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An arithmetic analogue of Clifford's Theorem

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Résumé. Nous considérons ici certains fibrés en droites métriques comme analogues des diviseurs sur les courbes. Van der Geer et Schoof ont défini une fonction h^0 sur les fibrés métriques dont les propriétés ressemblent à celles de la dimension de $H^0(X, \mathcal{L}(D))$, où D désigne un diviseur sur la courbe X. Ils obtiennent en particulier un analogue du théorème de Riemann-Roch. Nous proposons des analogues arithmétiques de trois théorèmes sur les courbes, notamment du théorème de Clifford.

Abstract. Number fields can be viewed as analogues of curves over fields. Here we use metrized line bundles as analogues of divisors on curves. Van der Geer and Schoof gave a definition of a function h^0 on metrized line bundles that resembles properties of the dimension $l(D)$ of $H^0(X, \mathcal{L}(D))$, where D is a divisor on a curve X. In particular, they get a direct analogue of the Riemann-Roch theorem. For three theorems of curves, notably Clifford's theorem, we will propose arithmetic analogues.

1. Introduction

A popular way to study number fields is to view them as analogues of curves over a field. The elements of the number field correspond with points on the curve. Divisors on curves find their analogue in Arakelov divisors for number fields or metrized line bundles.

Given a divisor D on a curve X, we have an associated line bundle $\mathcal{L}(D)$ and an integer $l(D)$, which is the dimension of the vector space $H^0(X, \mathcal{L}(D))$. One of the most well known theorems for curves is the Riemann-Roch theorem, which relates $l(D)$ to the degree $\deg D$ of a divisor. It states that there is a canonical divisor K such that for each divisor D we have

$$l(D) - \frac{1}{2}\deg D = l(D^\dagger) - \frac{1}{2}\deg(D^\dagger), \qquad \text{where } D^\dagger = K - D.$$

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Following Van der Geer and Schoof [3], this article presents a function h^0 such that for a metrized line bundle L of a number field, we have

$$h^0(L) - \frac{1}{2}\deg L = h^0(L^\dagger) - \frac{1}{2}\deg L^\dagger.$$

In this article we will find analogues for three theorems for curves, stated here.

Theorem 1.1.
(1) *Let D be a divisor on a curve. If $\deg D < 0$, then $l(D) = 0$.*
(2) *Let D be a divisor on a curve with $\deg D \geq 0$. Then $l(D) \leq 1 + \deg D$.*
(3) *(Clifford's theorem) Let D be a divisor on a curve such that $l(D) > 0$ and $l(D^\dagger) > 0$. Then $l(D) \leq \frac{1}{2}\deg D + 1$.*

Proof. For (1) and (3), see Hartshorne [4, lemma IV.1.2 and IV.5.4]. For (2), see Fulton [2, proposition 8.2.3]. □

Arithmetic analogues to the three theorems above are also considered in the preprint of Van der Geer and Schoof [3]. As for the first one, they prove that $h^0(L)$ tends doubly exponentially fast to 0 in terms of the degree of L when $\deg L$ becomes negative. Our result is basically the same, although the bound that we will prove is more explicit. As for the second statement, Van der Geer and Schoof have a conjecture for number fields that are Galois over $\mathbf{Q}$ or over an quadratic imaginary number field. The conjecture has been proven by P. Francini for quadratic number fields [1].

2. Statement of Clifford's theorem for number fields

We give a working definition of a metrized line bundle now, in order to state Clifford's theorem. For a full definition, see section 6.

Let K be a number field with ring of integers R. We can write $R \otimes_{\mathbf{Z}} \mathbf{R}$ as a product

$$R \otimes_{\mathbf{Z}} \mathbf{R} = \prod_{v \in S^\infty} K_v,$$

where S^∞ is the set of infinite primes of K. Each K_v is isomorphic to either $\mathbf{R}$ or $\mathbf{C}$, so it should be clear what it means to take the complex conjugate of an element in K_v and hence of an element in $R \otimes_{\mathbf{Z}} \mathbf{R}$. A metrized line bundle is a projective R-module L of rank 1 together with an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $L \otimes_{\mathbf{Z}} \mathbf{R}$, such that for $x, y \in L \otimes_{\mathbf{Z}} \mathbf{R}$ and $a \in R \otimes_{\mathbf{Z}} \mathbf{R}$, we have

$$\langle ax, y\rangle = \langle x, a^* y\rangle,$$

where a^* is the complex conjugate of a. The dual of a metrized line bundle L is given by $L^\dagger = \mathrm{Hom}(L, \mathbf{Z})$. The elements of $L^\dagger$ can be identified with the elements of $L \otimes_Z \mathbf{R}$ that have integer valued inner product with every element of L. The degree of a line bundle L is given by

$$\deg L = \log(\sqrt{|\Delta|}/\operatorname{vol} L),$$

where Δ is the discriminant of K and $\operatorname{vol} L$ is the covolume of the lattice L in $L \otimes_{\mathbf{Z}} \mathbf{R}$. Finally we define

$$k^0(L) = \sum_{x \in L} e^{-\pi\langle x,x\rangle} \qquad \text{and} \qquad h^0(L) = \log k^0(L).$$

These definitions give rise to the Riemann-Roch theorem from section 1.

The main goal is to give an analogue of Clifford's theorem, which we state here.

Theorem 2.1 (Clifford's theorem). *Let K be a number field of degree n over $\mathbf{Q}$ and let L be a metrized line bundle with $\deg L \geq 0$ and $\deg L^\dagger \geq 0$. Then we have*

$$h^0(L) \leq n \log \omega + n \log n + \frac{1}{2} \deg L,$$

where $\omega = \sum_{n \in \mathbf{Z}} e^{-\pi n^2}$.

3. Riemann-Roch for lattices

A Euclidean space E is a finite dimensional vector space over $\mathbf{R}$, equipped with a positive definite symmetric $\mathbf{R}$-bilinear map

$$\langle\,\cdot\,,\,\cdot\,\rangle \colon E \times E \to \mathbf{R},$$

which we call the inner product or Euclidean structure. A norm $\|\cdot\|$ on E is constructed in the obvious way by setting $\|x\| = \sqrt{\langle x,x\rangle}$ for $x \in E$. The norm uniquely determines the inner product by

$$\langle x,y\rangle = \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2}.$$

If we have $\langle x,y\rangle = 0$ for x,$y \in E$, then we say that x and y are perpendicular and we write $x \perp y$. Given a subspace V of E we write $V^\perp = \{\, x \in E : x \perp V \,\}$ for the orthogonal complement of V. Given a subset $S \subset E$ we write $\operatorname{span} S$ for the smallest linear subspace of E containing S.

A lattice in a Euclidean vector space is a discrete subgroup of E. A lattice has a $\mathbf{Z}$-basis and the rank is given by the cardinality of this basis. If the rank is equal to the dimension of the vector space E, it is said to have full rank. If L is of full rank and has basis $b_1, \ldots, b_n$, then the volume $\operatorname{vol} L$ of L is given by the volume of parallelepiped

$$\{\, \lambda_1 b_1 + \cdots + \lambda_n b_n : \lambda_i \in \mathbf{R}, 0 \leq \lambda_i < 1 \,\},$$

where the volume is measured by the Haar measure induced by the inner product. A lattice L has a dual lattice $L^\dagger$, defined by

$$L^\dagger = \{\, x \in \operatorname{span} L : \langle x, L\rangle \subset \mathbf{Z} \,\}.$$

The Riemann-Roch theorem for lattices is better known as the Poisson summation formula. If E is a Euclidean space and f is a C^∞-function $E \to \mathbf{C}$ such that for all m the function $x \mapsto |x|^m f(x)$ is bounded, we call such a function a *rapidly decreasing function.* We can take the Fourier transform of of a rapidly decreasing function f as follows. Let dx be the Haar measure on E induced by the inner product. Furthermore, define the function $[\,\cdot\,,\,\cdot\,]\colon E \times E \to \mathbf{T}$ to the circle $\mathbf{T}$ by

$$[x, y] = e^{-2\pi i\langle x,y\rangle}.$$

Then the Fourier transform $\hat{f}\colon E \to \mathbf{C}$ of f is defined by

$$\hat{f}(y) = \int_E f(x)[x, y]\, dx.$$

We can now state the Poisson summation formula.

Proposition 3.1. *Let L be a lattice of full rank in a Euclidean vector space E and let $L^\dagger$ be the dual lattice. Let f be a rapidly decreasing function on E. Then we have*

$$\sum_{x\in L} f(x) = \frac{1}{\operatorname{vol} L} \sum_{y\in L^\dagger} \hat{f}(y).$$

Proof. See Neukirch [5, VII.3.2]. □

Given a subset S of a Euclidean space, we define $k^0(S)$ as

$$k^0(S) = \sum_{x\in S} e^{-\pi\langle x,x\rangle}$$

if the sum converges. In particular, k^0 is well-defined on lattices and cosets of lattices. An application of the Poisson summation formula gives a multiplicative version of the Riemann-Roch theorem for lattices.

Theorem 3.2 (Riemann-Roch)**.** *For every lattice L, we have*

$$k^0(L)\sqrt{\operatorname{vol} L} = k^0(L^\dagger)\sqrt{\operatorname{vol} L^\dagger}.$$

Proof. The function $x \mapsto e^{-\pi\langle x,x\rangle}$ is self-dual with respect to taking Fourier transforms (see [5, VII.3.1]). Furthermore, we have $\operatorname{vol} L = (\operatorname{vol} L^\dagger)^{-1}$. The theorem now follows directly form the Poisson summation formula. □

4. Estimates for k^0

Let L be a lattice in a Euclidean space E. The *minimum* of L is the length of the shortest nonzero vector in L. A minimal vector is a vector with length equal to the minimum. We have the following lemma.

Lemma 4.1. *Let L be a lattice with minimum λ. Define α_t for $t \in \mathbf{R}_{\geq 0}$ as*

$$\alpha_t = \#\{\, x \in L : \langle x, x\rangle \leq \lambda^2 t\,\}.$$

Then we have

$$k^0(L) = \int_0^\infty \alpha_t \lambda^2 \pi e^{-\pi\lambda^2 t}\, dt.$$

Proof. We can write

$$\begin{aligned} k^0(L) = \sum_{x\in L} e^{-\pi\langle x,x\rangle} = \sum_{x\in L}\int_{\langle x,x\rangle}^\infty \pi e^{-\pi t}\, dt \\ = \int_0^\infty \#\{\, x\in L : \langle x,x\rangle \leq t\,\}\pi e^{-\pi t}\, dt. \end{aligned}$$

A substitution of $\lambda^2 t$ for t in the above expression yields the lemma. □

Lemma 4.2. *Let L be a lattice with minimum λ. Let α_t be defined as in lemma 4.1. Then we have*

$$\alpha_t \leq (2\sqrt{t}+1)^n.$$

Proof. Let t be any positive real number and let A_t be the set

$$A_t = \{\, x \in L : \langle x,x\rangle \leq \lambda^2 t\,\}.$$

The distance between any two points in A_t is at least λ. Hence, if x and y are two different points of A_t, the open balls $B_{\lambda/2}(x)$ and $B_{\lambda/2}(y)$ with radius $\lambda/2$ and center x and y are disjunct. The union of all balls $B_{\lambda/2}(x)$ for all $x \in A_t$ is a subset of a large ball with radius $\lambda\sqrt{t}+\lambda/2$. Hence, by taking the quotient of the volume of the large ball with radius $\lambda\sqrt{t}+\lambda/2$ and a small ball with radius $\lambda/2$ we get

$$\alpha_t = \#A_t \leq \Big(\frac{\lambda\sqrt{t}+\lambda/2}{\lambda/2}\Big)^n = (2\sqrt{t}+1)^n.$$

□

Corollary 4.3. *Let L be a lattice with minimum λ. Then we have*

$$k^0(L) \leq 1 + \int_1^\infty (2\sqrt{t}+1)^n \lambda^2 \pi e^{-\pi\lambda^2 t}\, dt.$$

Proposition 4.4. *Let L be a lattice of rank n and with minimum $\lambda \geq \sqrt{n}$. Then we have*

$$k^0(L) \leq 1 + \frac{3^n \pi}{\pi - \log 3} e^{-\pi\lambda^2}.$$

Proof. We have $2\sqrt{t} + 1 \leq 3^t$ for $t \geq 1$. Hence by corollary 4.3, we get

$$\begin{aligned} k^0(L) - 1 &\leq \pi\lambda^2 \int_1^\infty 3^{nt} e^{-\pi\lambda^2 t}\, dt = \pi\lambda^2 \int_1^\infty e^{(-\pi\lambda^2 + n\log 3)t}\, dt \\ &= \frac{\pi\lambda^2}{\pi\lambda^2 - n\log 3} e^{\pi\lambda^2 + n\log 3} = \frac{3^n \pi\lambda^2}{\pi\lambda^2 - n\log 3} e^{-\pi\lambda^2} \leq \frac{3^n\pi}{\pi - \log 3} e^{\pi\lambda^2}. \end{aligned}$$

This proves the proposition. □

5. Clifford's theorem for lattices

We write $\mathbf{Z}$ for the unit bundle of $\mathbf{Q}$ and we write $\omega = k^0(\mathbf{Z})$. We have $\omega \approx 1.086$.

Lemma 5.1. *For $\lambda > 0$, we have $k^0(\lambda\mathbf{Z}) \leq \omega \max\{1, \lambda^{-1}\}$.*

Proof. It is clear that for $\lambda \geq 1$, we have $k^0(\lambda\mathbf{Z}) \leq k^0(\mathbf{Z}) = \omega$. Now assume $\lambda \leq 1$. The dual lattice of $\lambda\mathbf{Z}$ is equal to $\lambda^{-1}\mathbf{Z}$ and by Riemann-Roch for lattices, we get $k^0(\lambda\mathbf{Z}) \leq k^0(\lambda^{-1}\mathbf{Z}) \operatorname{vol}(\lambda^{-1}\mathbf{Z}) \leq \omega\lambda^{-1}$. □

Lemma 5.2. *Let L be a lattice of full rank in a Euclidean vector space E. The function $E/L \to \mathbf{R}$ that sends a coset Z of L to $k^0(Z)$ attains a unique maximum in L.*

Proof. Recall that for y,$z \in E$, we have defined $[y, z]$ as $[y, z] = e^{-2\pi i\langle y,z\rangle}$. Let f be a rapidly decreasing function $E \to \mathbf{C}$ and for $z \in E$ let g equal f, translated over z, i.e., $g(x) = f(x + z)$. Then we can express the Fourier transform of g in terms of $\hat{f}$ as

$$\begin{aligned} \hat{g}(y) = \int_E f(x+z)[x, y]\, dx = \int_E f(x)[x-z, y]\, dx = \int_E f(x)[z, y]^{-1}[x, y]\, dx \\ = [z, y]^{-1}\hat{f}(y). \end{aligned}$$

The Poisson summation formula gives us

$$\sum_{x \in z+L} f(x) = \sum_{x \in L} g(x) = \frac{1}{\operatorname{vol} L^\dagger} \sum_{y \in L^\dagger} [z, y]^{-1}\hat{f}(y).$$

We specialize for the case $f(x) = \hat{f}(x) = e^{-\pi\langle x,x\rangle}$. Then this sum is maximal if $[z, y]$ equals 1 for all y, hence if z is in L. □

Lemma 5.3. *Let L be a lattice of full rank in a Euclidean vector space E. Let π be an orthogonal projection on a subspace of E such that the image πL is discrete. Let $L' \subset L$ be the kernel of π. This gives an exact sequence*

$$0 \longrightarrow L' \longrightarrow L \xrightarrow{\pi} \pi L \longrightarrow 0.$$

Then we have

$$k^0(L) \leq k^0(L')k^0(\pi L).$$

Equality holds if and only if L is equal to the direct sum $L' \oplus \pi L$.

Proof. As for $x \in L'$ and $y \in \pi L$, we have $e^{-\pi\langle x,x\rangle}e^{-\pi\langle y,y\rangle} = e^{-\pi\langle y+x,y+x\rangle}$ we have $k^0(L' \oplus \pi L) = k^0(L')k^0(\pi L)$. For each $x \in \pi L$ choose an element $l(x) \in \pi^{-1}(x)$. Then we have

$$k^0(L' \oplus \pi L) = \sum_{x\in\pi L} k^0(x + L') \qquad \text{and} \qquad k^0(L) = \sum_{x\in\pi L} k^0(l(x) + L').$$

Let E' be the subspace of E spanned by L' and let x be an element of πL. For $y \in E'$ we have $\langle x + y, x + y\rangle = \langle x, x\rangle\langle y, y\rangle$ and hence

$$k^0(x+L') = e^{-\pi\langle x,x\rangle}k^0(L') \quad \text{and} \quad k^0(l(x)+L') = e^{-\pi\langle x,x\rangle}k^0(l(x)-x+L').$$

Hence, by lemma 5.2, we have $k^0(x + L') \geq k^0(l(x) + L')$, where we have equality only if $l(x) \in x + L'$. □

Given a lattice L, we want to have some way of bounding $k^0(L)$ in terms of its minimum. By lemma 5.1, this is trivial if the rank of L is 1. For the higher rank case we will use orthogonal projection and lemma 5.3 to reduce to the 1-dimensional case.

In order to give these bounds, we use Hermite constants, so here is a quick reminder what they are. The ith Hermite constant γ_i is defined as the smallest real number such that any lattice of rank i and with volume 1 has a vector with length at most $\gamma_i^{1/2}$. In general, a lattice L of rank i has a vector of length at most $\gamma_i^{1/2}(\operatorname{vol} L)^{1/2}$. It follows from the Minkowski bound that the inequality $\gamma_i \leq i$ holds.

Proposition 5.4. *Let L be a lattice of rank n with minimum λ. Then we have*

$$k^0(L) \leq \omega^n \prod_{i=1}^{n} \max\{1, \gamma_i/\lambda\}.$$

Proof. Let L be contained in a Euclidean vector space E. We inductively choose $b_1, \ldots, b_n \in E$ as follows. The element b_1 is equal to a minimal vector of the dual lattice $L^\dagger$. Then we project the lattice L orthogonally on $b_1\mathbf{R}$. The projection map is given by

$$x \longmapsto \frac{\langle x, b_1\rangle}{\langle b_1, b_1\rangle} b_1.$$

Let L_1 be the kernel of the projection map. That is, L_1 consists of all elements of L perpendicular to b_1. Then b_2 is chosen as a minimal vector of $L_1^\dagger$. In general, b_i is chosen such that it is a minimal vector of the dual $L_i^\dagger$ of the sublattice L_i of L given by all elements of L perpendicular to $\operatorname{span}\{b_1, \dots, b_{i-1}\}$. The image of L_i under orthogonal projection on $b_i\mathbf{R}$ is

$$\frac{b_i}{\langle b_i, b_i\rangle}\mathbf{Z} \cong \frac{1}{\|b_i\|}\mathbf{Z}.$$

Hence, by lemma 5.3 we have

$$k^0(L) \leq \prod_{i=1}^{n} k^0(\|b_i\|^{-1}\mathbf{Z}).$$

We will now give bounds for $k^0(\|b_i\|^{-1}\mathbf{Z})$. If M is a lattice of rank i with minimum *at most* λ and if b is a minimal vector of the dual lattice $M^\dagger$, we have

$$\|b\| \leq \gamma_i^{1/2}(\operatorname{vol} M^\dagger)^{1/i} = \gamma_i^{1/2}(\operatorname{vol} M)^{-1/i} \leq \gamma_i/\lambda.$$

By lemma 5.1, we have

$$k^0(\|b_i\|^{-1}\mathbf{Z}) \leq \omega \max\{1, \|b_i\|\} \leq \max\{1, \gamma_i/\lambda\}.$$

This completes the proof. □

Proposition 5.5. *Let L be a lattice in E of rank n with minimum λ. Let W be the smallest subspace of E such that all points of L that are not in W have distance at least 1 to W. We write $l = \dim W$. Then we have*

$$k^0(L) \leq \omega^n \max\{1, n/\lambda\}^l n^{n-l}$$

Proof. Let l be the dimension of W and define L' as $L' = W \cap L$. Let π be orthogonal projection on $W^\perp$ and let the image of L be denoted πL. Then πL has minimum greater or equal to 1. Hence, by lemma 5.3, we have

$$k^0(L) \leq k^0(L')k^0(\pi L).$$

Applying proposition 5.4 twice yields

$$k^0(L) \leq \omega^n \prod_{i=1}^{l} \max\{1, \gamma_i/\lambda\} \prod_{i=1}^{n-l} \gamma_i.$$

We use $\gamma_i \leq n$ to get the wanted inequality. □

Lemma 5.6. *Let L be a lattice contained in a Euclidean vector space E and let W be the smallest subspace of E such that all points of L that are not in W have distance at least 1 to W. Similarly, let $W^\dagger$ be such a set for $L^\dagger$. Then W and $W^\dagger$ are perpendicular. In particular* $\dim W + \dim W^\dagger \leq \operatorname{rank} L$.

Proof. Suppose W and $W^\dagger$ are not perpendicular. Then there exists a $y \in L^\dagger \cap W^\dagger$ with $y \notin W^\dagger \cap W^\perp$. By minimality of $W^\dagger$ we can choose y with distance to $W^\dagger \cap W^\perp$ smaller than 1. Hence, there is a $y' \in W^\dagger \cap W^\perp$ with $\|y - y'\| < 1$. Similarly, there is a $z \in L \cap W$ with $z \notin W \cap (W^\dagger)^\perp$ and a $z' \in W \cap (W^\dagger)^\perp$ such that $\|z - z'\| < 1$. Hence we get

$$1 > \|z - z'\|\|y - y'\| \geq |\langle z - z', y - y'\rangle| = |\langle z, y\rangle|.$$

As $\langle z, y\rangle$ is an integer and z and y are not perpendicular, this is a contradiction. □

Combining proposition 5.5 and lemma 5.6, we get the following corollary. In section 7, Clifford's theorem for number fields follows directly from this corollary. Therefore, we see this corollary as an analogue of Clifford's theorem for lattices.

Corollary 5.7. *Let L be a lattice of rank n with minimum λ and let the dual $L^\dagger$ have minimum $\lambda^\dagger$. Then we have*

$$k^0(L) \leq \omega^n \max\{1, 1/\lambda\}^{n/2} n^n \qquad \text{or} \qquad k^0(L^\dagger) \leq \omega^n \max\{1, 1/\lambda^\dagger\}^{n/2} n^n.$$

6. Metrized line bundles

Now we turn to the number field case and prove arithmetic analogues of the geometric theorems mentioned in the introduction. As we shall see, almost all of the work is already done in the sections about lattices. We need a few facts about Euclidean spaces and finite étale algebras over $\mathbf{R}$ in order to define Hermitian modules and metrized line bundles.

We state the following lemma without proof.

Lemma 6.1.

(1) *If E_1 and E_2 are Euclidean spaces, the tensor product $E_1 \otimes E_2$ has a unique Euclidean structure such that for x,$y \in E_1$ and x',$y' \in E_2$, we have*

$$\langle x \otimes x', y \otimes y'\rangle = \langle x, y\rangle\langle x', y'\rangle.$$

(2) *Every quotient space D of a Euclidean space E, given by $\phi\colon E \to D$, has an induced Euclidean structure given by $(\ker\phi)^\perp \cong D$ such that $\|z\| = \inf_{\phi(x)=z} \|x\|$ for $z \in D$.*

(3) *The Endomorphism $\mathrm{End}_{\mathbf{R}}(E)$ of a Euclidean space E has a natural involution $\phi \mapsto \phi^*$, where the adjoint ϕ^* of ϕ is the unique element of $\mathrm{End}_{\mathbf{R}}(E)$ such that the relation $\langle \phi a, b\rangle = \langle a, \phi^* b\rangle$ holds.*

The category of finite étale algebras over $\mathbf{R}$ consists of the finite $\mathbf{R}$-algebras A such that the map $\phi\colon A \to \mathrm{Hom}_{\mathbf{R}}(A, \mathbf{R})$ given by $\phi(x)(y) = \mathrm{Tr}(xy)$ is an isomorphism. Here Tr is the trace map from A over $\mathbf{R}$. If we let v range over the points of $S = \operatorname{spec} A$, we get a decomposition $A = \prod_v A_v$, where A_v is the residue class field. Every A_v is isomorphic to **R** or **C**. For every $v \in \operatorname{spec} A$, we have a projection map $\phi\colon A \to A_v$. We contend that the identity functor on the category of the finite étale algebras over **R** has exactly one nontrivial automorphism. Indeed, suppose we have a functorial automorphism $x \mapsto x^*$ on every finite étale algebra over **R**. Then for all projections $\pi\colon A \to A_v$ and all elements $x \in A$ we have $\pi(x^*) = \phi(x)^*$. Hence, on each étale algebra our nontrivial automorphism is complex conjugation on the factors A_v that are isomorphic to **C** and is trivial on factors isomorphic to **R**. When we talk about the involution of a finite étale algebra over **R** we mean this map.

Let M be a module over an étale algebra A over **R**, with a Euclidean structure. Then M is called *Hermitian* if the natural map $A \to \operatorname{End}_{\mathbf{R}}(M)$ preserves involutions. This is equivalent to the condition that for all $a \in A$ and $m_1, m_2 \in M$ we have

$$\langle am_1, m_2\rangle = \langle m_1, a^* m_2\rangle.$$

If we are given two Hermitian modules M and N over A, then $M \otimes_{\mathbf{R}} N$ is a Euclidean space and the quotient space $M \otimes_A N$ has a natural Euclidean structure. Furthermore, we can view A as a module over itself and give it the unique Euclidean structure such that the inner product on A and the induced inner product on $A \otimes_A A$ is compatible with the map $A \otimes_A A \xrightarrow{\sim} A$. This is the canonical inner product for A. A trace of the definitions results in the following lemma.

Lemma 6.2.

(1) *Let M and N be Hermitian modules, free of rank 1 over a field A, algebraic over* **R**. *For $m \in M$, $n \in N$ and $m \otimes n \in M \otimes_A N$, we have*

$$\|m \otimes n\| = [A : \mathbf{R}]^{-1/2} \|m\| \cdot \|n\|.$$

(2) *Let A be a finite étale algebra over* **R** *and let M and N be Hermitian modules over A, free of rank 1. Then, for $v \in \operatorname{spec} A$, we have an isomorphism*

$$(M \otimes_A N)_v \cong M_v \otimes_{A_v} N_v$$

as Hermitian modules.

(3) *Let A be a finite étale algebra, viewed as a Hermitian module over itself with the canonical inner product. Let v be an element of* $\operatorname{spec} A$ *and let $\|\cdot\|_v$ be the restriction of $\|\cdot\|$ to A_v. Then we have $\|1\|_v = [A_v : \mathbf{R}]^{1/2}$.*

We are now ready to give the definition of a metrized line bundle. Let K be a number field and let R be its ring of integers. A line bundle on R is a projective R-module L of rank 1. Now $R \otimes_{\mathbf{Z}} \mathbf{R}$ is a finite étale algebra over **R** and $L \otimes_{\mathbf{Z}} \mathbf{R}$ is a module of rank 1 over $R \otimes_{\mathbf{Z}} \mathbf{R}$. We call L a *metrized line bundle* over R if $L \otimes_{\mathbf{Z}} \mathbf{R}$ is given a Euclidean structure such that it becomes an Hermitian module over $R \otimes_{\mathbf{Z}} \mathbf{R}$.

Two metrized line bundles are isomorphic if there is an R-module isomorphism that preserves the inner product. Given two metrized line bundles L_1,L_2 over R, their product L_1L_2 is given by the module $L_1 \otimes_R L_2$. The inner product on $(L_1 \otimes_R L_2) \otimes_{\mathbf{Z}} \mathbf{R}$ is given by the canonical isomorphism

$$(L_1 \otimes_R L_2) \otimes_{\mathbf{Z}} \mathbf{R} \cong (L_1 \otimes_{\mathbf{Z}} \mathbf{R}) \otimes_{R\otimes\mathbf{R}} (L_2 \otimes_{\mathbf{Z}} \mathbf{R}).$$

The set of isomorphism classes of metrized line bundles over R is denoted $\operatorname{Pic} K$ and with this multiplication it is a group. The unit element is equal to R with the canonical Euclidean structure on $R \otimes_{\mathbf{Z}} \mathbf{R}$. We call R with this structure the *unit bundle.*

Let L be a metrized line bundle over R and let S^∞ be the set of infinite primes of K. Then we have a decomposition $L \otimes_{\mathbf{Z}} \mathbf{R} = \prod_{v\in S^\infty} L_v$, where $L_v = L \otimes K_v$ is a 1-dimensional K_v-vector space. The factors L_v are perpendicular and we write $\|\cdot\|_v$ for the restriction of the norm to L_v. For instance, if R is the unit bundle and $\|\cdot\|_v$ is the restricted norm on $R_v = K_v$, we have

$$\|1\|_v = \sqrt{[K_v : \mathbf{R}]}.$$

We define the *norm* of a metrized line bundle L as

$$N(L) = \frac{\operatorname{vol} R}{\operatorname{vol} L} = \frac{\sqrt{|\Delta|}}{\operatorname{vol} L},$$

where Δ is the discriminant of K. The *degree* is defined as $\deg L = \log N(L)$.

Proposition 6.3.
(1) *The norm function is a group homomorphism* $\operatorname{Pic} K \to \mathbf{R}_{>0}$.
(2) *If L is a metrized line bundle and $t \in L$ is any nonzero element, then*

$$N(L) = \#(L/Rt) \Big/ \prod_{v\in S^\infty} \frac{\|t\|^{[K_v:\mathbf{R}]}}{[K_v : \mathbf{R}]}.$$

Proof. Define M_t by

$$M_t = \prod_{v\in S^\infty} \frac{\|t\|_v^{[K_v:\mathbf{R}]}}{[K_v : \mathbf{R}]}.$$

As we have $\|1\|_v = \sqrt{[K_v : \mathbf{R}]}$, we can also write

$$M_t = \prod_{v\in S^\infty} \Big(\frac{\|t\|_v}{\|1\|_v}\Big)^{[K_v:\mathbf{R}]}.$$

Consider the map $R \otimes_{\mathbf{Z}} \mathbf{R} \to L \otimes_{\mathbf{Z}} \mathbf{R}$ given by multiplication with t. It blows up the measure by a factor M_t. Hence, we have

$$M_t \operatorname{vol} R = \operatorname{vol} R_t = (\operatorname{vol} L)/[L : Rt].$$

This proves (2). To prove (1) one uses (2) together with some explicit calculations. □

Let L be a metrized line bundle over R. Then it can be viewed as a lattice in $L \otimes_{\mathbf{Z}} \mathbf{R}$. Hence, we have a definition for $k^0(L)$, given as

$$k^0(L) = \sum_{x \in L} e^{-\pi \langle x,x \rangle}.$$

Furthermore, we define $h^0(L)$ as

$$h^0(L) = \log k^0(L).$$

Both k^0 and h^0 induce functions from $\operatorname{Pic} K$ to $\mathbf{R}$. In order to state the Riemann-Roch theorem, we need the notion of the dual of a metrized line bundle L. Consider the map

$$\begin{array}{rcl} L \otimes_{\mathbf{Z}} \mathbf{R} & \longrightarrow & \operatorname{Hom}_{\mathbf{R}}(L \otimes_{\mathbf{Z}} \mathbf{R}, \mathbf{R}) \\ x & \longmapsto & \langle x, \cdot \rangle. \end{array}$$

This map is an isomorphism, giving

$$\operatorname{Hom}_{\mathbf{R}}(L \otimes_{\mathbf{Z}} \mathbf{R}, \mathbf{R}) = \operatorname{Hom}_{\mathbf{Z}}(L, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R}$$

a canonical Euclidean structure. We let $L^\dagger$ be $\operatorname{Hom}(L, \mathbf{Z})$ with this structure.

Proposition 6.4 (Riemann-Roch). *Let L be a metrized line bundle. Then we have*

$$h^0(L) - \frac{1}{2} \deg L = h^0(L^\dagger) - \frac{1}{2} \deg L^\dagger.$$

Proof. This follows directly from the Riemann-Roch theorem for lattices. This formula also appears in [3, Proposition 1] in a different form. □

7. Analogues of theorems for curves

We have set up everything to prove in quick succession the analogues of the geometric theorems mentioned in section 1. The only lemma we need to tie the results for lattices to metrized line bundles is the following lemma, that relates the minimum of a lattice to the norm of the line bundle.

Lemma 7.1. *Let n be the degree of a number field K, and let L be a metrized line bundle. Then for all elements $x \in L$, we have*

$$\|x\|^2 \geq n N(L)^{-2/n}.$$

Proof. The geometric-arithmetic mean inequality and proposition 6.3 give for nonzero $x \in L$ the estimate

$$\begin{aligned}\|x\|^2 = \sum_{v \in S^\infty} \|x\|_v^2 = \sum_{v \in S^\infty} [K_v : \mathbf{R}] \frac{\|x\|_v^2}{[K_v : \mathbf{R}]} \\ \geq n \Big(\prod_{v \in S^\infty} \Big(\frac{\|x\|_v^2}{[K_v : \mathbf{R}]} \Big)^{[K_v : \mathbf{R}]} \Big)^{1/n} = n \Big(\prod_{v \in S^\infty} \frac{\|x\|^{[K_v : \mathbf{R}]}}{[K_v : \mathbf{R}]} \Big)^{2/n} \\ \geq n \Big(\frac{\#(L/Rx)}{N(L)} \Big)^{2/n} \geq n N(L)^{-2/n}.\end{aligned}$$

□

First, we will prove the analogue of the geometric fact that $l(D) = 0$ if the degree of a divisor D is negative. The proposition states that $h^0(L)$ tends doubly exponentially fast to zero in terms of the degree of L when the degree becomes negative. This was already noted by Van der Geer and Schoof [3, Corollary 1 to Proposition 2].

Proposition 7.2. *Let K be a number field of degree n over $\mathbf{Q}$ and let L be a metrized line bundle of degree at most 0. Then we have*

$$h^0(L) < \frac{3^n \pi}{\pi - \log 3} e^{-\pi n e^{-\frac{2}{n} \deg L}}.$$

Proof. Immediate from proposition 4.4 and lemma 7.1 and the fact that $h^0(L) \leq k^0(L) - 1$. □

Second, we prove the analogue of the geometric theorem that $l(D) \leq 1 + \deg D$ if D is effective.

Proposition 7.3. *Let K be a number field of number field of degree n over $\mathbf{Q}$ and let L be a metrized line bundle with $\deg L \geq 0$. Then we have*

$$h^0(L) \leq n \log \omega + \frac{1}{2} n \log n + \deg L.$$

Proof. Let λ be the minimum of the lattice L. The assumption $\deg L \geq 0$ translates into $N(L) \geq 1$. Using proposition 5.4, the fact that $\gamma_i \leq n$ and lemma 7.1, we get

$$k^0(L) \leq \omega^n \max\{1, \Big(\frac{n}{\lambda}\Big)^n\} \leq \omega^n \max\{1, \frac{n^n}{n^{n/2}} N(L)\} = \omega^n n^{n/2} N(L).$$

Finally, take the logarithm to prove the proposition. □

The third analogue is Clifford's theorem for number fields, of which a sneak preview was given in section 2, theorem 2.1.

Theorem 7.4 (Clifford's theorem). *Let K be a number field of degree n over $\mathbf{Q}$ and let L be a metrized line bundle with $\deg L \geq 0$ and $\deg L^{\dagger} \geq 0$. Then we have*

$$h^0(L) \leq n \log \omega + n \log n + \frac{1}{2} \deg L.$$

Proof. By corollary 5.7, we have $k^0(M) \leq \omega^n n^n \max\{1, 1/\mu\}^{n/2}$, where M is either L or $L^{\dagger}$ and μ is the minimum of M. Using $1/\mu \leq N(M)^{1/n}$, we get

$$k^0(M) \leq \omega^n n^n N(M)^{1/2}.$$

and hence

$$h^0(M) \leq n \log \omega + n \log n + \frac{1}{2} \deg M.$$

Using Riemann-Roch, we also have

$$h^0(M^{\dagger}) \leq n \log \omega + n \log n + \frac{1}{2} \deg M^{\dagger}.$$

As we have $L = M$ or $L = M^{\dagger}$, this proves the theorem. □

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