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On the distribution in the arithmetic progressions of reducible quadratic polynomials in short intervals, II

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RÉSUMÉ. Ce texte donne de nouveaux résultats sur la répartition dans les progressions arithmétiques (modulo un produit de deux nombres premiers) des valeurs $(an + b)(cn + d)$ prises par un polynôme quadratique réductible lorsque l'entier n varie dans des intervalles courts $n \in [x, x + x^\vartheta]$, où $\vartheta \in (0, 1]$. Nous utilisons ici la méthode de dispersion, pour obtenir un niveau de répartition au delà du niveau classique θ. Nous obtenons pour niveau $3\vartheta/2$, améliorant en cela la valeur $3\vartheta - 3/2$ obtenue par le grand crible. Nous terminons par une comparaison graphique des deux approches.

ABSTRACT. This paper gives further results about the distribution in the arithmetic progressions (modulo a product of two primes) of reducible quadratic polynomials $(an + b)(cn + d)$ in short intervals $n \in [x, x + x^\vartheta]$, where now $\vartheta \in (0, 1]$. Here we use the Dispersion Method instead of the Large Sieve to get results beyond the classical level ϑ, reaching $3\vartheta/2$ (thus improving also the level of the previous paper, *i.e.* $3\vartheta - 3/2$), but our new results are different in structure. Then, we make a graphical comparison of the two methods.

1. Introduction and statement of the results

In this paper we continue the study of the distribution in the arithmetic progressions of the polynomial sequence $n(n + 2)$ and, also, of sequences of reducible quadratic polynomials, in short intervals that we started with [2]. Here, instead of the Large Sieve (see [1]), we use the Dispersion method (see [6]) to get results which are independent (even if from some point of view stronger) of the ones in [2].

We first briefly recall the arguments thereby treated.

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We want to show the heuristically reasonable estimate (d is odd and square-free):

$$\sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0 \bmod d}} 1 = \frac{h}{d} \sum_{\substack{t|d\\ \frac{d}{x+h}<t<x+h}} 1 + R_d(x,h),$$

where h is the length of the interval and $R_d(x,h)$ is "good" on average.

When this holds on average over $d \sim D$, *i.e.* over $D < d \le 2D$, with $D = x^{\alpha-\varepsilon}$ ($\varepsilon > 0$ is "small"), we say that the **distribution level** of the sequence (here $n(n+2)$), in the arithmetic progressions, is (at least) α.

For example, standard arguments give the distribution level ϑ for $n(n+2)$, when $n \in [x, x+x^{\vartheta}]$. For instance, we refer to [3], where classical sieve methods are used.

A higher distribution level is reached using bilinear forms; in the ones we consider we use the bounded coefficients γ_q, δ_r, where $q \sim Q$ and $r \sim R$. Here the level of distribution is obviously $\log(QR)/\log(x)$.

(In the sequel we will write $s \equiv a(m)$ for $s \equiv a \bmod m$).

A non-trivial treatment of the bilinear form of the error in

$$\sum_{q\sim Q} \sum_{r\sim R} \gamma_q \delta_r \sum_{\substack{n\le x\\ n^2+1\equiv 0(qr)}} 1$$

allowed Iwaniec to show in 1978 [5] that the level of distribution in this case is 16/15; he then used this to prove that $n^2+1 = P_2$ for infinitely many n (here P_k denotes an integer with at most k prime factors), whilst the trivial level of distribution, that is 1, enables one to get only P_3.

We study the bilinear form over prime moduli, *i.e.* (here $QR > 2h+2$)

$$\sum_{q\sim Q} \sum_{r\sim R} \gamma_q \delta_r \sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}} 1,$$

where q and r are distinct primes.

This restriction is connected with the arithmetical nature of the problem and is essential for our results. We hope in the future to manage also the case of general moduli.

We expect that the following estimate holds

$$\sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q \delta_r \sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}} 1 = \sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q \delta_r \left(\frac{2h}{qr} + \sum_{\substack{x<n\le x+h\\ n\equiv 0(qr)}} 1 + \sum_{\substack{x<n\le x+h\\ n\equiv -2(qr)}} 1 \right)$$

with a "good" error term, that is $\mathcal{O}(h^{1-\varepsilon})$.

In this paper we use elementary methods (that is, essentially the Dispersion method) to get level of distribution $3\vartheta/2$, *i.e.* $3/2$ in large intervals (see our Corollary 1). As we saw in [2], the Large Sieve gives different ranges for Q and R, even if in the case $\vartheta < 1$ the level of distribution is lower (*i.e.* $3\vartheta - 3/2$).

Our results are the following.

Theorem 1. *Let* $x > 4,\ 0 < \vartheta \le 1,\ x^{\vartheta} \le h \le x,\ 0 < \varepsilon \le \frac{1}{8}$*; let* Q, R *be in* $[1, h/2[$ *and* γ_q, δ_r *be bounded arithmetical functions with support on the primes of* $]Q, 2Q]$, $]R, 2R]$ *(respectively). Then, for* $(q, r) = 1$

$$\sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q\delta_r \sum_{\substack{x<n\le x+h\\ n(n+2)\equiv 0(qr)}} 1 = \sum_{\substack{q\sim Q\\ r\sim R}} \gamma_q\delta_r \left(\frac{2h}{qr} + \sum_{\substack{x<n\le x+h\\ n\equiv 0(qr)}} 1 + \sum_{\substack{x<n\le x+h\\ n\equiv -2(qr)}} 1 \right) + \mathcal{O}(h^{1-\varepsilon}), \tag{1}$$

provided $R \le h^{1-2\varepsilon}$, $Q \le R^{1/2}h^{-\varepsilon}$, $QR > 2h + 2$.

As an application we get the level of distribution $3\vartheta/2$

Corollary 1. *Let* x, ϑ, ε, q, r, γ_q, δ_r *be as above. Then the estimate (1) of Theorem 1 holds for* $QR = h^{3/2-4\varepsilon}$.

As in [7] we generalize Theorem 1 and Corollary 1 to get (here we set $[A, B, C] = l.c.m.(A, B, C)$)

Corollary 2. *Let* $(an+b)(cn+d)$ *be a polynomial without fixed divisors. Let* x, ϑ, h, ε, q, r, γ_q *and* δ_r *be as above, with* qr *coprime with* $[a, c, ad - bc]$*. Then the same conclusions of Theorem 1 and Corollary 1 hold true with* $(an + b)(cn + d)$ *in place of* $n(n + 2)$.

We point out that the Dispersion method improves on the trivial level of distribution ϑ for any $\vartheta > 0$, whilst with the Large Sieve method (see [2]) this is true only for $\vartheta > 3/4$.

In fact, the level of distribution given by the Large Sieve is lower then the one given by the Dispersion.

Nevertheless, the main Theorems (like, also, the respective Corollaries) are independent, due to different ranges of Q and R covered by the two results.

The situation is depicted in section 3.

The paper is organized as follows:

- in section 2 we give the "elementary" proof of Theorem 1, based on the Dispersion method;
- in section 3 we compare the present results with the results given in [2], showing the feasible regions for the quantities $\log Q/\log x$ and $\log R/\log x$ in both cases.

2. Proof of Theorem 1 (and of the Corollaries)

Let q, r be distinct primes (as in the sequel) and define

$$I \equiv \sum_{q\sim Q}\sum_{r\sim R}\gamma_q\delta_r \sum_{\substack{x<n\le x+h \\ n(n+2)\equiv 0 \bmod qr}} 1.$$

We get

$$I = \sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r \left(\sum_{\substack{x<n\le x+h \\ n\equiv 0(r) \\ n\equiv -2(q)}} 1 + \sum_{\substack{x<n\le x+h \\ n\equiv 0(q) \\ n\equiv -2(r)}} 1 + \sum_{\substack{x<n\le x+h \\ n\equiv 0(qr)}} 1 + \sum_{\substack{x<n\le x+h \\ n\equiv -2(qr)}} 1 \right).$$

Thus to prove (1) it will suffice to prove

$$\sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r \sum_{\substack{x<n\le x+h \\ n\equiv 0(r) \\ n\equiv -2(q)}} 1 = h\sum_{\substack{q\sim Q \\ r\sim R}}\frac{\gamma_q\delta_r}{qr} + \mathcal{O}(h^{1-\varepsilon}). \tag{2}$$

Clearly, it is sufficient to show that

$$\Sigma \equiv \sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r \sum_{\substack{x<n\le x+h \\ n\equiv 0(r) \\ n\equiv -2(q)}} 1 - \sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r \sum_{\substack{x<n\le x+h \\ n\equiv 0(r)}} \frac{1}{q} \ll h^{1-\varepsilon}; \tag{3}$$

in fact the difference between the main terms of (2) and (3) is negligible, because

$$\sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r \sum_{\substack{x<n\le x+h \\ n\equiv 0(r)}} \frac{1}{q} - h\sum_{\substack{q\sim Q \\ r\sim R}}\frac{\gamma_q\delta_r}{qr} = \sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r\,\mathcal{O}(1/q) \ll R$$

and $R \ll h^{1-\varepsilon}$ by hypothesis.

Since $qr > 2h+2$ and q is prime

$$\Sigma = \sum_{\substack{q\sim Q \\ r\sim R}}\gamma_q\delta_r \left(\sum_{\substack{\frac{x}{r}<m\le\frac{x+h}{r} \\ mr\equiv -2(q)}} 1 - \sum_{\substack{\frac{x}{r}<m\le\frac{x+h}{r} \\ (m,q)=1}} \frac{1}{q} \right) + \mathcal{O}(R), \tag{4}$$

$\mathcal{O}(R)$ being negligible as before; we then get by Cauchy inequality

$$\Sigma \ll \sqrt{R}\sqrt{\Delta},$$

where, say, $\Delta = \Delta(x,h,Q,R)$ equals

$$\Delta \equiv \sum_{r\sim R}\left|\sum_{q\sim Q}\gamma_q\left(\sum_{\substack{\frac{x}{r}<m\le\frac{x+h}{r}\\ mr\equiv -2(q)}} 1 - \sum_{\substack{\frac{x}{r}<m\le\frac{x+h}{r}\\ (m,q)=1}}\frac{1}{q}\right)\right|^2.$$

Thus, to prove that $\Sigma \ll h^{1-\varepsilon}$, we shall prove the bound

$$\Delta \ll h^{2-2\varepsilon}/R. \tag{5}$$

In order to do this we apply the Dispersion method on Δ; we expand the square and exchange the sum over r and the inner sums to get

$$\Delta = \sum_{q_1,q_2\sim Q}\gamma_{q_1}\overline{\gamma_{q_2}}\sum_{\substack{\frac{x}{2R}<m_1\le\frac{x+h}{R}\\ (m_1,q_1)=1}}\ \sum_{\substack{\frac{x}{2R}<m_2\le\frac{x+h}{R}\\ (m_2,q_2)=1}} E,$$

where $E = E(m_1,m_2,q_1,q_2,x,h,R)$ is defined as

$$E \equiv \sum_{\substack{X<r\le Y\\ rm_1\equiv -2(q_1)\\ rm_2\equiv -2(q_2)}} 1 - \sum_{\substack{X<r\le Y\\ rm_1\equiv -2(q_1)}}\frac{1}{q_2} - \sum_{\substack{X<r\le Y\\ rm_2\equiv -2(q_2)}}\frac{1}{q_1} + \sum_{X<r\le Y}\frac{1}{q_1q_2},$$

with $X = X(m_1,m_2,x,h,R)$ defined as $X = \max(R, x/m_1, x/m_2)$ and $Y = Y(m_1,m_2,x,h,R)$ as $Y = \min(2R,(x+h)/m_1,(x+h)/m_2)$; here all the sums over r have $(r,q_1q_2)=1$.

Hence, by these definitions and the hypothesis on R, we see that each one of the sums over m_1 and m_2 has length $\mathcal{O}(h/R)$.

We will implicitly assume this in the following estimates.

First of all, we evaluate the diagonal of Δ, *i.e.* Δ', say

$$\Delta' \equiv \sum_{q\sim Q}|\gamma_q|^2\sum_{\substack{\frac{x}{2R}<m_1\le\frac{x+h}{R}\\ (m_1,q_1)=1}}\ \sum_{\substack{\frac{x}{2R}<m_2\le\frac{x+h}{R}\\ (m_2,q_2)=1}} E,$$

where this time $E = E(m_1,m_2,q,q,x,h,R)$ is, say, $E \equiv E_1+E_2+E_3+E_4$:

$$\sum_{\substack{X<r\le Y\\ rm_1\equiv -2(q)\\ rm_2\equiv -2(q)}} 1 - \sum_{\substack{X<r\le Y\\ rm_1\equiv -2(q)}}\frac{1}{q} - \sum_{\substack{X<r\le Y\\ rm_2\equiv -2(q)}}\frac{1}{q} + \sum_{X<r\le Y}\frac{1}{q^2}.$$

The contribute of E_1 to Δ' is bounded by

$$\sum_{q\sim Q}\sum_{\frac{x}{2R}<m_1\le\frac{x+h}{R}}\sum_{\substack{R<r\le 2R\\ \frac{x}{m_1}<r\le\frac{x+h}{m_1}\\ rm_1\equiv -2(q)}}\sum_{\substack{\frac{x}{r}<m_2\le\frac{x+h}{r}\\ rm_2\equiv -2(q)}}1\ll\sum_{q\sim Q}\sum_{\frac{x}{2R}<m_1\le\frac{x+h}{R}}\left(\frac{h}{m_1q}+1\right)\ll$$

$$\ll h\frac{h}{R}\frac{R}{x}+Q\frac{h}{R}\ll h+\frac{Qh}{R}$$

(having exchanged the sum over r with the sum over m_2), since, by our hypotheses, $QR>h$ and $h\le x$.

The contribute of E_2 (that of E_3 is analogous) is bounded by (since $QR>h$)

$$\sum_{q\sim Q}\frac{1}{q}\sum_{r\sim R}\sum_{\frac{x}{r}<m_2\le\frac{x+h}{r}}\sum_{\substack{\frac{x}{r}<m_1\le\frac{x+h}{r}\\ m_1r\equiv -2(q)}}1\ll\sum_{q\sim Q}\frac{1}{q}\sum_{r\sim R}\frac{h}{R}\ll h,$$

after the exchange of the sum over r with the double sum over m_1,m_2.

Finally, the contribute of E_4 is (again by $QR>h$)

$$\sum_{q\sim Q}\frac{1}{q^2}\sum_{\frac{x}{2R}<m_1\le\frac{x+h}{R}}\sum_{\frac{x}{2R}<m_2\le\frac{x+h}{R}}\sum_{X<r\le Y}1\ll\frac{1}{Q}\sum_{r\sim R}\frac{h^2}{R^2}\ll\frac{h^2}{QR}\ll h.$$

Thus

$$\Delta'\ll h+\frac{hQ}{R}\ll\frac{h^{2-2\varepsilon}}{R},$$

as required in (5) (our hypotheses on Q,R imply that $R\le h^{1-2\varepsilon}$ and that $Q\le h^{1-2\varepsilon}$).

Now we estimate $\Delta-\Delta'$ (the non-diagonal terms of Δ).

We first show that we can drop the condition $(r,q_1q_2)=1$ from the last three sums of E in $\Delta-\Delta'$. In fact the sums in E which have $(r,q_1q_2)>1$ are (since q_1 and q_2 are distinct primes), say

$$E'\equiv-\sum_{\substack{X<r\le Y\\ rm_1\equiv -2(q_1)\\ q_2|r}}\frac{1}{q_2}+\sum_{\substack{X<r\le Y\\ (r,q_1)=1\\ q_2|r}}\frac{1}{q_1q_2}-\sum_{\substack{X<r\le Y\\ rm_2\equiv -2(q_2)\\ q_1|r}}\frac{1}{q_1}+\sum_{\substack{X<r\le Y\\ (r,q_2)=1\\ q_1|r}}\frac{1}{q_1q_2}$$

and, say,

$$E''\equiv\sum_{\substack{X<r\le Y\\ q_1q_2|r}}\frac{1}{q_1q_2}.$$

Their contribute to $\Delta-\Delta'$ is negligible, *i.e.* is $\mathcal{O}(h^{2-2\varepsilon}/R)$, as required in (5); in fact exchanging back the sums on r and m_1, m_2 we get

$$\sum_{\substack{\frac{x}{2R}<m_1\le\frac{x+h}{R}\\(m_1,q_1)=1}}\ \sum_{\substack{\frac{x}{2R}<m_2\le\frac{x+h}{R}\\(m_2,q_2)=1}} E' =$$

$$=\frac{1}{q_2}\sum_{\substack{r\sim R\\(r,q_1)=1\\q_2|r}}\ \sum_{\frac{x}{r}<m_2\le\frac{x+h}{r}}\mathcal{O}(1)+\frac{1}{q_1}\sum_{\substack{r\sim R\\(r,q_2)=1\\q_1|r}}\ \sum_{\frac{x}{r}<m_1\le\frac{x+h}{r}}\mathcal{O}(1),$$

and this gives a contribution $\mathcal{O}(h)$ to $\Delta-\Delta'$, which is negligible, as we saw (we have used the hypothesis $QR>h$).

In the same way, E'' contributes to $\Delta-\Delta'$ as (since $4Q^2<R$)

$$\sum_{q_1,q_2\sim Q}\frac{1}{q_1q_2}\sum_{\substack{r\sim R\\q_1q_2|r}}\ \sum_{\substack{\frac{x}{r}<m_1\le\frac{x+h}{r}\\\frac{x}{r}<m_2\le\frac{x+h}{r}}}1\ll\sum_{q_1,q_2\sim Q}\frac{1}{q_1^2q_2^2}\frac{h^2}{R}\ll\frac{h^2}{Q^2R}\ll h,$$

which is also negligible (again, we have used $QR>h$).

Hence, by a direct calculation, E in $\Delta-\Delta'$ equals

$$\left(\left[\frac{Y-c}{q_1q_2}\right]-\left[\frac{X-c}{q_1q_2}\right]\right)-\frac{1}{q_2}\left(\left[\frac{Y+2\overline{m_1}}{q_1}\right]-\left[\frac{X+2\overline{m_1}}{q_1}\right]\right)$$
$$-\frac{1}{q_1}\left(\left[\frac{Y+2\overline{m_2}}{q_2}\right]-\left[\frac{X+2\overline{m_2}}{q_2}\right]\right)+\frac{Y-X}{q_1q_2}$$

where

$$c\equiv-2\overline{m_2}q_1\overline{q_1}-2\overline{m_1}q_2\overline{q_2}\bmod q_1q_2$$

with

$$\overline{m_1}m_1\equiv1\bmod q_1\qquad\overline{m_2}m_2\equiv1\bmod q_2$$
$$\overline{q_1}q_1\equiv1\bmod q_2\qquad\overline{q_2}q_2\equiv1\bmod q_1,$$

which is $\mathcal{O}(1)$, because the main terms cancel and the fractional parts give $\mathcal{O}\left(1+\frac{1}{q_2}+\frac{1}{q_1}\right)$.

Thus we get

$$\Delta-\Delta'\ll\sum_{q_1,q_2\sim Q}\ \sum_{\frac{x}{2R}<m_1\le\frac{x+h}{R}}\ \sum_{\frac{x}{2R}<m_2\le\frac{x+h}{R}}1\ll Q^2\frac{h^2}{R^2}$$

and this is $\mathcal{O}(h^{2-2\varepsilon}/R)$, since $Q^2\le Rh^{-2\varepsilon}$, as required in the hypotheses of our Theorem 1.

As regards the proof of Corollary 1, it is immediate from Theorem 1, choosing $R = h^{1-2\varepsilon}$ and $Q = h^{1/2-2\varepsilon}$. Corollary 2 can be proved along the same lines of the proof of Corollary 1.2 in [7].

3. Graphics comparing the two methods

Here we give four different examples (depending on the value of ϑ) of feasible regions, respectively the shaded triangle for the standard classical estimate (which gives the "trivial" level of distribution, ϑ), the triangle for the Dispersion method (with the level of distribution $3\vartheta/2$) and the rectangle for the Large Sieve (giving the level of distribution $3(\vartheta - 1/2)$); in all the following graphs we have ignored all the ε-contributes.

The "classical" region is the one due to the standard classical estimates (see section 1). The two "non-classical" regions are the ones due to the Dispersion method and to the Large Sieve (see section 1 and our previous paper). In the legends "Both methods region" indicates the intersection in which the two methods give the same results. The Dispersion-region is always non-empty (for any $\vartheta > 0$), while the Large Sieve-region is non-empty for any $\vartheta > 1/2$.

The graphic below shows the first case, $\vartheta = 5/8$.

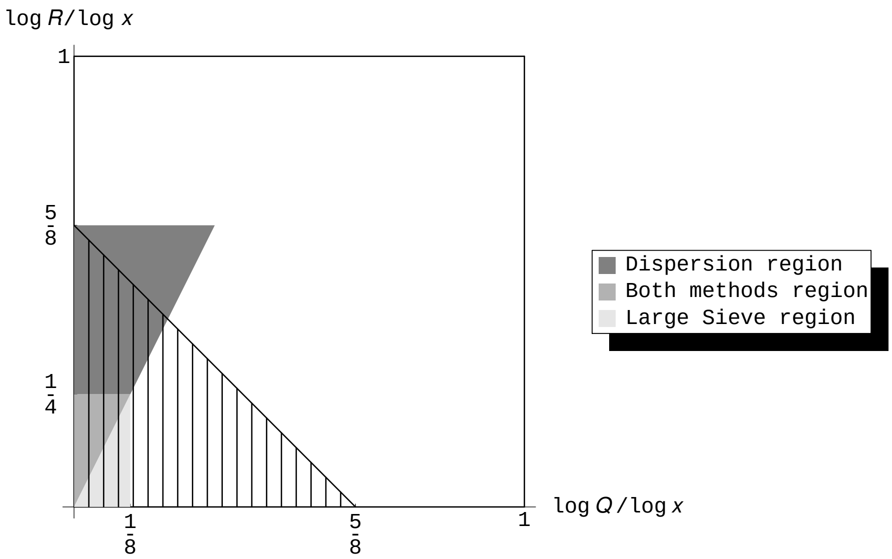

Figure 1. Case $\vartheta = 5/8$

The second example is $\vartheta = 3/4$. This a limit case, since the level reached by the Large sieve method is the trivial one; in fact, the results given in our first paper are non-trivial when $\vartheta > 3/4$ (while our present result is always non-trivial).

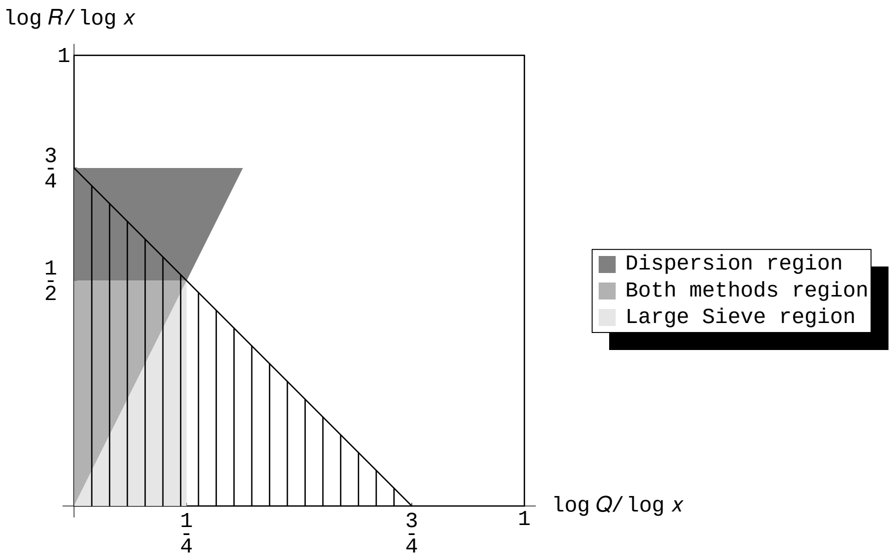

FIGURE 2. Case $\vartheta = 3/4$

The graphic below describes the third case, namely $\vartheta = 7/8$.

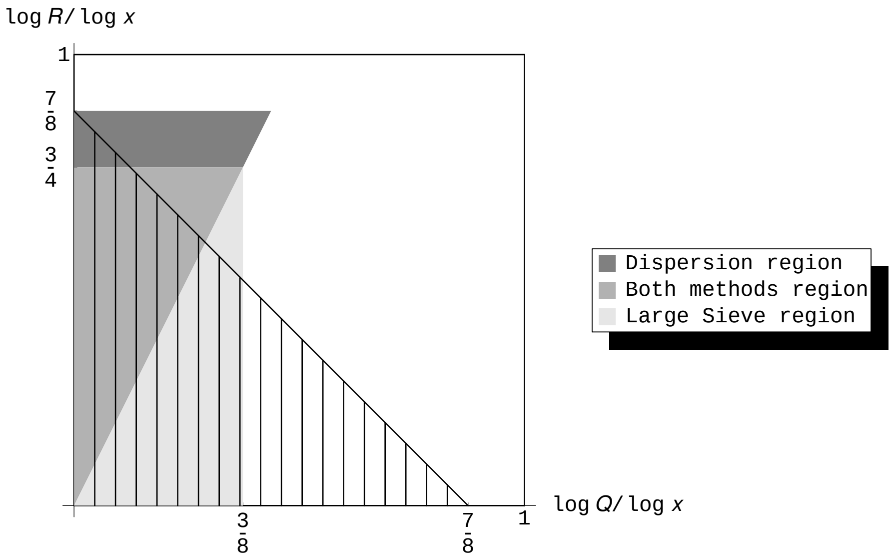

FIGURE 3. Case $\vartheta = 7/8$

The last particular value of ϑ we give is $\vartheta = 1$, which describes the important case of "long intervals".

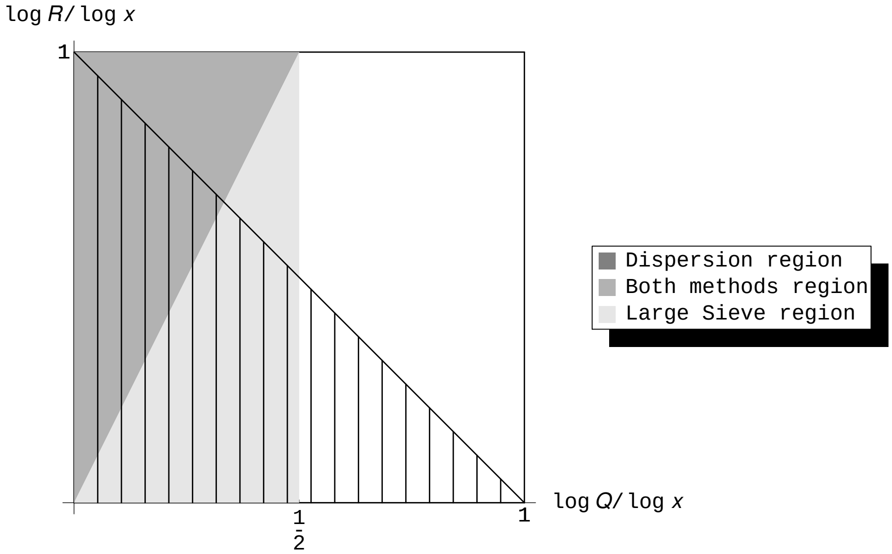

Figure 4. Case $\vartheta = 1$

This graphic shows clearly that, although the distribution level is the same, 3/2, the Large Sieve is stronger than the Dispersion method on long intervals (and, in fact, the Dispersion-region is all contained in the Large Sieve-region).

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