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Arakelov computations in genus 3 curves

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Résumé. Les invariants d'Arakelov des surfaces arithmétiques sont bien connus pour le genre 1 et 2 ([4], [2]). Dans cette note, nous étudions la hauteur modulaire et la self-intersection d'Arakelov pour une famille de courbes de genre 3 possédant beaucoup d'automorphismes, à savoir

$$C_n : Y^4 = X^4 - (4n-2)\, X^2 Z^2 + Z^4 \,.$$

La théorie d'Arakelov fait intervenir à la fois des calculs arithmétiques et des calculs analytiques. Nous exprimons les périodes de C_n en termes d'intégrales elliptiques. Les substitutions utilisées dans les intégrales fournissent une décomposition de la jacobienne de C_n en produit de trois courbes elliptiques. En utilisant l'isogénie correspondante, nous déterminons le modèle stable de la surface arithmétique définie par C_n.

Une fois calculés les périodes et le modèle stable de C_n, nous sommes en mesure de déterminer la hauteur modulaire et la self-intersection du modèle canonique. Nous donnons une bonne estimation de cette hauteur modulaire, traduite par son comportement logarithmique. Nous donnons également une minoration de la self-intersection qui montre qu'elle peut être arbitrairement grande.

Nous présentons ici nos calculs presque sans démonstrations. Les détails peuvent être lus dans [5].

Abstract. Arakelov invariants of arithmetic surfaces are well known for genus 1 and 2 ([4], [2]). In this note, we study the modular height and the Arakelov self-intersection for a family of curves of genus 3 with many automorphisms:

$$C_n : Y^4 = X^4 - (4n-2) X^2 Z^2 + Z^4.$$

Arakelov calculus involves both analytic and arithmetic computations. We express the periods of the curve C_n in terms of elliptic integrals. The substitutions used in these integrals provide a splitting of the jacobian of C_n as a product of three elliptic curves. Using the corresponding isogeny, we determine the stable model of the arithmetic surface given by C_n. Once we have the periods and the stable model of C_n, we can study the modular height and

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the self-intersection of the canonical sheaf. We can give a good estimate for the modular height, which reflects its logarithmic behaviour. We provide a lower bound for the self-intersection, which shows that it can be arbitrarily large.

We present here all our calculations on the curves C_n, almost without proofs. Details can be found in [5].

1. A family of curves of genus 3

We will study the curves given by the equation:

$$C_{(a)} : Y^4 = (X^2 - a^2Z^2)(X^2 - a^{-2}Z^2), \tag{1}$$

for $a \in \mathbb{R}, a \neq 0, \pm 1$. These are non-singular curves of genus 3. We note that each of the four values of the parameter $a, -a, 1/a, -1/a$ gives rise to the same curve. Equation (1) is particularly well-suited to study the geometry of the curves $C_{(a)}$, but to study their arithmetic we should use the equation

$$C_n : Y^4 = X^4 - (4n-2)X^2Z^2 + Z^4,$$

obtained from (1) through the change of parameter $a = \sqrt{n} + \sqrt{n-1}$. In either equation one can observe that these curves have many symmetries. More specifically, we prove:

Proposition 1. *Let $a \neq 0, \pm 1, \pm 1 \pm \sqrt{2}$. The automorphisms of $C_{(a)}$ are the restrictions of the following 16 projectivities of $\mathbb{P}^2(\mathbb{C})$:*

$$\varphi_{0k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_{1k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^k & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$\varphi_{2k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i^k & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varphi_{3k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i^k & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$k = 0, 1, 2, 3 \qquad .$$

The group $\mathrm{Aut}(C_{(a)})$ is isomorphic to a semidirect product $\mathbb{Z}/4\mathbb{Z} \ltimes V_4$. The automorphisms

$$\begin{aligned} \alpha(x,y,z) &= \varphi_{01}(x,y,z) = (x, iy, z), \\ \beta(x,y,z) &= \varphi_{12}(x,y,z) = (-x, y, z), \\ \gamma(x,y,z) &= \varphi_{20}(x,y,z = (z,y,x) \end{aligned}$$

form a system of generators for $\mathrm{Aut}(C_{(a)})$.

Proposition 2. *The curve $\mathcal{C}_{\pm 1 \pm \sqrt{2}}$ is isomorphic to the Fermat curve of fourth degree, $F_4 = \{Y^4 = X^4 + Z^4\}$. The group $\mathrm{Aut}(F_4)$ has order 96, and is isomorphic to a semidirect product $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \ltimes \mathcal{S}_3$.*

2. Geometry of the curves $C_{(a)}$

The determination of the period lattice of the curves $C_{(a)}$ is possible thanks to their many symmetries. First of all, we can find a basis for the space of holomorphic differentials on $C_{(a)}$:

Proposition 3. *Let $x = X/Z, y = Y/Z$. The differential forms*

$$\omega_1 = \frac{dx}{y^3}, \qquad \omega_2 = \frac{xdx}{y^3}, \qquad \omega_3 = \frac{dx}{y^2}$$

yield an orthogonal basis for $H^0(C_{(a)}, \Omega^1)$.

We can also determine a basis for the singular homology of the curves. We regard the curve $C_{(a)}$ as a 4-sheeted covering of the complex projective line, through the map:

$$\begin{aligned} \pi : C_{(a)} &\longrightarrow \mathbb{P}^2(\mathbb{C}) \\ (x : y : z) &\longrightarrow (x : z). \end{aligned}$$

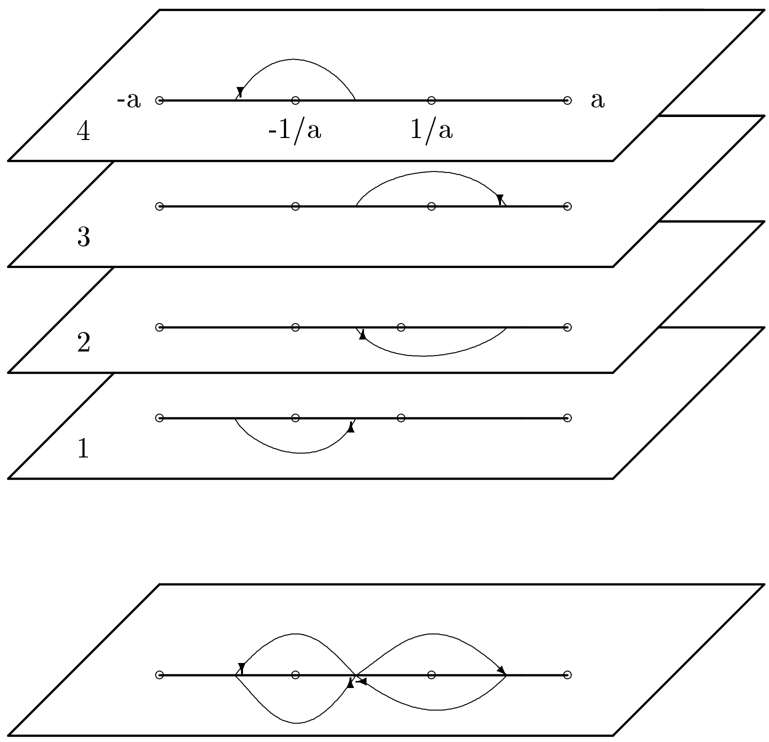


Figure 1

We can explicitly describe the monodromy of this map. For instance, a path on $\mathbb{P}^2(\mathbb{C})$ crossing the segment $[-1/a, 1/a]$, when lifted to $C_{(a)}$ through

π goes from a sheet n to the sheet $n+2$ (mod 4). The action of the automorphism α of $C_{(a)}$ is easily described in terms of the covering π: it lifts points one sheet. With all these data, we can proceed to build closed paths on $C_{(a)}$, lifting closed paths in $\mathbb{P}^2(\mathbb{C})$, as seen in figure 1. We refer to the path sketched there as G. We proceed analogously to build the path F indicated in figure 2 (numbers indicate the sheet on which each part of the path lies).

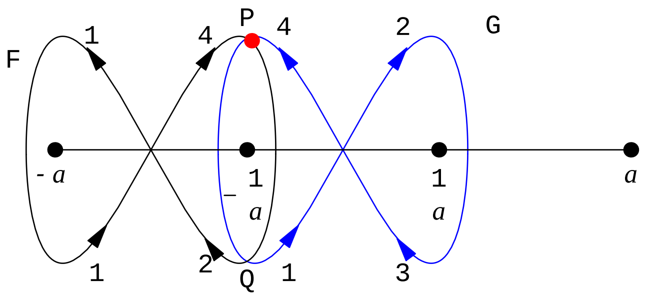


Figure 2

Proposition 4. *The homology classes of the paths*

$$\begin{array}{lll} e_1 = F, & e_2 = \alpha^2(F), & e_3 = \alpha(G) + \alpha^2(G) + F - \alpha^2(F), \\ e_4 = G, & e_5 = \alpha^2(G), & e_6 = \alpha^2(G) - G + \alpha(F), \end{array}$$

form a symplectic basis for $H_1(C_{(a)}, \mathbb{Z})$.

Once we have bases for the spaces $H^0(C_{(a)}, \Omega^1)$ and $H_1(C_{(a)}, \mathbb{Z})$ we compute the period lattice of the curve $C_{(a)}$. The Abelian integrals that we must compute can be reduced to elliptic integrals, and can thus be expressed in terms of elliptic functions. We obtain:

Proposition 5. *A normalized period matrix for the curves* $C_{(a)}$ *is* $(I|Z_a)$, *where*

$$Z_a = \begin{pmatrix} \frac{1+i}{2} + i\tau_a & -\frac{1+i}{2} + i\tau_a & -\frac{1+i}{2} \\ -\frac{1+i}{2} + i\tau_a & \frac{1+i}{2} + i\tau_a & \frac{1+i}{2} \\ -\frac{1+i}{2} & \frac{1+i}{2} & 1+i \end{pmatrix},$$

$$\tau_a = \frac{a^2}{2(a^2+1)} \frac{K\left(\frac{a-1/a}{a+1/a}\right)}{K(a^{-2})},$$

and $K(k) = \int_0^1 dX/\sqrt{(X^2-1)(1-k^2X^2)}$ *is the complete elliptic integral of the first kind.*

For instance, for the Fermat curve of fourth degree, isomorphic to the curve $C_{1+\sqrt{2}}$, we re-obtain its well-known period matrix:

$$Z_{1+\sqrt{2}} = \begin{pmatrix} \frac{1}{2}+i & -\frac{1}{2}+i & -\frac{1+i}{2} \\ -\frac{1}{2} & \frac{1}{2}+i & \frac{1+i}{2} \\ -\frac{1+i}{2} & \frac{1+i}{2} & 1+i \end{pmatrix}.$$

The fact that all the Abelian integrals on $C_{(a)}$ can be reduced to elliptic integrals tells us that the jacobian $J(C_{(a)})$ of the curve should split as a product of elliptic curves. We can show the situation exactly. Let $\mu = i\sqrt[4]{4n(n-1)}$, $\zeta = \frac{1+i}{\sqrt{2}}$, $K_0 = \mathbb{Q}(\sqrt{n}+\sqrt{n-1}, \zeta\mu)$.

Proposition 6. *Let E be the elliptic curve*

$$Y^2Z = X^3 - XZ^2,$$

and let E_n be the elliptic curve

$$Y^2Z = X(X-Z)(X-nZ)$$

which is a Weierstrass model of the curve

$$Y^2Z^2 = X^4 - (4n-2)X^2Z^2 + Z^4.$$

We have isomorphisms defined over K_0:

$$\begin{aligned} C_n/ <\beta\alpha^2> &\simeq E, \\ C_n/ <\beta> &\simeq E, \\ C_n/ <\alpha^2> &\simeq E_n. \end{aligned}$$

The quotient maps induce an isogeny of degree 8

$$\Psi : J(C_n) \longrightarrow E \times E \times E_n,$$

defined over K_0.

3. Arithmetic of the curves C_n

From now on, we will assume that $n \in \mathbb{N}$ is a natural number, and, for technical reasons only, that $n \not\equiv 0, 1 \pmod{2^5}$. The isogeny described in the previous section is the key to the study of reduction of the curves C_n. It is well known that the the reduction of a curve is intimately related with that of its jacobian. Moreover, at the level of Abelian varieties, the type of reduction is invariant under isogenies. Thus, studying the reduction of the elliptic curves E, E_n we can achieve the reduction of the curves C_n. First of all, we determine the locus of good reduction:

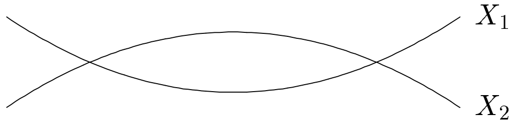
Proposition 7. *The curve C_n has good reduction outside the primes dividing $n(n-1)$.*

Let $K_n = K_0(\alpha, \alpha_n)$, where $\alpha = \sqrt[3]{18-6\sqrt{3}}$ and $\alpha_n^3(\alpha_n^3-24)^3 - j(E_n)$ $(\alpha_n^3-27) = 0$. Let $\mathcal{O}_n$ be the ring of integers of K_n. The elliptic curves E, E_n acquire good reduction at 2 over the field K_n, since their Deuring models are defined over this field. The curve E_n has multiplicative reduction at odd primes dividing $n(n-1)$. From this, we conclude:

Proposition 8. *a) The curve C_n has a stable model $\mathcal{C}_n^{st}$ and a minimal regular model $\mathcal{C}_n^{reg}$ over $\mathcal{O}_n$.*
b) At the primes $\mathfrak{p}$ in $\mathcal{O}_n$ which divide 2, $J(C_n)$ has Abelian reduction over $\mathcal{O}_n$.
c) At the odd primes $\mathfrak{p}$ in $\mathcal{O}_n$ which divide $n(n-1)$, $J(C_n)$ has semi-Abelian reduction over $\mathcal{O}_n$, and the toric part of its Néron model has dimension 1.

We now describe the bad fibres of the stable model $\mathcal{C}_n^{st}$ of the curves C_n. At the odd primes, Prop. 8 gives enough information to describe the bad fibres.

Theorem 9. *a) The special fibre of $\mathcal{C}_n^{st}$ at an odd prime $\mathfrak{p}$ in $\mathcal{O}_n$ dividing $n(n-1)$ has two elliptic components, intersecting at two different points.*

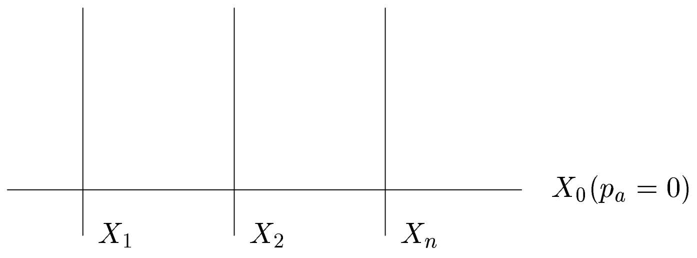


b) The intersection pairing on this fibre is given by

$$(X_1, X_2) = -X_1^2 = -X_2^2 = 2.$$

The situation at the primes dividing 2 is much more complicated, and requires a detailed study of the reduction of the automorphisms of C_n. Once this is done, we obtain that:

Theorem 10. *a) The special fibre of $\mathcal{C}_n^{st}$ at a prime $\mathfrak{p}$ in $\mathcal{O}_n$ dividing 2 consists of three elliptic components, which intersect a rational component at three different points. These intersection points map to 4-torsion points in the elliptic quotients of $\mathcal{C}_n^{st}$.*



b) The intersection pairing on this fibre is given by

$$(X_0, X_1) = (X_0, X_2) = (X_0, X_3) = 1,$$
$$(X_1, X_2) = (X_1, X_3) = (X_2, X_3) = 0,$$
$$X_0^2 = -3, X_1^2 = X_2^2 = X_3^2 = -1.$$

4. Arakelov invariants for arithmetic surfaces

Having a great deal of geometric and arithmetic information about a curve, one can proceed to study it from the Arakelov viewpoint. We will now do so with the curves C_n. We will study the two basic Arakelov invariants for curves: the modular height and the self-intersection. We briefly recall their definitions (details can be found in [4]).

Given a curve X of genus g, defined over a number field K, consider an extension L/K where X acquires stable reduction, and let $\mathcal{X}^{est} \longrightarrow T = \mathrm{Spec}(\mathcal{O}_L)$ be its stable model. The relative dualizing sheaf $\omega_{\mathcal{X}^{est}/T}$ can be provided with an admissible metric, giving rise to an *Arakelov line sheaf* $\overline{\omega}_{\mathcal{X}^{est}/T}$.

Definition 11. *The modular height of the curve X is*

$$h(X) = \deg_{Ar} \overline{\omega}_{\mathcal{X}^{est}/T}.$$

The Arakelov self-intersection of X is

$$e(X) = \frac{1}{[L:K]} (\overline{\omega}_{\mathcal{X}^{est}/T}, \overline{\omega}_{\mathcal{X}^{est}/T})_{Ar},$$

where $(\cdot,\cdot)_{Ar}$ denotes the Arakelov intersection pairing ([4]*).*

For genus $g = 1$, it is known that $e(X) = 0$ and $h(X)$ can be expressed as a function $h(X) = h(\Delta, \tau)$ in terms of the periods of the curve, its discriminant and its locus of bad reduction. For genus $g = 2$, Bost - Mestre - Moret-Bailly ([3])give explicit formulas for both invariants, where again the periods and the reduction of the curve play an important role. But there is not much more information about these two invariants in general. Ullmo ([8]) has proved that $e(X) > 0$ for $g > 1$, and Abbes and Ullmo ([1]) have found upper and lower bounds for the self-intersection of modular curves $X_0(N)$, N squarefree.

5. Modular height of the curves C_n

Theorem 12. *Let $\tau_n = iK(\sqrt{1-1/n})/K(1/\sqrt{n})$. The height of the curve C_n is given by*

$$h(C_n) = 2\log\frac{\Gamma(3/4)\sqrt{2}}{\Gamma(1/4)\sqrt{\pi}} + \frac{1}{6}\log\frac{n(n-1)}{2^{v_2(n(n-1))}} - \frac{1}{12}\log(|\Delta(\tau_n)|(Im\tau_n)^6) + k\log 2,$$

for $k \in \{0, \pm 1\}$.

Sketch of the proof: The modular height of a curve coincides with that of its jacobian as an Abelian variety. A result of Raynaud ([7]) controls the behaviour of the modular height of Abelian varieties through isogenies $A \xrightarrow{\phi} B$ of degree a prime power p^e: we have that $h(A) = h(B) + k\log p$, with $k \in \mathbb{Z}, |k| \leq e/2$. Moreover, the modular height of a product of Abelian varieties is the sum of the heights of the factors. We thus obtain:

$$h(C_n) = 2h(E) + h(E_n) + k\log 2, \qquad k \in \{0, \pm 1\}.$$

Now we use the explicit formula for the height of the elliptic curves given in ([4]), to determine the terms $h(E), h(E_n)$. Note that τ_n is the fundamental period of the elliptic curve E_n. □

6. A lower bound for $e(C_n)$

Lemma 13 ([6]). *Let $\mathcal{X} \longrightarrow S = Spec(\mathcal{O}_K)$ be a stable arithmetic surface of genus $g \geq 2$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the primes in $\mathcal{O}_K$ where the fibres of $\mathcal{X}$ are reducible. We have that*

$$(\overline{\omega}_{\mathcal{X}/S}, \overline{\omega}_{\mathcal{X}/S})_{Ar} \geq \frac{1}{6(g-1)}\sum_{t=1}^{r}\log N_{K/\mathbb{Q}}(\mathfrak{p}_i).$$

We can apply this result to the curves C_n, since we have a precise description of the primes where their stable model has reducible fibres: they are the primes which divide $n(n-1)$. We thus obtain:

Proposition 14.

$$e(C_n) \geq \frac{1}{12[K_n : K]}\log\prod_{p|n(n-1)} p.$$

As the degree $[K_n : K]$ is always less than or equal to 1152, this result implies that the Arakelov self-intersection of a curve can be arbitrarily large.

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