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The size function h^0 for quadratic number fields

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Résumé. Nous étudions le cas quadratique d'une conjecture énoncée par Van der Geer et Schoof prédisant le comportement de certaines fonctions définies sur le groupe des diviseurs d'Arakelov d'un corps de nombres. Ces fonctions correspondent à la fonction usuelle h^0 relative aux diviseurs de courbes algébriques. Nous montrons qu'elles atteignent leur maximum en les diviseurs d'Arakelov principaux, et nulle part ailleurs. De plus, nous introduisons une fonction $\widetilde{k^0}$, qui est l'analogue de $\exp h^0$ sur le groupe de classes, et on montre que cette fonction atteint elle-aussi son maximum en la classe triviale.

Abstract. We study the quadratic case of a conjecture made by Van der Geer and Schoof about the behaviour of certain functions which are defined over the group of Arakelov divisors of a number field. These functions correspond to the standard function h^0 for divisors of algebraic curves and we prove that they reach their maximum value for principal Arakelov divisors and nowhere else. Moreover, we consider a function $\widetilde{k^0}$, which is an analogue of $\exp h^0$ defined on the class group, and we show it also assumes its maximum at the trivial class.

Introduction

Given a number field F, let R_F be the set of its real places and let C_F be the set of its complex places, *i.e.*, C_F is given by picking one of each two distinct conjugate embeddings $F \hookrightarrow \mathbb{C}$. An *Arakelov divisor* is a couple $D = (J_D, (\alpha, \beta))$ where J_D is a fractional ideal of the ring of integers $\mathcal{O}_F$ and $(\alpha, \beta) \in \mathbb{R}^{R_F} \times \mathbb{R}^{C_F}$. We call J_D the ideal, or even the finite, part of D; the vector (α, β) is said to be the infinite part of D, as its components correspond to the infinite primes of F. We denote by $\mathcal{N}(J_D)$ the ordinary norm of J_D as a fractional ideal, and we define the *norm* of D in the following way:

$$\mathcal{N}(D) = \frac{(\prod_{\rho \in R_F} e^{\alpha_\rho})(\prod_{\gamma \in C_F} e^{\beta_\gamma})}{\mathcal{N}(J_D)}$$

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where we write $\alpha = (\alpha_\rho)_{\rho\in R_F}$ and $\beta = (\beta_j)_{\gamma\in C_F}$, and we define the *degree* of D as $\deg(D) = \log \mathcal{N}(D)$. One should observe that the Arakelov divisors of a number field form a group, which will be denoted by $\mathcal{D}iv(F)$, and that zero-degree Arakelov divisors are a subgroup $\mathcal{D}iv^0(F)$ of $\mathcal{D}iv(F)$. Another subgroup of $\mathcal{D}iv(F)$ is $\mathcal{PD}iv(F)$, the set of *principal* Arakelov divisors:

$$(x) = \left(x^{-1}\mathcal{O}_F, ((-\log|\rho(x)|)_{\rho\in R_F}, (-2\log|\gamma(x)|)_{\gamma\in C_F})\right) \qquad x \in F^*$$

where $|z|$ stands for the standard absolute value of the complex number z. The product formula for number fields says that the inclusion $\mathcal{PD}iv(F) \subset \mathcal{D}iv^0(F)$ holds. We define the Picard group of F as $\mathcal{P}ic(F) = \dfrac{\mathcal{D}iv(F)}{\mathcal{PD}iv(F)}$ and then we put $\mathcal{P}ic^0(F) = \dfrac{\mathcal{D}iv^0(F)}{\mathcal{PD}iv(F)}$. We observe that $\mathcal{P}ic^0(F)$ fits into the exact sequence

$$(\star) \qquad 0 \longrightarrow T \longrightarrow \mathcal{P}ic^0(F) \xrightarrow{\pi} \mathcal{C}l(F) \longrightarrow 0$$

where T is a real torus of dimension $\#(R_F) + \#(C_F) - 1$. More details about this sequence are explained in [3].

One can view the ring $\mathcal{O}_F$, as well as any fractional ideal, as a real lattice in $\mathbb{R}^{R_F} \times \mathbb{C}^{C_F}$, via the embeddings corresponding to the infinite primes. When S is a subset of a lattice in a Euclidean space, we attach to it the real number

$$k^0(S) = \sum_{x\in S} e^{-\pi\|x\|^2}.$$

Some properties of this function have been studied in [1]. Following this direction, one can consider an Arakelov divisor D just as J_D endowed with a metric specified by the infinite part of D, in the following way: given $x \in J_D$, we put

$$\|x\|^2 = \sum_{\rho\in R_F} \frac{|\rho(x)|^2}{e^{2\alpha_\rho}} + 2\sum_{\gamma\in C_F} \frac{|\gamma(x)|^2}{e^{\beta_\gamma}}.$$

We now define $k^0(D)$ as the number $k^0(J_D)$. An equivalent definition is the following: for $x \in F$ and $D = (J_D, (\alpha, \beta)) \in \mathcal{D}iv(F)$, put

$$\Phi_D(x) = \left(\left(\frac{\rho(x)}{e^{\alpha_\rho}}\right)_{\rho\in R_F}, \left(\sqrt{\frac{2}{e^{\beta_\gamma}}}\gamma(x)\right)_{\gamma\in C_F}\right)$$

and consider the lattice $\widetilde{D} = \{\Phi_D(x) \mid x \in J_D\}$. Then we set $k^0(D) = k^0(\widetilde{D})$. In this sense, changing the infinite part of D produces different lattices lying in the same, fixed, Euclidean space $\mathbb{R}^{R_F} \times \mathbb{C}^{C_F}$, with the standard inner product given by

$$\|(a,b)\|^2 = \|((a_\rho),(b_\gamma))\|^2 = \sum_{\rho\in R_F} a_\rho^2 + \sum_{\gamma\in C_F} b_\gamma\overline{b_\gamma}.$$

This will be our favourite point of view. An important remark to be made is that the value of the function k^0 on the Arakelov divisor D depends just upon the class of D in $\mathcal{P}ic(F)$. We shall be concerned with the following conjecture, which is stated in [3]:

Conjecture. *Let F be number field which is Galois over $\mathbb{Q}$ or over an imaginary quadratic number field. Then the function k^0 on $\mathcal{P}ic^0(F)$ assumes its maximum on the trivial class $\mathcal{O}_F$.*

We recall from [3] that the function $h^0 = \log k^0$ can be seen as an analogue of the classical function h^0 over the divisors of a curve: indeed, even in this setting, a Riemann-Roch theorem holds. Under this analogy, the above conjecture corresponds to the standard property of algebraic curves, which says that $h^0(D) = 0$ for any non principal zero-degree divisor D and $h^0(D) = 1$ when D is principal. We shall prove this fact in the quadratic case.

Theorem 1. *Let F be a quadratic number field. The function k^0 on $\mathcal{P}ic^0(F)$ has a unique global maximum at the identity element.*

When F is a complex quadratic field, the proof reduces to estimate the number of representations of a given integer by binary quadratic forms. In the real case, we look at the lattices $\widetilde{D}$, recalling that the main contributions to k^0 come from the shortest lattice vectors. The key point is that, for $x \in \mathcal{O}_F - \{0\}$, we have $\|\Phi_D(x)\|^2 \geqslant 2$ and, when x is not a unit, $\|\Phi_D(x)\|^2 \geqslant 4$. From this we get some bounds for the contributions to $k^0(D)$ from distant poins of $\widetilde{D}$, which permit to reduce the problem to a local one.

In the last part, we define a function $\widetilde{k^0}$ on $\mathcal{C}l(F)$ as follows:

$$\widetilde{k^0}([J]) = \int_{\pi^{-1}([J])} k^0(D)\, dD$$

where π is the projection appearing in the exact sequence $(\star)$, the symbol $[J]$ denotes the class of the ideal J and integration is made with respect to the Haar measure on $\pi^{-1}([J])$ normalized to $\mathcal{R}$, the regulator of F. This is a natural analogue of k^0 on the class group and it should be reasonable to expect it still has the maximum value at the trivial class. Indeed we prove the following:

Theorem 2. *Let F be a quadratic number field. Then the function $\widetilde{k^0}$ on $\mathcal{C}l(F)$ assumes its maximum at the trivial class and nowhere else.*

About this subject we also would like to mention [4], where special emphasis is put on computational aspects, and [2], which is focused on a harmonic analysis approach. Moreover, the author wishes to thank Prof. René Schoof for his help.

1. Complex quadratic fields

When F is a complex quadratic field, the group $\mathcal{P}ic^0(F)$ is finite and coincides with the class group $\mathcal{C}l(F)$. This follows from the sequence $(\star)$. Given a fractional ideal J of $\mathcal{O}_F$, we have $k^0(J) = \sum_{x\in J} e^{-2\pi \frac{\mathcal{N}(x)}{\mathcal{N}(J)}}$.

Proposition 1.1. *Let F be a complex quadratic number field. Then the function k^0 attains its maximum at the identity element of $\mathcal{P}ic^0(F) = \mathcal{C}l(F)$ and nowhere else.*

Proof. As a first step, observe that, given an ideal J in $\mathcal{O}_F$, we have that

$$k^0(J) = \sum_{m,n\in\mathbb{Z}} e^{-2\pi\varphi(m,n)}$$

where $\varphi(m,n) = \dfrac{\mathcal{N}(\alpha m+\beta n)}{\mathcal{N}(J)}$, with $\{\alpha,\beta\}$ a $\mathbb{Z}$-basis for J, is a representative for the $\mathbf{SL}_2(\mathbb{Z})$-orbit of quadratic forms associated to the ideal class of J. The form φ is primitive, positive definite and has the same discriminant Δ as the field F. It represents the integer 1 if and only if it is equivalent to the principal form, *i.e.*, when J is a principal ideal.

Clearly, we have $k^0(\mathcal{O}_F) > 1+\sum_{n\geqslant 1}\dfrac{2}{e^{2n^2\pi}} > 1+\dfrac{2}{e^{2\pi}}$. Hence it is enough to show that, for a form φ which is not equivalent to the principal one, we have $\sum_{m,n\in\mathbb{Z}} e^{-2\pi\varphi(m,n)} < 1+\dfrac{2}{e^{2\pi}}$. Moreover, when $\mathcal{O}_F$ is a principal ideal domain, the statement is trivial, so we may assume $\Delta \leqslant -15$. If we put $\ell = e^{-2\pi}$, we may write the above series as

$$k^0(J) = 1+\sum_{h\geqslant 2} R(h)\ell^h,$$

where $R(h)$ stands for the number of representations of the number h by the integral binary form φ. Now, in order to write an upper bound for $R(h)$, let $PR(h)$ denote the number of *proper* representations of h by φ, *i.e.*, those by coprime integers. Since $\Delta_F \leqslant -15$, we have from [5, 9.3] that $PR(h) \leqslant 2s$, where

$$s = \#\big\{x|\ x^2 \equiv \Delta_F \pmod{4h} \text{ with } 0\leqslant x < 2h\big\}.$$

So we have $PR(h) \leqslant 2h$ and therefore

$$R(h) \leqslant \sum_{n\geqslant 1} PR(\frac{h}{n^2}) < \frac{\pi^2}{3}h.$$

Hence, we obtain that

$$k^0(J) < 1+\frac{\pi^2}{3}\sum_{h\geqslant 2} h\ell^h = 1+\frac{\pi^2\ell^2(2-\ell)}{3(1-\ell)^2} < 1.00003 < 1+2\ell,$$

as required. □

2. Real quadratic fields

Let F be a real quadratic field of class number h and regulator $\mathcal{R}$. The connected component of the identity of the group $\mathcal{P}ic^0(F)$, will be called the *principal component*, which is the subgroup whose representatives in $\mathcal{D}iv^0(F)$ are Arakelov divisors D with J_D a principal ideal . We notice that this group has h component and that the principal component may be identified with a circle $\frac{\mathbb{R}}{\mathcal{R}\mathbb{Z}}$. Indeed, for each of these elements, there is a unique $t \in [0, \mathcal{R})$ such that $E_t = (\mathcal{O}_F, (t, -t))$ lies in the same $\mathcal{P}ic^0(F)$-class. We can summarize the above considerations by writing the exact sequence $(\star)$ for the real quadratic case:

$$0 \longrightarrow \frac{\mathbb{R}}{\mathcal{R}\mathbb{Z}} \longrightarrow \mathcal{P}ic^0(F) \xrightarrow{\pi} \mathcal{C}l(F) \longrightarrow 0.$$

We remark that, for the lattice $\widetilde{E_t}$, we have $\widetilde{E_t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \widetilde{E_0}$, and therefore

$$\widetilde{E_{\mathcal{R}}} = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \widetilde{E_0} = \begin{cases} \widetilde{E_0} & \text{if } \mathcal{N}(\varepsilon) = 1 \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \widetilde{E_0} & \text{if } \mathcal{N}(\varepsilon) = -1\,, \end{cases}$$

where ε is a fundamental unit of $\mathcal{O}_F$. Moreover, we have that

$$\widetilde{E_{-t}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{E_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{E_t}\,,$$

since quadratic extensions are normal. This gives a picture of the $\mathcal{R}$-periodicity and the symmetry of the function $t \mapsto k^0(E_t)$.

Lemma 2.1. *Let $S \neq \{(0,0)\}$ be a subset of a lattice in $\mathbb{R}^2$ which is symmetric with respect to the origin, such that, for any pair of distinct points $x, y \in S$, we have $\|x\| \geqslant 2$ and $\|x - y\| \geqslant \sqrt{2}$. Then we have $k^0(S) < \frac{3}{10^5}$ and $\sum_{x \in S} \|x\|^4 e^{-\pi \|x\|^2} < \frac{5.1}{10^4}$.*

Proof. Given $r \geqslant 2$ and $\delta < \sqrt{2}$, let us consider the set

$$A_{r,\delta} = \big\{x \in S |\ r \leqslant \|x\| \leqslant r + \delta\big\}.$$

Now, the angle between two points in $A_{r,\delta}$, seen as two vectors in $\mathbb{R}^2$, has to be at least $2 \arcsin \sqrt{\frac{2 - \delta^2}{4r(r+\delta)}}$. Indeed, this angle corresponds to a pair of points x and y such that $\|x - y\| = \sqrt{2}$, with $\|x\| = r$ and $\|y\| = r + \delta$. As the set S is symmetric with respect to the origin, and since the inequality $2 \arcsin \sqrt{\frac{2 - \delta^2}{4r(r+\delta)}} > \sqrt{\frac{2 - \delta^2}{r(r+\delta)}}$ holds, the number of elements in $A_{r,\delta}$ is at most $2(\lceil \nu_{r,\delta} \rceil - 1)$, where $\nu_{r,\delta} = \pi\sqrt{\dfrac{r(r+\delta)}{2-\delta^2}}$. Indeed, the number of points of $A_{r,\delta}$ which are contained in a half-plane cannot exceed π divided by the minimal angle. In order to have $\nu_{2,\delta} = 5$, we choose $\delta = \dfrac{-\pi^2 + \sqrt{\pi^4 - 100\pi^2 + 1250}}{25} \simeq 0.3646$, so that the contribution to $k^0(S)$ from the points in $A_{2,\delta}$ is at most $\dfrac{8}{e^{4\pi}}$. Therefore we obtain:

$$k^0(S) < \frac{8}{e^{4\pi}} + 2\sum_{c \geqslant 1} \frac{\lceil \nu_{2+c\delta,\delta} \rceil - 1}{e^{\pi(2+c\delta)^2}} < \frac{3}{10^5} \ ,$$

and, as the function $t \mapsto \dfrac{t^2}{e^{\pi t}}$ decreases for $t \geqslant 4$,

$$\sum_{x \in S} \|x\|^4 e^{-\pi\|x\|^2} < \frac{8 \cdot 16}{e^{4\pi}} + 2\sum_{c \geqslant 1} \frac{(\lceil \nu_{2+c\delta,\delta} \rceil - 1)(2+c\delta)^4}{e^{\pi(2+c\delta)^2}} < \frac{5.1}{10^4}.$$

□

To simplify the notation, we now fix an embedding $F \hookrightarrow \mathbb{R}$ and we look at the elements of F as real numbers. Moreover, for each $x \in F$, we denote by $\bar{x}$ its Galois conjugate.

Corollary 2.2. *Let F be a real quadratic number field and $D \in \mathcal{D}iv^0(F)$. If J_D is not a principal ideal, then we have $k^0(D) < k^0(\mathcal{O}_F)$.*

Proof. Let $D = (J_D, (a,b)) \in \mathcal{D}iv^0(F)$. The condition $\deg(D) = 0$ means, by definition, that $e^a e^b = \mathcal{N}(J_D)$. Then we have

$$\begin{aligned} k^0(D) &= \sum_{x \in J_D} e^{-\pi(e^{-2a}x^2 + e^{-2b}\bar{x}^2)} \\ &= \sum_{x \in J_D} e^{-\frac{\pi}{\mathcal{N}(J_D)}\left(\left(\frac{\sqrt{\mathcal{N}(J_D)}}{e^a}x\right)^2 + \left(\frac{\sqrt{\mathcal{N}(J_D)}}{e^b}\bar{x}\right)^2\right)} = \sum_{x \in J_D} e^{-\frac{\pi}{\mathcal{N}(J_D)}(y_x^2 + y_{\bar{x}}^2)}, \end{aligned}$$

where we put $y_x = \dfrac{\sqrt{\mathcal{N}(J_D)}}{e^a}x$ and $y_{\bar{x}} = \dfrac{\sqrt{\mathcal{N}(J_D)}}{e^b}\bar{x}$. For every non-zero $x \in J_D$, we have that $|y_x\, y_{\bar{x}}| = |\mathcal{N}(x)| \geqslant 2\mathcal{N}(J_D)$, since J_D is not a principal ideal. Hence $y_x^2 + y_{\bar{x}}^2 \geqslant 4\mathcal{N}(J_D)$ for all non-zero $x \in \mathcal{O}_F$, so that $\widetilde{D}$ is a lattice whose non-zero shortest vector has length at least 2. In particular, we can apply lemma 2.1 and conclude that $k^0(D) < 1 + \frac{3}{10^5} < 1 + \frac{2}{e^{2\pi}} < k^0(\mathcal{O}_F)$. □

We observe that we also could have deduced proposition 1.1 from lemma 2.1, just in the same way as for corollary 2.2. We preferred to give the other proof, since it contains essentially the same argument we shall use in proposition 3.1.

Corollary 2.3. *Let F be a real quadratic number field, let $\mathcal{R}$ be its regulator and $E_t = (\mathcal{O}_F, (t,-t)) \in \mathcal{D}iv^0(F)$. If $t \in [\frac{1}{2}\log\frac{27}{25}, \frac{\mathcal{R}}{2}]$, then we have $k^0(E_t) < k^0(\mathcal{O}_F)$.*

Proof. Let Δ_F be the discriminant of F and ε be the fundamental unit of $\mathcal{O}_F$ larger than 1. For $s \in [1,\varepsilon]$, consider the set

$$B_s = \left\{ w_x = \left(\sqrt{s}x, \frac{\bar{x}}{\sqrt{s}}\right) \mid 0 < \|w_x\| < 2,\, x \in \mathcal{O}_F \right\} \subset \mathbb{R}^2.$$

Observe that if $w_x \in B_s$, then $|\mathcal{N}(x)| = 1$, *i.e.*, $x \in \mathcal{O}_F^*$. By Dirichlet's unit theorem, these element are, up to sign, powers of the fundamental unit ε. Hence, to determine the set B_s amounts to finding the solutions of the inequality

$$s\varepsilon^{2m} + \frac{1}{s\varepsilon^{2m}} < 4 \qquad \text{with } s \in [1,\varepsilon].$$

For $\Delta_F \geqslant 8$, we have $\varepsilon \geqslant 1+\sqrt{2}$, so there exist solutions for this only if $m \in \{0,-1\}$. The case $m = 0$ corresponds to the shortest vector of the lattice $\widetilde{E_t}$, for small values of t. Moreover, we have that $w_{\frac{1}{\varepsilon}} \in B_s$ if and only if $s > (2-\sqrt{3})\varepsilon^2$.

Now, for $\Delta_F \neq 5, 8, 13$, we have $\varepsilon \geqslant 2+\sqrt{3}$, so that $(2-\sqrt{3})\varepsilon^2 \geqslant \varepsilon$, hence $B_s = \{w_1, w_{-1}\}$ for all $s \in [1,\varepsilon]$. Therefore, in this case, we can apply lemma 2.1 to $\widetilde{E_t}$ for $t \in [\frac{1}{2}\log\frac{27}{25}, \frac{\mathcal{R}}{2}]$ and obtain:

$$k^0(E_t) < 1 + \frac{2}{e^{\pi(s+\frac{1}{s})}} + \frac{3}{10^5} \leqslant 1 + \frac{2}{e^{(\frac{27}{25}+\frac{25}{27})\pi}} + \frac{3}{10^5} < 1 + \frac{2}{e^{2\pi}} < k^0(\mathcal{O}_F).$$

For $\mathbb{Q}(\sqrt{2})$, with $\varepsilon = 1+\sqrt{2}$, and for $\mathbb{Q}(\sqrt{13})$, with $\varepsilon = \dfrac{3+\sqrt{13}}{2}$, we have that $s + \dfrac{1}{s} \geqslant 12 - 4\sqrt{6}$ and $\frac{\varepsilon^2}{s} + \frac{s}{\varepsilon^2} \geqslant 2\sqrt{2}$ for every $s \in [(2-\sqrt{3})\varepsilon^2, \varepsilon]$. Therefore lemma 2.1 implies that, for all $t \in [\frac{1}{2}\log\frac{27}{25}, \frac{\mathcal{R}}{2}]$,

$$k^0(E_t) < 1 + \frac{2}{e^{(12-4\sqrt{6})\pi}} + \frac{2}{e^{(2\sqrt{2})\pi}} + \frac{3}{10^5} < 1 + \frac{2}{e^{2\pi}} < k^0(\mathcal{O}_F).$$

At last, we deal with $\mathbb{Q}(\sqrt{5})$, where $\varepsilon = \dfrac{1+\sqrt{5}}{2}$. As a first step, we point out that, in this case, for all $s \in [1,\varepsilon]$, we have that $B_s = \{w_{\frac{1}{\varepsilon}}, w_1, w_\varepsilon, w_{-\frac{1}{\varepsilon}}, w_{-1}, w_{-\varepsilon}\}$ and hence

$$k^0(B_s) = 2\left(\frac{1}{e^{\pi(\frac{\varepsilon^2}{s}+\frac{s}{\varepsilon^2})}} + \frac{1}{e^{\pi(s+\frac{1}{s})}} + \frac{1}{e^{\pi(s\varepsilon^2+\frac{1}{s\varepsilon^2})}} \right).$$

As a second step, we notice that $s \mapsto k^0(B_s)$ is a decreasing function when $s \in [1,\varepsilon]$: this fact can be verified by direct calculations on the derivative.

Then we obtain that for all $t \in [\frac{1}{2}\log\frac{27}{25}, \frac{\mathcal{R}}{2}]$

$$k^0(E_t) < 1 + k^0(B_{e^{2t}}) + \frac{3}{10^5} \leqslant 1 + k^0(B_{\frac{27}{25}}) + \frac{3}{10^5} < k^0(\mathcal{O}_{\mathbb{Q}(\sqrt{5})}).$$

Our assertion is proved. □

At this point, in order to finish the proof of theorem 1, it is enough to show the following:

Lemma 2.4. *Let F be a real quadratic number field and, for every $t \in \mathbb{R}$, set $E_t = (\mathcal{O}_F, t, -t) \in \mathcal{D}iv^0(F)$. The function $k^0(E_t)$ is strictly decreasing when $t \in [0, \frac{1}{2}\log\frac{27}{25}]$.*

Proof. Define, for $s \in \mathbb{R}_+$, the function

$$f(s) = \sum_{x \in \mathcal{O}_F} e^{-\pi\left(s\bar{x}^2 + \frac{x^2}{s}\right)}.$$

Our next task is to see that f is decreasing on the interval $[1, \frac{27}{25}]$. Observe that, since $f(s) = f(\frac{1}{s})$, we have $f'(1) = 0$, hence it is enough to check that $f''(s) < 0$ for all $s \in [1, \frac{27}{25}]$. Therefore, we have to prove that

$$f''(s) = -\pi \sum_{x \in \mathcal{O}_F} \left(\frac{2x^2}{s^3} - \pi\left(\bar{x}^2 + \frac{x^2}{s}\right)^2\right) e^{-\pi\left(s\bar{x}^2 + \frac{x^2}{s}\right)} < 0,$$

that amounts to showing that

$$\sum_{x \in \mathcal{O}_F} \left(s^2\bar{x}^2 - x^2\right)^2 e^{-\pi\left(s\bar{x}^2 + \frac{x^2}{s}\right)} < \frac{2\,s}{\pi} \sum_{x \in \mathcal{O}_F} x^2\, e^{-\pi\left(s\bar{x}^2 + \frac{x^2}{s}\right)},$$

which may be rewritten as:

$$\begin{aligned} g(s) &= s \sum_{x \in \mathcal{O}_F} \left(s\bar{x}^2 + \frac{x^2}{s}\right)^2 e^{-\pi\left(s\bar{x}^2 + \frac{x^2}{s}\right)} \\ &< 2 \sum_{x \in \mathcal{O}_F} \left(\frac{x^2}{\pi} + 2s\mathcal{N}(x^2)\right) e^{-\pi\left(s\bar{x}^2 + \frac{x^2}{s}\right)} = h(s). \end{aligned}$$

Here, the left hand side may also be viewed be as

$$g(s) = \sum_{x \in L_s} \|x\|^4 e^{-\pi\|x\|^2}$$

where L_s is the lattice $\widetilde{E_{\frac{\log(s)}{2}}}$. As in the proof of lemma 2.3, we have $w_x \in B_s$ only when $x \in \mathcal{O}_F^*$.

Now, for $F \neq \mathbb{Q}(\sqrt{5})$, we have $B_s = \{w_1, w_{-1}\}$ for each $s \in [1, \frac{27}{25}]$. Therefore, for every $s \in [1, \frac{27}{25}]$, it happens that

$$g(s) < 2s(s + \frac{1}{s})^2\, e^{-\pi(s + \frac{1}{s})} + \frac{5.1}{10^4}.$$

Thus, since we clearly have $h(s) > 4(\frac{1}{\pi} + 2s)e^{-\pi(s+\frac{1}{s})}$, it is enough to see that, for all $s \in [1, \frac{27}{25}]$,

$$2s(s - \frac{1}{s})^2 \, e^{-\pi(s+\frac{1}{s})} + \frac{5.1}{10^4} < \frac{4}{\pi} \, e^{-\pi(s+\frac{1}{s})},$$

which is readily checked.

When $F = \mathbb{Q}(\sqrt{5})$, there are minor complications due to the fact that, in this case, $B_s = \{w_{\frac{1}{\varepsilon}}, w_1, w_\varepsilon, w_{-\frac{1}{\varepsilon}}, w_{-1}, w_{-\varepsilon}\}$. Hence we have

$$\begin{aligned} g(s) &< 2s\left(\frac{(s+\frac{1}{s})^2}{e^{\pi(s+\frac{1}{s})}} + \frac{(\frac{\varepsilon^2}{s}+\frac{s}{\varepsilon^2})^2}{e^{\pi(\frac{\varepsilon^2}{s}+\frac{s}{\varepsilon^2})}} + \frac{(s\varepsilon^2+\frac{1}{s\varepsilon^2})^2}{e^{\pi(s\varepsilon^2+\frac{1}{s\varepsilon^2})}}\right) + \frac{5.1}{10^4} \\ h(s) &> \frac{4+8s}{\pi e^{\pi(s+\frac{1}{s})}} + \frac{4(\frac{\varepsilon^2}{\pi}+2s)}{e^{\pi(\frac{s}{\varepsilon^2}+\frac{\varepsilon^2}{s})}} + \frac{4(\frac{1}{\pi\varepsilon^2}+2s)}{e^{\pi(\varepsilon^2 s+\frac{1}{\varepsilon^2 s})}} \ . \end{aligned}$$

where $\varepsilon = \frac{1+\sqrt{5}}{2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{5})$. Thus, it is sufficient to see that

$$\begin{aligned} &\frac{2s(s-\frac{1}{s})^2}{e^{\pi(s+\frac{1}{s})}} + 2s\left(\frac{(\frac{\varepsilon^2}{s}+\frac{s}{\varepsilon^2})^2}{e^{\pi(\frac{\varepsilon^2}{s}+\frac{s}{\varepsilon^2})}} + \frac{(s\varepsilon^2+\frac{1}{s\varepsilon^2})^2}{e^{\pi(s\varepsilon^2+\frac{1}{s\varepsilon^2})}}\right) + \frac{5.1}{10^4} \\ &\quad < \frac{4}{\pi e^{\pi(s+\frac{1}{s})}} + \frac{4(\frac{\varepsilon^2}{\pi}+2s)}{e^{\pi(\frac{s}{\varepsilon^2}+\frac{\varepsilon^2}{s})}} + \frac{4(\frac{1}{\pi\varepsilon^2}+2s)}{e^{\pi(\varepsilon^2 s+\frac{1}{\varepsilon^2 s})}} \end{aligned}$$

for all $s \in [1, \frac{27}{25}]$. Such inequality can be obtained from direct computation, for instance by splitting $[1, \frac{27}{25}]$ as $[1, \frac{103}{100}] \cup [\frac{103}{100}, \frac{53}{50}] \cup [\frac{53}{50}, \frac{27}{25}]$ and, for each interval, evaluating maxima of terms on the left and minima of terms on the right. $\square$

3. A function on $\mathcal{C}l(F)$

When F is a complex quadratic field, $\mathcal{C}l(F) = \mathcal{P}ic^0(F)$, so the function $\widetilde{k^0}$ coincides with k^0. Indeed, the properties of this function are very similar in both the real and the complex case.

Let F be a real quadratic field having discriminant Δ and let J be an ideal in the ring of integers $\mathcal{O}_F$. For $h \in \mathbb{Z}$, we define the set

$$S(h) = \big\{x \in J |\ \mathcal{N}(x) = h\mathcal{N}(J)\big\},$$

which is either empty or infinite. In this second case, the group $\mathcal{O}_F^+$ of units with positive norm acts on $S(h)$ by multiplication and the number $Or(h)$ of orbits is finite. Elements in $S(h)$ correspond to the various representations of h by the integral quadratic form $\varphi(m,n) = \dfrac{\mathcal{N}(m\alpha + n\beta)}{\mathcal{N}(J)}$, where $\{\alpha, \beta\}$ is any $\mathbb{Z}$-basis of J. To write an upper bound for $Or(h)$, we first consider the number $POr(h)$ of *proper* orbits, *i.e.*, those which do not contain elements of the kind mx, with $x \in J$ and $1 < m \in \mathbb{Z}$. This amounts to considering the orbits of proper solutions of the equation $\varphi(m,n) = h$. Following [5, 9.3], each orbit is characterized by an integer x which is a solution to the congruence $x^2 \equiv \Delta \pmod{4|h|}$ such that $0 \leqslant x < 2|h|$. Therefore we have $POr(h) \leqslant |h|$ and

$$Or(h) \leqslant \sum_{n \geqslant 1} POr(\frac{h}{n^2}) < \sum_{n \geqslant 1} \frac{|h|}{n^2} = \frac{\pi^2}{6}|h|.$$

We can now prove theorem 2 by showing the following.

Proposition 3.1. *Let F be a real quadratic field. Then the function $\widetilde{k^0}$ attains its maximum at the trivial class and nowhere else.*

Proof. Let ε be the fundamental unit of $\mathcal{O}_F$. Set $\gamma = 1$ if $\mathcal{N}(\varepsilon) = 1$ and $\gamma = 2$ if $\mathcal{N}(\varepsilon) = -1$. Let J be an ideal of $\mathcal{O}_F$. We have that:

$$\begin{aligned}
\widetilde{k^0}([J]) &= \int_0^{\mathcal{R}} \Big(\sum_{x \in J} \exp\Big(-\frac{\pi}{\mathcal{N}(J)} \Big(\frac{\mathcal{N}(J)}{e^{2t}} \bar{x}^2 + \frac{e^{2t}}{\mathcal{N}(J)} x^2 \Big)\Big)\Big)\, dt \\
&= \sum_{x \in J} \int_{\frac{1}{\sqrt{\mathcal{N}(J)}}}^{\frac{\varepsilon}{\sqrt{\mathcal{N}(J)}}} \frac{1}{s} \exp\Big(-\frac{\pi}{\mathcal{N}(J)} \Big(\frac{\bar{x}^2}{s^2} + s^2 x^2 \Big)\Big)\, ds \\
&= \mathcal{R} + \sum_{h \geqslant 1} \sum_{\frac{|\mathcal{N}(x)|}{\mathcal{N}(J)} = h} \int_{\frac{1}{\sqrt{\mathcal{N}(J)}}}^{\frac{\varepsilon}{\sqrt{\mathcal{N}(J)}}} \frac{1}{s} \exp\Big(-h\pi \Big(\frac{\bar{x}^2}{|\mathcal{N}(x)| s^2} + \frac{s^2 x^2}{|\mathcal{N}(x)|} \Big)\Big)\, ds.
\end{aligned}$$

Hence, setting $\psi_h(z) = \dfrac{1}{s} \exp\big(-h\pi(z^2 + \frac{1}{z^2})\big)$, $\rho(x) = \sqrt{|\mathcal{N}(x)|\ \mathcal{N}(J)}$, and using the same notation as in the above discussion, we obtain:

$$\begin{aligned}
\widetilde{k^0}([J]) - \mathcal{R} &= \sum_{h \geqslant 1} \sum_{\frac{|\mathcal{N}(x)|}{\mathcal{N}(J)} = h} \int_{\frac{x}{\rho(x)}}^{\frac{\varepsilon x}{\rho(x)}} \psi_h(z)\, dz \\
&= \frac{1}{\gamma} \sum_{h \geqslant 1} \sum_{\frac{|\mathcal{N}(x)|}{\mathcal{N}(J)} = h} \int_{\frac{x}{\rho(x)}}^{\frac{\varepsilon^\gamma x}{\rho(x)}} \psi_h(z)\, dz \\
&= \frac{2}{\gamma} \sum_{h \geqslant 1} (Or(h) + Or(-h)) \int_0^\infty \psi_h(z)\, dz \\
&= \frac{4}{\gamma} \sum_{h \geqslant 1} (Or(h) + Or(-h)) \int_1^\infty \psi_h(z)\, dz.
\end{aligned}$$

Therefore we have that

$$\widetilde{k^0}([\mathcal{O}_F]) - \mathcal{R} > 2 \int_0^\infty \psi_1(z)\, dz = 4 \int_1^\infty \psi_1(z)\, dz.$$

When J is not a principal ideal, from the discussion above, we have that

$$\begin{aligned}\widetilde{k^0}([J]) - \mathcal{R} &< \frac{4\pi^2}{3\gamma}\sum_{h\geqslant 2} h\int_1^\infty \psi_h(z)\,dz \\ &< \frac{4\pi^2}{3\gamma}\sum_{h\geqslant 2}\frac{(\sqrt{h}-1)h}{e^{2\pi h}} + \frac{h}{2\pi(h^2-1)e^{\pi(h^2+1)}}\ .\end{aligned}$$

The last inequality comes from writing

$$\int_1^\infty \psi_h(z)\,dz < \int_1^{\sqrt{h}} e^{-2\pi h}\,dz + \frac{h}{h^2-1}\int_{\sqrt{h}}^\infty (s-\frac{1}{s^3})\,e^{-h\pi(s^2+\frac{1}{s^2})}\,dz.$$

The fact that $4\int_1^\infty \psi_1(z)\,dz > \frac{1.8}{10^3}$ and $\frac{4\pi^2}{3\gamma}\sum\limits_{h\geqslant 2}\frac{(\sqrt{h}-1)h}{e^{2\pi h}} + \frac{h}{2\pi(h^2-1)e^{\pi(h^2+1)}} < \frac{4}{10^5}$ concludes the proof. □

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