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Zhang-Zagier heights of perturbed polynomials

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Résumé. Dans un précédent article, nous étudions le spectre de la hauteur de Zhang-Zagier [2]. Les progrès accomplis reposaient sur un algorithme qui donnaient des polynômes possédant une petite hauteur. Ici, nous décrivons un nouvel algorithme qui produit des hauteurs encore plus petites. Ceci nous a permis de mettre en évidence un point d'accumulation inférieur à 1,289735. Cette borne est meilleure que la précédente qui était 1,2916674. Après quelques définitions nous détaillons le principe de l'algorithme, les résultats obtenus et la construction explicite qui mène à cette nouvelle borne.

Abstract. In a previous article we studied the spectrum of the Zhang-Zagier height [2]. The progress we made stood on an algorithm that produced polynomials with a small height. In this paper we describe a new algorithm that provides even smaller heights. It allows us to find a limit point less than 1.289735 *i.e.* better than the previous one, namely 1.2916674. After some definitions we detail the principle of the algorithm, the results it gives and the construction that leads to this new limit point.

1. Introduction

Let P be a polynomial in $\mathbb{Z}[z_1, \dots, z_n]$. The Mahler measure of P is then defined by

$$M(P) = \exp\left\{\int_0^1 \dots \int_0^1 \log|P(e^{2i\pi t_1}, \dots, e^{2i\pi t_n})|\, dt_1 \dots dt_n\right\}. \tag{1}$$

If P is a one variable polynomial, $P(z) = a_0 \prod_{j=1}^d (z - \alpha_j)$, Jensen's formula ensures that

$$M(P) = |a_0| \prod_{j=1}^d \max(1, |\alpha_j|). \tag{2}$$

In this case we denote the absolute Mahler measure of P *i.e.* $M(P)^{1/d}$ by $\mathfrak{M}(P)$. If $\alpha \in \overline{\mathbb{Q}}$, we agree that $M(\alpha)$ and $\mathfrak{M}(\alpha)$ represent respectively the Mahler measure and the absolute Mahler measure of the irreducible

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polynomial of α with coefficients in $\mathbb{Z}$. The *Zhang-Zagier height* or simply the *height* of α, denoted by $\mathfrak{H}(\alpha)$, is then defined as $\mathfrak{H}(\alpha) = \mathfrak{M}(\alpha)\mathfrak{M}(1-\alpha)$. From results of Zhang and of Zagier (cf. [10], [9]), we know that if α is an algebraic number different from the roots of $(z^2 - z)(z^2 - z + 1)$,

$$\mathfrak{H}(\alpha) \geqslant \sqrt{\frac{1+\sqrt{5}}{2}} = 1.2720196\ldots \tag{3}$$

In fact Zagier gives also the algebraic numbers which lead to an equality in (3). They are exactly the roots of $\Phi_{10}(z)\Phi_{10}(1-z)$ where $\Phi_{10}(z)$ represents the 10th cyclotomic polynomial.

Our contribution to the study of the Zhang-Zagier height consisted firstly of an improvement of (3), namely

Theorem A. *Let α be an algebraic number. We suppose that α is different from the roots of $(z^2 - z)(z^2 - z + 1)\Phi_{10}(z)\Phi_{10}(1-z)$. Then*

$$\mathfrak{H}(\alpha) \geqslant 1.2817770214.$$

Secondly, we proved that $\mathcal{V} = \{\mathfrak{H}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ admitted a limit point less than 1.2916674. Finally, we discovered an algebraic integer whose height, $1.2875274\ldots$, is less than the smallest previously known *i.e.* $1.2903349\ldots$ [4]. All this work is described in [2].

From this, we are able to do a little better with the help of a new algorithm. In fact the main result of this paper is the following.

Theorem. *The smallest limit point of $\mathcal{V} = \{\mathfrak{H}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ is less than* 1.289735.

Before we prove this theorem, we display some strange relations.

Let

$$\begin{aligned}
A_1(z) &= z,\\
A_2(z) &= z^2 - z + 1,\\
A_3(z) &= \Phi_{10}(z) = z^4 - z^3 + z^2 - z + 1,\\
A_4(z) &= z^8 - 3z^7 + 4z^6 - 2z^5 + x^4 - 2z^3 + 3z^2 - 2z + 1,\\
A_5(z) &= z^{16} - 7z^{15} + 23z^{14} - 45z^{13} + 57z^{12} - 46z^{11} + 19z^{10}\\
&+ 5z^9 - 9z^8 - 10z^7 + 39z^6 - 56z^5 + 52z^4 - 34z^3 + 16z^2 - 5z + 1.
\end{aligned}$$

These polynomials are remarkable according to their height. Indeed A_1 and A_2 have a trivial height, $\mathfrak{H}(A_3)$ is the second point of the spectrum, A_4 has the smallest height for the degree 8 and $\mathfrak{H}(A_5) = 1.2875274\ldots$ is the smallest *known* height greater than 1.2720196... We verify that

$$\begin{aligned}
A_2(z) &= -A_1(z)A_1(1-z)+1,\\
A_3(z) &= A_2^2(z)+A_1(z)A_1^2(1-z),\\
A_4(z) &= A_3(z)A_3(1-z)+A_1(z)A_1^2(1-z)A_2^2(z),\\
A_5(z) &= A_4(z)A_4(1-z)+A_1(z)A_1^2(1-z)A_2^2(z)A_3(z)A_3(1-z).
\end{aligned}$$

So we make the hypothesis that small heights come from "perturbed" polynomials of small height.
The phenomenon seems quite general since it occurs also for the Mahler measure [6] and for the spectrum of $\mathfrak{M}(\alpha)\mathfrak{M}(1/(1-\alpha))\mathfrak{M}(1-1/\alpha)$ [3]. Therefore it is quite reasonable to test polynomials $A_6(z)$ such that

$$\begin{aligned}
A_6(z) = {} & A_5(z)A_5(1-z)\\
& \pm A_1^{a_1}(z)A_1^{a_2}(1-z)A_2^{a_3}(z)A_3^{a_4}(z)A_3^{a_5}(1-z)A_4(z)^{a_6}A_4^{a_7}(1-z)
\end{aligned}$$

with the a_i's in $\mathbb{N}$ verifying $a_1+a_2+2a_3+4(a_4+a_5)+8(a_6+a_7)\leqslant 31$. The results are quite disappointing since we get only a handful of good polynomials; the best corresponding to the choice

$$A_6(z)=A_5(z)A_5(1-z)-A_1^5(z)A_1^5(1-z)A_2^5(z)A_3(z)A_3(1-z)$$

having $\mathfrak{H}(A_6)=1.2906235\ldots$ We also notice that $A_6(z)$ is invariant under the map $z\mapsto 1-z$. Now, every polynomial of even degree symmetric under $z\mapsto 1-z$ can be expressed in terms of

$$X=z(1-z).$$

For instance $\Phi_{10}(z)\Phi_{10}(1-z)=X^4-2X^3+4X^2-3X+1$.

After many unfruitful computations we decided to consider only polynomials in the new variable X and no longer in z. As a consequence we can increase the number of perturbing factors. So we put

$$\begin{aligned}
P_1(X) &= X,\\
P_2(X) &= 1-X,\\
P_3(X) &= X^3+X^2-2X+1,\\
P_4(X) &= X^4-2X^3+4X^2-3X+1,
\end{aligned}$$

$$
\begin{aligned}
P_5(X) &= X^8 - 2X^7 + 4X^6 - 7X^5 + 13X^4 - 16X^3 + 12X^2 - 5X + 1,\\
P_6(X) &= X^8 - 3X^7 + 8X^6 - 16X^5 + 26X^4 - 27X^3 + 17X^2 - 6X + 1,\\
P_7(X) &= X^{12} - 3X^{11} + 8X^{10} - 18X^9 + 36X^8 - 62X^7 + 97X^6\\
&\qquad -123X^5 + 114X^4 - 73X^3 + 31X^2 - 8X + 1,\\
P_8(X) &= X^{12} - 3X^{11} + 7X^{10} - 14X^9 + 30X^8 - 58X^7 + 96X^6\\
&\qquad -123X^5 + 114X^4 - 73X^3 + 31X^2 - 8X + 1,\\
P_9(X) &= X^{16} - 4X^{15} + 10X^{14} - 17X^{13} + 26X^{12} - 47X^{11} + 119X^{10}\\
&\quad -298X^9 + 592X^8 - 878X^7 + 963X^6 - 780X^5 + 464X^4\\
&\qquad -199X^3 + 59X^2 - 11X + 1,
\end{aligned}
$$

and $d_i = \deg P_i$. Note that $P_1(X) = A_1(z)A_1(1-z)$, $P_2(X) = A_2(z)$, $P_4(X) = A_3(z)A_3(1-z)$, $P_5(X) = A_4(z)A_4(1-z)$, $P_9(X) = A_5(z)A_5(1-z)$.

Then we choose a starting polynomial $P(X)$ of degree d, we take integers $(a_i)_{1\leqslant i\leqslant 9}$ and ask simply that

$$\sum_{i=1}^{9} a_i d_i \leqslant d - 1. \tag{4}$$

At this point, for each combination of $(a_i)_{1\leqslant i\leqslant 9}$ verifying (4) we estimate

$$\mathfrak{H}\Big(P\big(z(1-z)\big) \pm P_1^{a_1}P_2^{a_2}P_3^{a_3}P_4^{a_4}P_5^{a_5}P_6^{a_6}P_7^{a_7}P_8^{a_8}P_9^{a_9}\big(z(1-z)\big)\Big)$$

by the method of Graeffe [1]. If this evaluation is rather small we compute precisely its Zhang-Zagier height. Before we did this search, we knew only 9 polynomials whose height is less than 1.29. Now, just for the degree 28 we have more than 120 polynomials with a height less than 1.29. Figures 1 and 2 help us to see the gap we filled.

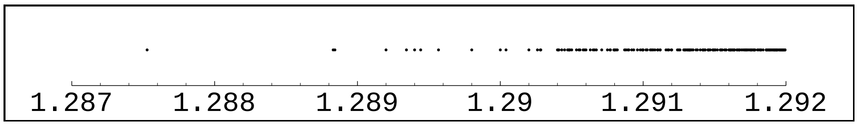

FIGURE 1. Known heights previously.

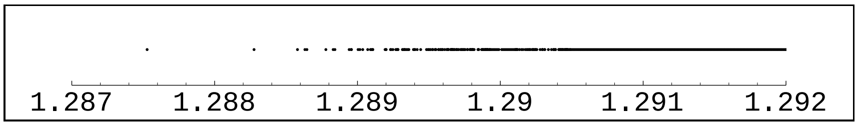

FIGURE 2. Known heights at present.

Table 1 shows the improvements made, if any, for each degree.

Degree	Previous record	New record
10	1.2945155 ...	—
11	1.2939545 ...	1.2916397 ...
12	1.2888421 ...	—
13	1.2926938 ...	—
14	1.2917134 ...	1.2911770 ...
15	1.2914361 ...	—
16	1.2875274 ...	—
17	1.2907680 ...	1.2905866 ...
18	1.2913799 ...	1.2903115 ...
19	1.2926006 ...	—
20	1.2893428 ...	1.2885499 ...
21	1.2904063 ...	1.2898748 ...
22	1.2913747 ...	1.2893658 ...
23	1.2917477 ...	—
24	1.2888365 ...	—
25	1.2893561 ...	1.2890842 ...
26	1.2909655 ...	1.2899285 ...
27	1.2901873 ...	—
28	1.2895016 ...	1.2882759 ...
32	1.2907082 ...	1.2893407 ...
Total	1.2875274 ...	—

TABLE 1.

It is hard to explain precisely why this algorithm gives good results. Nevertheless here is a kind of heuristic argument. In [2], we observed the importance of the resultant for the Zhang-Zagier height. Namely, a polynomial P with $\mathfrak{H}(P)$ small usually has a small resultant with each of the P_i. Now it is obvious that

$$\operatorname{Res}\left(P \pm P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7} P_8^{a_8} P_9^{a_9}, P_i\right) = \operatorname{Res}(P, P_i)$$

for any P_i. So $P \pm P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7} P_8^{a_8} P_9^{a_9}$ are good candidates.

These results convince us of the existence of a limit point of $\mathcal{V}$ less than 1.29 which leads us to the Theorem. To prove it, we use the same techniques as in [2].

2. Proof of the Theorem

With the above notations, we shall use $P_1(X)$, $P_2(X)$, $P_4(X)$, $P_6(X)$ and $P_8(X)$. We introduce also $Q(X) = Q_1(X)Q_2(X)$ with

$$\begin{aligned}Q_1(X) = X^{28} - 7X^{27} + 30X^{26} - 97X^{25} + 269X^{24} - 679X^{23} + 1612X^{22}\\ - 3618X^{21} + 7646X^{20} - 15180X^{19} + 28457X^{18} - 50741X^{17} + 86189X^{16}\\ - 138288X^{15} + 206152X^{14} - 279897X^{13} + 339335X^{12} - 360911X^{11}\\ + 331775X^{10} - 260367X^{9} + 172556X^{8} - 95554X^{7} + 43677X^{6}\\ - 16221X^{5} + 4786X^{4} - 1084X^{3} + 178X^{2} - 19X + 1,\end{aligned}$$

and

$$\begin{aligned}Q_2(X) = X^{28} - 7X^{27} + 30X^{26} - 96X^{25} + 255X^{24} - 586X^{23} + 1212X^{22}\\ - 2360X^{21} + 4573X^{20} - 9148X^{19} + 18749X^{18} - 37783X^{17}\\ + 71770X^{16} - 124910X^{15} + 195848X^{14} - 273368X^{13} + 335981X^{12}\\ - 359545X^{11} + 331349X^{10} - 260271X^{9} + 172542X^{8} - 95553X^{7}\\ + 43677X^{6} - 16221X^{5} + 4786X^{4} - 1084X^{3} + 178X^{2} - 19X + 1.\end{aligned}$$

We verify that

$$\begin{aligned}&\mathfrak{H}(P_1) = \mathfrak{H}(P_2) = 1, && \mathfrak{H}(P_4) = 1.272019650\dots,\\ &\mathfrak{H}(P_6) = 1.297431163\dots, && \mathfrak{H}(P_8) = 1.289442541\dots,\\ &\mathfrak{H}(Q_1) = 1.288275954\dots, && \mathfrak{H}(Q_2) = 1.288646007\dots.\end{aligned}$$

Finally, we need the following lemma proved in [2].

Lemma. *Let P be a polynomial in two variables y and z, such that $\deg_z P > 0$. Let ζ_n be $e^{\frac{2i\pi}{n}}$ and assume that for all n and all k, $P(\zeta_n^k, z)$ is not identically zero. We then have*

$$M(P(y,z))^{(1/\deg_z P)} = \lim_{n\to\infty} \mathfrak{M}\left(\prod_{k=1}^{n} P(\zeta_n^k, z)\right).$$

At present let $(q_1, q_2, \dots, q_5) \in \mathbb{Q}_+^5$ and b a denominator of the q_i's. Then it is clear that

$$\mathfrak{H}\Big(\Big(P_1^{bq_1}P_2^{bq_2}P_4^{bq_3}P_6^{bq_4}P_8^{bq_5}\big(z(1-z)\big)\Big)^n - \Big(Q^b\big(z(1-z)\big)\Big)^n\Big) \tag{5}$$

gives rise to a limit point of $\{\mathfrak{H}(P) \mid P \in \mathbb{Z}[z]\}$ when n tends to infinity. Factorizing the polynomial over $\mathbb{C}[z]$, we see that the limit of (5) when n

tends to infinity is equal to

(6)
$$\lim_{n\to\infty} \mathfrak{M}\left(\prod_{k=1}^{n}\left(P_1^{bq_1}P_2^{bq_2}P_4^{bq_3}P_6^{bq_4}P_8^{bq_5}\big(z(1-z)\big)-\zeta_n^k Q^b\big(z(1-z)\big)\right)\right)^2 .$$

As the degree in z of the polynomial in the product of (6) is

$$D(b) = 2b\max(56, q_1+q_2+4q_3+8q_4+8q_5),$$

the Lemma asserts that (6) equals

$$M\left(P_1^{bq_1}P_2^{bq_2}P_4^{bq_3}P_6^{bq_4}P_8^{bq_5}\big(z(1-z)\big)-yQ^b\big(z(1-z)\big)\right)^{2/D(b)} . \tag{7}$$

Now recall (1) and put $\chi(s) = e^{2i\pi s}(1-e^{2i\pi s})$ so that (7) is

$$\Big[\exp\Big\{\int_0^1\!\!\int_0^1 \log\Big|\Big(P_1^{bq_1}P_2^{bq_2}P_4^{bq_3}P_6^{bq_4}P_8^{bq_5}\big(\chi(s)\big) - e^{2i\pi t}Q^b\big(\chi(s)\big)\Big)\Big|\,dsdt\Big\}\Big]^{2/D(b)}$$

and finally by b changes of variable $t\mapsto t+j/b$, for $j\in [\![1,b]\!]$, we get

$$\Big[\exp\Big\{\int_0^1\!\!\int_0^1 \log\Big|\Big(P_1^{q_1}P_2^{q_2}P_4^{q_3}P_6^{q_4}P_8^{q_5}\big(\chi(s)\big) - e^{2i\pi t}Q\big(\chi(s)\big)\Big)\Big|\,dsdt\Big\}\Big]^{2b/D(b)} \tag{8}$$

The trick of considering z as a simple parameter together with formula (2) imply that (8) is nothing but

$$\Big[M\left(Q\big(z(1-z)\big)\right) \times\exp\Big\{\int_0^1 \log^+\Big|\frac{P_1^{q_1}P_2^{q_2}P_4^{q_3}P_6^{q_4}P_8^{q_5}\big(\chi(s)\big)}{Q\big(\chi(s)\big)}\Big|\,ds\Big\}\Big]^{2b/D(b)}, \tag{9}$$

where $\log^+(w)$ is $\max(0,\log|w|)$ as usual. Then we search the best q_i's in order to have the smallest limit point possible. As we can easily compute the simple integral of (9) by a Riemann sum, we test several choices of $q_i's$. Thanks to these computations, we produce a limit point less than 1.29 thus better than the previously known *i.e.* 1.2916674. The best combination found is

$$q_1 = 13.1,\ q_2 = 10.6,\ q_3 = 3.2,\ q_4 = 1.15,\ q_5 = 0.24$$

which yields a limit point less than $\ell = 1.289735$. Of course this limit point ℓ concerns only polynomials but fortunately we can deduce from them a subsequence of irreducible polynomials with a height less than ℓ (see [2, Lemmas 3, 4, 5]).

Let us say a few words about the polynomials involved in (9). While the selection of P_1, P_2, P_4, P_6, P_8 among $P_1, \dots, P_9$ is forced by the search of the minimum of (9), the choice of the polynomials to be perturbed is more arbitrary. In fact, we take $Q_1(X)$ and $Q_2(X)$ according to their very small height and also because we remarked that a product of P_i's, namely $P_1^7(X)P_2^5(X)P_4(X)$, divides $Q_1(X) - Q_2(X)$. This point seems determinant, however we do not understand why it is so important.

The results confirm that our previous algorithm was far from being exhaustive since a great number of heights were missed. Nevertheless, no new height less than 1.2875274 appears, so that we still do not know the second non-trivial point, if any, of $\mathcal{V}$. It would be desirable to improve our knowledge of the set of $\mathfrak{H}(\alpha)$ as it was done for the Mahler measure of non-reciprocal polynomials [7] and for the height of totally real algebraic integers [8] and [5], but we have the feeling that this will be difficult without new ideas.

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