L} = \text{the Fourier transform of } \ C_c^\infty(O_F^*, \overline{\mathbf{Z}}_\ell).$$

We have $\mathcal{L} \subset C_c^\infty(\hat{O}_F^*, \overline{\mathbf{Z}}_\ell)$ and $\mathcal{L} = C_c^\infty(\hat{O}_F^*, \overline{\mathbf{Z}}_\ell)$ if and only if $q \not\equiv 1 \bmod \ell$ [V2]. In general, we separate the ℓ-regular part X of O_F^* from the ℓ-part Y of O_F^* which is a cyclic group of order $m = \ell^a$. The volume of X for dx should be a unit in $\mathbf{Z}_\ell^*$; we can suppose it is equal to 1. The group of $\overline{\mathbf{Q}}_\ell$-characters satisfy $\hat{O}_F^* \simeq \hat{X} \times \hat{Y}$. A general character in $\hat{O}_F^*$ is now written as $\chi\mu$ where $\chi \in \hat{X}$ and $\mu \in \hat{Y}$, and a function $v : \hat{O}_F^* \to \overline{\mathbf{Q}}_\ell$ is thought as a function $v : \hat{X} \to C(\hat{Y}, \overline{\mathbf{Q}}_\ell)$ with $v(\chi)(\mu) := v(\chi\mu)$.

The $\overline{\mathbf{Z}}_\ell$-module $\mathcal{L}$ consists of all functions $v : \hat{X} \to L$ with compact support, where

$$L \subset C_c^\infty(\hat{Y}, \overline{\mathbf{Z}}_\ell)$$

is the free $\overline{\mathbf{Z}}_\ell$-module with basis the characters $\underline{y} : \mu \to \mu(y^{-1})$ of $\hat{Y}$ for all $y \in Y$.

We need some elementary linear algebra. The $\overline{\mathbf{Z}}_\ell$-module L is the set of functions $v \in C_c^\infty(\hat{Y}, \overline{\mathbf{Q}}_\ell)$ such that

$$y \mapsto < v, y > := |Y|^{-1} \sum_{\mu \in \hat{Y}} v(\mu)\mu(y)$$

belongs to $C(Y, \overline{\mathbf{Z}}_\ell)$. The orthogonality formula of characters gives

$$v = \sum_{y \in Y} < v, y > \underline{y}$$

for all $v \in C(\hat{Y}, \overline{\mathbf{Q}}_\ell)$. For the usual product, $C_c^\infty(\hat{Y}, \overline{\mathbf{Q}}_\ell)$ is an algebra.

Lemma. *Let $v \in C_c^\infty(\hat{Y}, \overline{\mathbf{Q}}_\ell)$.*
(i) *The inclusion $vL \subset L$ is equivalent to $v \in L$.*
(ii) *The equality $vL = L$ is equivalent to $v \in L$ and $v(\mu) \in \overline{\mathbf{Z}}_\ell^*$ for all $\mu \in \hat{Y}$.*
(iii) *The inclusion $vL \subset \Lambda L$ is equivalent to $< v, y > \in \Lambda$ for all $y \in Y$*
(Λ is the maximal ideal of $\overline{\mathbf{Z}}_\ell$).

Proof. (i) The inclusion $vL \subset L$ is equivalent to $< v\underline{z}, z' > = < v, z^{-1}z' > \in \overline{\mathbf{Z}}_\ell$ for all $z, z' \in Y$, which is equivalent to $v \in L$.

(ii) $vL = L$ means that $v\underline{z}$ for $z \in Y$ is a basis of L. We have $v\underline{z} = \sum_{z' \in Y} < v, z^{-1}z' > \underline{z'}$, hence $vL = L$ means that

$$(< v, z^{-1}z' >)_{z,z'} \in SL(m, \overline{\mathbf{Z}}_\ell).$$

The Dedekind determinant $\det(< v, z^{-1}z' >)_{z,z'}$ is equal to $\prod_{\mu \in \hat{Y}} v(\mu)$ (see [L] exercise 28 page 495).

(iii) see the proof of (i). $\square$

Let $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral. As $\pi(w)$ is an isomorphism of the integral Kirillov model, the function

$$c(\pi \otimes \chi) : \ \mu \in \hat{Y} \to c(\pi \otimes \chi\mu) \in \overline{\mathbf{Q}}_\ell$$

satisfies $c(\pi \otimes \chi)L = L$ for all character $\chi \in \hat{X}$. We apply the lemma to $c(\pi \otimes \chi)$. We define **new epsilon factors**

$$\varepsilon(\pi, y) :=< c(\pi), y > X^{f(\pi)}, \qquad < c(\pi), y >= |Y|^{-1} \sum_{\mu \in \hat{Y}} c(\pi \otimes \mu)\mu(y),$$

for all $y \in Y$. As have $f(\pi) \geq 2$ for $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$, we have $f(\pi) = f(\pi \otimes \mu) \geq 2$ for all $\mu \in \hat{Y}$. When Y is trivial (i.e. $q \not\equiv 1 \bmod \ell$), they are simply the usual ones.

Theorem. (1) *Let $\pi \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral. Then the constant of the epsilon factor is a unit $c(\pi) \in \overline{\mathbf{Z}}_\ell^*$ and the new constants $< c(\pi), y > \in \overline{\mathbf{Z}}_\ell$ are integral, for all $y \in Y$.*
(2) *Let $\pi, \pi' \in \mathrm{Cusp}_{\overline{\mathbf{Q}}_\ell} G$ integral with central characters $\omega_\pi, \omega_{\pi'}$. Then $r_\ell \pi = r_\ell \pi'$ if and only if $r_\ell \omega_\pi = r_\ell \omega_{\pi'}$ and their new epsilon factors have the same reduction modulo ℓ: the conductors $f(\pi \otimes \chi) = f(\pi' \otimes \chi)$ are equal, and the new constants have the same reduction modulo ℓ :*

$$r_\ell < c(\pi \otimes \chi), y >= r_\ell < c(\pi' \otimes \chi), y >$$

for all $y \in Y$, and all $\overline{\mathbf{Q}}_\ell$-characters $\chi \in \hat{X}$.

Proof. With the last corollary of the paragraph (1), $r_\ell \pi = r_\ell \pi'$ if and only if $r_\ell \omega_\pi = r_\ell \omega_{\pi'}$ and

$$(*) \qquad c(\pi \otimes \chi) \hat{f}_m(\chi^{-1}\omega_\pi^{-1}) = c(\pi' \otimes \chi) \hat{f}_{m'}(\chi^{-1}\omega_{\pi'}^{-1}) \quad \mathrm{modulo}\, \Lambda\mathcal{L}$$

for all $f_n \in C_c^\infty(O_F^*, \overline{\mathbf{Z}}_\ell)$ and all $n \in \mathbf{Z}$. With the lemma, we deduce the theorem. □

We apply now the theorem to representations over $\overline{\mathbf{F}}_\ell$. Any $\pi \in \mathrm{Cusp}_{\overline{\mathbf{F}}_\ell} G$ lifts to $\overline{\mathbf{Q}}_\ell$ and we can define epsilon factors

$$\varepsilon(\pi \otimes \chi, y) :=< c(\pi \otimes \chi), y > X^{f(\pi \otimes \chi)}$$

for all $y \in Y$ and all $\chi \in \mathrm{Hom}(O_F^*, \overline{\mathbf{F}}_\ell^*) = \mathrm{Hom}(X, \overline{\mathbf{F}}_\ell^*)$, by reduction modulo ℓ. They are not zero for any (y, χ).

Corollary. *$\pi, \pi' \in \mathrm{Cusp}_{\overline{\mathbf{F}}_\ell} G$ are isomorphic if and only if they have the same central character and the same epsilon factors*

$$\varepsilon(\pi \otimes \chi, y) = \varepsilon(\pi' \otimes \chi, y)$$

for all $y \in Y$, and for all character $\chi \in \mathrm{Hom}(O_F^*, \overline{\mathbf{F}}_\ell^*)$.

Final remarks. a) When $n > 2$, the groups $GL(m, O_F)^*$ for $m \leq n-1$ replace O_F^*.

b) Using the explicit description for the irreducible representations of dimension n of W_F [V3], one could try to prove a similar theorem for the irreducible integral $\overline{\mathbf{Q}}_\ell$-representations of W_F of dimension n. To my knowledge this is a known and harder problem, which is not solved in the complex case.

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