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# Families of modular forms

par Kevin BUZZARD

Résumé. Nous donnons une introduction terre à terre de la théorie des familles de formes modulaires, et discutons des démonstrations élémentaires de résultats suggérant que les formes modulaires apparaissent sous forme de familles.

Abstract. We give a down-to-earth introduction to the theory of families of modular forms, and discuss elementary proofs of results suggesting that modular forms come in families.

## 1. What is a family?

Let $\Gamma$ be the group $\mathrm{SL}_2(\mathbb{Z})$, or more generally a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $k \geq 1$ be an integer. A modular form of weight $k$ for $\Gamma$ is a holomorphic function $f$ on the upper half plane satisfying

$$f\big((az+b)/(cz+d)\big) = (cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and some boundedness conditions, which ensure that for fixed $k$ and $\Gamma$, the space of such forms is finite-dimensional. If $\Gamma = \Gamma_1(N)$ for some positive integer $N$, then we say that $f$ has weight $k$ and level $N$.

Any enquiring mind seeing the precise definition for the first time would surely wonder whether any non-constant modular forms exist at all. But of course they do—and we shall begin our study of families with the forms that are frequently the first non-trivial examples of modular forms given in an introductory course, namely the Eisenstein series $E_k$. Here $k \geq 4$ is an even integer, and $E_k$ is a modular form of weight $k$ and level 1 which has, at least up to a constant, the following power series expansion:

$$E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

As is standard, we use $q$ to denote $\exp(2\pi i z)$, and $\sigma_d(n)$ to denote the sum of the $d$th powers of the (positive) divisors of $n$.

Note that although these forms are constructed only for $k \geq 4$ and even, the coefficients of the power series above happen to make sense for any

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non-zero $k \in \mathbb{C}$. Indeed, if we were asked to come up with a natural way of interpolating the Fourier coefficients of the $E_k$, then the choice given above would surely be a natural one. Almost unwittingly, we have created our first family of modular forms.

Actually, this article is really concerned with $p$-adic families of modular forms, so before we continue, we shall modify the $E_k$ slightly. For simplicity, let us assume for the rest of this section that $p$ is odd. Recall that there is a $p$-adic analogue of the zeta function, so called because it (essentially) $p$-adically interpolates the values of the zeta function at negative integers. We say "essentially" because there are two caveats. The first is that one has to restrict oneself to evaluating the zeta function at negative integers congruent to 1 modulo $p-1$. The second is that one has to drop an Euler factor—the correct classical object to $p$-adically interpolate is $(1-p^{-s})\zeta(s) = \prod_{q\neq p}(1-q^{-s})^{-1}$.

Similarly one has to drop an Euler factor when attempting to $p$-adically interpolate the Eisenstein series $E_k$. For $k \geq 4$ an even integer, define

$$E_k^*(z) = E_k(z) - p^{k-1}E_k(pz) = \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1}^*(n)q^n,$$

where $\sigma_d^*(n)$ is the sum of the $d$th powers of the divisors of $n$ that are prime to $p$. Then $E_k^*(z)$ is an oldform of level $p$, and it is these oldforms that will interpolate beautifully, at least if we restrict to $k$ in some fixed conjugacy class modulo $p-1$.

Let us for example consider the set $S$ consisting of positive even integers $k$ which are at least 4 and are congruent to 0 mod $p-1$. Then this set is dense in $\mathbb{Z}_p$. It turns out that the Fourier expansions for $E_k^*(z)$ are $p$-adically continuous as $k$ varies through $S$, where now we think of $S$ as having the $p$-adic topology. In fact, one can show more. Consider the functions

$$G_k(z) = \frac{2E_k^*(z)}{(1-p^{k-1})\zeta(1-k)} = 1 + \frac{2}{(1-p^{k-1})\zeta(1-k)}\sum_{n\geq 1}\sigma_{k-1}^*(n)q^n.$$

Then we have the following well-known theorem.

**Theorem 1.** *There are unique $p$-adic analytic functions $A_n : \mathbb{Z}_p \to \mathbb{Q}_p$ for $n \geq 0$, with $A_0 = 1$, such that for all $k$ in $S$ we have the following equality of formal sums:*

$$G_k(z) = \sum_{n\geq 1} A_n(k)q^n.$$

□

This theorem is little more than the existence of the $p$-adic zeta function, and the fact that it has no zeros. One thing which is not immediately obvious, but comes out of a more general study of Eisenstein series, is that for all integers $k \geq 0$, the power series $\sum_{n\geq 1} A_n(k)q^n$ is actually the Fourier expansion of a modular form of level $p$ and weight $k$.

Motivated by this theorem, we shall now give a definition of a $p$-adic family of modular forms. Firstly, let $\mathbb{C}_p$ denote the completion of the algebraic closure of $\mathbb{Q}_p$, and fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. Our families will live over $p$-adic discs in $\mathbb{Z}_p$, so let us define, for $c \in \mathbb{Z}_p$ and $r \in \mathbb{R}_{>0}$, the disc $B(c, r)$ to be the set of elements $k \in \mathbb{Z}_p$ with $|k - c| < r$.

**Definition.** Let $N$ be an integer prime to $p$, and fix a disc $B(c, r)$ as above. A *$p$-adic family of modular forms* of level $N$ is a formal power series

$$\sum_{n\geq 0} F_n q^n$$

where each $F_n : B(c, r) \to \mathbb{C}_p$ is a $p$-adic analytic function, and with the property that for all sufficiently large (rational) integers $k$ in $B(c, r)$, the formal sum $\sum_n F_n(k)q^n$ is the Fourier expansion of a classical modular form of weight $k$ and level $Np$.

Note that the family of Eisenstein series that we have constructed is indeed a family in the above sense. We also remark that the restriction to $p$ odd that we made a while ago is entirely unnecessary and that one can still construct a family of Eisenstein series when $p = 2$. Henceforth, $p$ will be an arbitrary prime.

## 2. Forms live in families.

Given what we know, it is now not too difficult to show that any modular form of level $Np$ lives in a family of level $N$—one can just multiply the form in question by the family of Eisenstein series already constructed (which fortuitously goes through the constant modular form 1 when $k = 0$). A much more challenging question is:

**Question.** Let $c$ be a rational integer, and let $f$ be a classical *eigenform* of level $Np$ and weight $c$. Does $f$ live in a $p$-adic family of eigenforms?

By a $p$-adic family of eigenforms, we simply mean a family of forms whose specialisations to weight $k$, for all sufficiently large rational integers $k \in B(c, r)$, is an eigenform. We remark that the family of Eisenstein series is a family of eigenforms. However, the trick of multiplying an arbitrary eigenform $f$ by this family will not produce a family of eigenforms, because the product of two eigenforms is not necessarily still an eigenform (and indeed frequently will not be).

Before we continue, we introduce one more piece of notation. If $f$ is a classical eigenform of level $Np$, then in particular it is an eigenvector for the $U_p$ operator. We say that the *slope* of $f$ is the $p$-adic valuation of the corresponding eigenvalue. Note that it is a classical result that eigenforms of level $Np$ always have finite slope. We say that a $p$-adic family of eigenforms has *constant slope* $s$ if, for all sufficiently large rational integers $k \in B(c,r)$, the corresponding weight $k$ eigenform has slope $s$.

The question stated above has in fact been answered affirmatiely by Coleman, in the paper [C]. Inspired by this work, Coleman and Mazur constructed in [CM] a geometric object, the so-called "Eigencurve", whose very *existence* implies that eigenforms come in families! Points on the eigencurve correspond to certain $p$-adic eigenforms. So assuming that the eigencurve exists, the points in a small open disc on this curve will correspond to a family of eigenforms.

The existence of the eigencurve also explains more conceptually what is going on when one specialises a $p$-adic family of forms at a point where the resulting $q$-expansion is not that of a classical modular form—although we shall not go into the details here. But one thing that it did not initially shed light on was the question of how big one could expect the radius $r$ of the family to be—or equivalently, how big a disc could you fit round a point in the eigencurve?

Understanding the nature of the radius has in fact turned out to be a tricky problem. In [GM], Gouvea and Mazur make some precise conjectures about what $r$ might be expected to be, at least when the level of the forms in question is $\Gamma_0(Np)$, and the optimist can easily extend these conjectures to cover the case of $\Gamma_1(Np)$. These conjectures were based on a lot of numerical examples computed by Mestre. What seemed to be going on in Mestre's computations was that the radius of a family passing through an eigenform $f$ of level $Np$ and slope $s$ was, broadly speaking, $p^{-s}$. The reader who wants to know the precise form of the conjecture (which is in fact rather easy to explain) is referred to [GM].

As well as this computational evidence, the paper [W] showed, using Coleman's work, that an eigenform $f$ of slope $s$ should live in a family of eigenforms with radius $p^{-t}$, where $t = O(s^2)$. Unfortunately, no-one has so far been able to "close the gap", with the result that we still seem to be unsure whether Mestre's computational results are misleading or whether Wan's work can be strengthened.

The fact that eigenforms should come in families has classical consequences. For example, the results above imply that there should be a lower bound on the Newton Polygon of $U_p$ acting on the space of modular forms of level $Np$ and weight $k$, and that this bound should be uniform in $k$. This question was raised explicitly by Ulmer in [U], who produced a number of interesting results in this direction. We should perhaps remark at this stage that Coleman's work used a lot of machinery from the theory of rigid-analytic geometry, and Ulmer's ideas appealed a lot to crystalline methods. However, in the unpublished [T], Richard Taylor noted that in fact a more elementary approach could perhaps be used to attack these problems. Inspired by these ideas, we shall here present a completely elementary proof of

**Theorem 2.** *The Newton Polygon of $U_p$ acting on the space of classical modular forms of weight $k$ and level $Np$ is bounded below by an explicit quadratic lower bound which is independent of $k$.*

As a corollary, we get explicit bounds for the number of forms of slope $\alpha$, weight $k$ and level $Np$, and moreover these bounds are independent of $k$.

We shall state and prove a precise version of the theorem, and its corollaries, in the next section. As far as the author knows, the first proof of this theorem was given in [W] and made essential use of Coleman's rigid-analytic methods. The proof we present uses nothing more than group cohomology.

## 3. The proofs.

**Some notation.**

Recall that $N$ is a positive integer, and $p$ is a prime not dividing $N$.

Let $R = \mathbb{Z}_p[\xi]$ denote the polynomial ring in one variable over $\mathbb{Z}_p$. Let $L$ be a finite free $\mathbb{Z}_p$-module of rank $t$, equipped with a $\mathbb{Z}_p$-linear endomorphism with non-zero determinant. Then $L$ can be regarded as a $\mathbb{Z}_p[\xi]$-module, where $\xi$ acts via this endomorphism.

We recall the definition of the Newton Polygon of $\xi$ acting on $L$. Write the characteristic polynomial of $\xi$ on $L \otimes \mathbb{Q}_p$ as $\sum_{i=0}^{t} c_{t-i}X^i$. Then let $v_p$ denote the usual valuation on $\mathbb{Z}_p$, and plot the points $(i, v_p(c_i))$ in $\mathbb{R}^2$, for $0 \leq i \leq t$, ignoring the $i$ for which $c_i = 0$. Let $Z$ denote the convex hull of these points. The Newton Polygon of $\xi$ on $L$ is the lower faces of $Z$, that is the union of the sides forming the lower of the two routes from $(0,0)$ to $(t, v_p(c_t))$ on the boundary of $Z$. It is well-known that this graph encodes the $p$-adic valuations of the eigenvalues of $\xi$. In fact, if the Newton Polygon has a side of slope $\alpha$ and whose projection onto the $x$ axis has length $n$, then there are precisely $n$ eigenvectors of $\xi$ with $p$-adic valuation equal to $\alpha$.

It is our intention to say something about this polygon when $\xi$ is the operator $U_p$ acting on a certain lattice in a space of modular forms. As a result we shall recover results about the $p$-adic valuations of the eigenvalues of $U_p$. We shall first establish a general result, after setting up some more notation.

Let $L$ be as above, and let $K$ be a sub-$R$-module of $L$ such that the quotient $Q = L/K$ is a finite $p$-group. Say

$$Q \cong \prod_{i=1}^{r} (\mathbb{Z}/p^{a_i}\mathbb{Z}),$$

with $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_r > 0$. Write $a_i = 0$ for $r < i \leq t$.

Now assume that there exists a non-negative integer $n$, greater than or equal to all the $a_i$, and such that $\xi(K) \subseteq p^n L$. Write $b_i = n - a_i$ for $1 \leq i \leq t$ and let $B(j) = \sum_{i=1}^{j} b_i$.

**Lemma 1.** *The Newton Polygon of $\xi$ on $L$ lies on or above the points $(j, B(j))$ for $0 \leq j \leq t$.*

*Proof.* The proof is very easy, and we shall just sketch it. First choose a $\mathbb{Z}_p$-basis $(e_i)_{1 \leq i \leq t}$ for $L$ such that $(p^{a_i} e_i)_{1 \leq i \leq t}$ is a $\mathbb{Z}_p$-basis for $K$. With respect to this basis, let $\xi : L \to L$ be represented by the matrix $(u_{i,j})$. One sees that the assumption $\xi(K) \subseteq p^n L$ implies that $p^{b_j}$ divides $u_{i,j}$. Then, from the definition of the characteristic polynomial as $\det(X.1 - \xi) = \sum c_{t-i} X^i$, and from the fact that the $b_i$ are increasing, we see that $p^{B(j)}$ divides $c_j$ for $0 \leq j \leq t$. Using once more that the $b_i$ are increasing, the lemma follows easily. $\square$

We now show how to apply this in the case of classical modular forms. Write $\Gamma$ for $\Gamma_1(Np)$, and for $k \geq 2$ an integer, let $S_k(\Gamma)$ denote the space of classical cusp forms of weight $k$ and level $Np$. Note that if $Np \geq 5$ then $\Gamma$ is torsion-free and hence free. In fact, if $Np \geq 5$ then $\Gamma$ is free on $2g(X(\Gamma)) + c - 1$ generators, where $g(X(\Gamma))$ is the genus of the compactified modular curve associated to $\Gamma$, and $c$ is the number of cusps added to make the compactification. For general $Np$, we let $m(\Gamma)$ denote the minimal number of generators of $\Gamma$ as a group. Hence $m(\Gamma) = 2g(X(\Gamma)) + c - 1$ if $Np \geq 5$, and $m(\Gamma)$ can easily be worked out explicitly in the other cases.

Let $g = k - 2$. For a commutative ring $A$, let $V_g(A) = A^{g+1}$, considered as column matrices of length $g + 1$. Let $\mathbf{M}$ denote the monoid consisting of two by two matrices with integer entries and positive determinant. There is a unique left action of $\mathbf{M}$ on $V_g(\mathbb{Z})$ with the following property: for all $y, z \in \mathbb{Z}$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (y^g, y^{g-1}z, y^{g-2}z^2, \ldots, yz^{g-1}, z^g)^T$$

$$= ((ay + bz)^g, (ay + bz)^{g-1}(cy + dz), \ldots, (ay + bz)(cy + dz)^{g-1}, (cy + dz)^g)^T,$$

where the superscript $T$ denotes transpose. In fact, this action can be constructed explicitly thus: Consider elements of $\mathbb{Z}^2$ as row vectors. Then there is a natural right action of $\mathbf{M}$ on $\mathbb{Z}^2$ and hence on $S^g(\mathbb{Z}^2)$, its symmetric $g$th power. The left action we are interested in is the natural induced left action of $\mathbf{M}$ on the $\mathbb{Z}$-dual of $S^g(\mathbb{Z}^2)$.

Define an action of $\mathbf{M}$ on $V_g(A)$ for any commutative ring $A$ by extending the action on $V_g(\mathbb{Z})$ $A$-linearly. Note that $\Gamma \subseteq \mathbf{M}$ and hence $V_g(A)$ has the structure of a left $\Gamma$-module.

There is a well-known isomorphism (see Theorem 8.4 of [S]) due to Eichler and Shimura:

$$ES : S_k(\Gamma) \to H^1_P(\Gamma, V_g(\mathbb{R}))$$

where $H^1_P$ denotes parabolic cohomology. Composing $ES$ with the natural inclusion $H^1_P(\Gamma, V_g(\mathbb{R})) \to H^1(\Gamma, V_g(\mathbb{R}))$ gives us an inclusion $ES : S_k(\Gamma) \to H^1(\Gamma, V_g(\mathbb{R}))$.

The Hecke operator $U_p$ acts ($\mathbb{C}$-linearly) on $S_k(\Gamma)$, and we shall now recall the definition of an operator on $H^1(\Gamma, V_g(\mathbb{R}))$ that extends $U_p$.

Let $A$ be any ring, and define $\xi : H^1(\Gamma, V_g(A)) \to H^1(\Gamma, V_g(A))$, an $A$-linear map, thus. Firstly note that

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma = \coprod_{i=0}^{p-1} \Gamma \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix},$$

and write $\alpha_i$ for $\begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}$. For $\gamma$ in $\Gamma$ and $0 \le i \le p-1$ there exists a unique $0 \le j \le p-1$ and $\gamma_i \in \Gamma$ such that $\alpha_i \gamma = \gamma_i \alpha_j$. First we shall define the map $\xi$ on 1-cocycles. If $u : \Gamma \to V_g$ is a 1-cocycle, define $v(\gamma) \in V_g$ by

$$v(\gamma) = \sum_{i=0}^{p-1} \begin{pmatrix} p & -i \\ 0 & 1 \end{pmatrix} u(\gamma_i).$$

One can check that $v$ is a 1-cocycle and moreover that the association $u \mapsto v$ induces a map from $H^1(\Gamma, V_g)$ to itself, which we define to be $\xi$.

**Lemma 2.** *The diagram*

$$\begin{array}{ccc} S_k(\Gamma) & \xrightarrow{ES} & H^1(\Gamma, V_g(\mathbb{R})) \\ \downarrow{\scriptstyle U_p} & & \downarrow{\scriptstyle \xi} \\ S_k(\Gamma) & \xrightarrow{ES} & H^1(\Gamma, V_g(\mathbb{R})) \end{array}$$

*commutes.*

*Proof.* This is Proposition 8.5 of [S]. □

Let $\alpha(X)$ denote the characteristic polynomial of $U_p$ acting on the complex vector space $S_k(\Gamma)$. It is the roots of this polynomial that we ultimately wish to study. Note first that the coefficients of $\alpha$ are rational. Hence $\alpha^2$ is the characteristic polynomial of $U_p$ on $S_k(\Gamma)$, considered as a real vector space, and so if $\beta(X)$ is the characteristic polynomial of $\xi$ acting on the real vector space $H^1(\Gamma, V_g(\mathbb{R}))$ then we see that $\alpha^2$ divides $\beta$, as $ES$ is an injection. Note that $\beta$ is also the characteristic polynomial of $\xi$ acting on the $K$-vector space space $H^1(\Gamma, V_g(K))$ for any field $K$ of characteristic 0.

We now pass to the case $K = \mathbb{Q}_p$. Firstly note that, although the group $H^1(\Gamma, V_g(\mathbb{Z}_p))$ may have some torsion, its image in $H^1(\Gamma, V_g(\mathbb{Q}_p))$ is a lattice. It is this lattice that we shall use for our $L$.

Let $\epsilon_i \in V_g(\mathbb{Z}_p)$ denote the column vector with 1 in its $i$th row and zeros elsewhere, and let $W_g(\mathbb{Z}_p)$ denote the $\mathbb{Z}_p$-subspace of $V_g(\mathbb{Z}_p)$ generated by the vectors

$$\{\epsilon_1, p\epsilon_2, p^2\epsilon_3, \ldots, p^g\epsilon_{g+1}\}.$$

It is easily checked that $W_g(\mathbb{Z}_p)$ is preserved by $\Gamma$. Let $I$ denote the quotient $V_g(\mathbb{Z}_p)/W_g(\mathbb{Z}_p)$ with its induced $\Gamma$-action. Taking the long exact sequence in group cohomology associated to the short exact sequence

$$0 \to W_g(\mathbb{Z}_p) \to V_g(\mathbb{Z}_p) \to I \to 0$$

gives us an exact sequence of finitely-generated $\mathbb{Z}_p$-modules

$$H^0(\Gamma, I) \to H^1(\Gamma, W_g(\mathbb{Z}_p)) \to H^1(\Gamma, V_g(\mathbb{Z}_p)) \to H^1(\Gamma, I),$$

where the first and last terms in this sequence are finite. It follows that there is an exact sequence

$$0 \to H^1(\Gamma, W_g(\mathbb{Z}_p))^{TF} \to H^1(\Gamma, V_g(\mathbb{Z}_p))^{TF} \to Q \to 0$$

where $Q$ is some subquotient of $H^1(\Gamma, I)$ and the superscript $TF$ denotes the maximal torsion-free quotient. In particular $Q$ is finite (it would be good to have a more explicit description of $Q$). Now write $L$ for $H^1(\Gamma, V_g(\mathbb{Z}_p))^{TF}$ and $K$ for $H^1(\Gamma, W_g(\mathbb{Z}_p))^{TF}$. The action of $\xi$ on $H^1(\Gamma, V_g(\mathbb{Z}_p))$ induces an action of $\xi$ on $L$ and hence on $K$. Now we have

**Lemma 3.** *$\xi(K) \subseteq p^g L$.*

*Proof.* Let $\kappa : \Gamma \to W_g(\mathbb{Z}_p)$ be a 1-cocycle. It suffices to prove that the cocycle $\xi\kappa$ is divisible by $p^g$. From the definition of $\xi\kappa$, this will follow if we can show that

$$\begin{pmatrix} p & -i \\ 0 & 1 \end{pmatrix} W_g(\mathbb{Z}_p) \subseteq p^g V_g(\mathbb{Z}_p)$$

for $0 \leq i \leq p-1$. But in fact one can check that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} W_g(\mathbb{Z}_p) \subseteq p^g V_g(\mathbb{Z}_p)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}$ such that $p$ divides both $a$ and $c$, and we are home. $\square$

We are now in a position to apply Lemma 1, with $n = g$. The lemma will give us an explicit lower bound for the Newton polygon of $\xi$ on $L$, which encodes the slopes of the eigenvalues of the polynomial $\beta$. But we remarked above that $\alpha^2$ divides $\beta$ and so we are also getting explicit lower bounds for the Newton Polygon $U_p$ on $S_k(\Gamma)$.

First of all we make some easy observations. Because $L$ is a quotient of the group of cocycles $c : \Gamma \to V_g(\mathbb{Z}_p)$, we see that the rank $t$ of $L$ is at most $(g+1)m$. We do not know $Q$ explicitly, but we know that it is a subquotient of $H^1(\Gamma, I)$, which is itself a quotient of the group of 1-cocycles from $\Gamma$ to $I$. This group of cocycles is non-canonically isomorphic to a subgroup of $I^m$, where we recall that $m = m(\Gamma)$ is the minimal number of generators of $\Gamma$. Moreover, $I$ itself is isomorphic to $\prod_{i=1}^{g}(\mathbb{Z}/p^i\mathbb{Z})$.

If $x \in \mathbb{R}$, write $\lfloor x \rfloor$ for the largest integer which is at most $x$, and $\lceil x \rceil$ for the smallest integer which is at least $x$. The arguments above show that if we write $Q = \prod_{i=1}^{r}(\mathbb{Z}/p^{a_i}\mathbb{Z})$, with $a_i$ decreasing, then $r \leq mg$. Set $a_i = 0$ for $r < i \leq t$. Then we can deduce that $a_i \leq \lfloor n + 1 - i/m \rfloor$ for $1 \leq i \leq t$ and hence $b_i = n - a_i \geq \lfloor (i-1)/m \rfloor$ (with notation as before Lemma 1.) A simple application of Lemma 1 then gives:

**Proposition 1.** *The Newton polygon of $\xi$ acting on $L$ is bounded below by the function which goes through $(0,0)$, has slope $0$ for $0 \leq x \leq m$, slope $1$ for $m \leq x \leq 2m$, slope $2$ for $2m \leq x \leq 3m$, and so on.*

□

Note that this bound depends only on $m$ and in particular only on $N$ and $p$. Because of the relationship between $S_k(\Gamma)$ and $H^1(\Gamma, V_g(\mathbb{R}))$ explained above, we deduce

**Theorem 3.** *The Newton polygon of $U_p$ acting on $S_k(\Gamma)$ is bounded below by the function which goes through $(0,0)$, has slope $0$ for $0 \leq x \leq m/2$, slope $1$ for $m/2 \leq x \leq m$, and in general has slope $r$ for $mr/2 \leq x \leq (mr+m)/2$.*

□

This is, of course, a more precise form of the theorem that we were aiming for. An easy consequence is

**Corollary 1.** *If $\alpha \in \mathbb{Q}$, the space of cusp forms of slope $\alpha$, weight $k$ and level $Np$ has dimension at most $\lceil \alpha + 1/2 \rceil m$, independent of $k$.*

We remind the reader that $m$ is the minimal number of generators of $\Gamma = \Gamma_1(Np)$ as a group. As concrete examples, we could perhaps mention that If $N = 1$ and $p \geq 5$ then $m$ can be taken to be $\frac{p^2+11}{12}$. In general, $m$ is about $O((Np)^2)$. Moreover, it is not difficult to show that in fact $m$ can be taken to be any integer such that $\Gamma_1(Np)$ has a subgroup of finite index prime to $p$ which is generated by $m$ elements. We remark also that our analysis can easily be generalised to other congruence subgroups $\Gamma$ of the form $\Gamma_0 \cap \Gamma_1(p)$ or $\Gamma_0 \cap \Gamma_0(p)$, where $\Gamma_0$ is any congruence subgroup of level prime to $p$. Finally, we should mention that for explicit values of $N$ and $p$ (typically $N = 1$ and $p \leq 3$), more precise results have been obtained by Coleman, Stevens, Teitelbaum, and Emerton and Smithline using rigid-analytic methods. It would be interesting to know whether there were more down-to-earth techniques that could establish their results.

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Kevin BUZZARD
Department of Maths
Huxley Building
Imperial College
London, SW7 2BZ
England
*E-mail* : k.buzzard@ic.ac.uk