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Metaplectic covers of GL_n and the Gauss–Schering Lemma

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Résumé. Le lemme de Gauss-Schering est une identité impliquant le symbole de Legendre utilisé dans des preuves élémentaires de la loi de réciprocité quadratique. Dans cet article nous montrons comment ce lemme peut-être généralisé pour donner une formule sur un 2-cocycle correspondant à une plus grande extension métaplectique de GL_n/k où k est un corps global. Dans le cas où la caractéristique de k est non nulle, la formule fournit une construction complète du groupe métaplectique, et par suite donne une nouvelle preuve de la loi de réciprocité pour le symbole de Legendre supérieur.

Abstract. The Gauss-Schering Lemma is a classical formula for the Legendre symbol commonly used in elementary proofs of the quadratic reciprocity law. In this paper we show how the Gauss Schering Lemma may be generalized to give a formula for a 2-cocycle corresponding to a higher metaplectic extension of GL_n/k for any global field k. In the case that k has positive characteristic, our formula gives a complete construction of the metaplectic group and consequently an independent proof of the power reciprocity law for k.

1. Introduction

Let k be a global field with adele ring $\mathbb{A}$ and let G/k be an affine algebraic group. A *metaplectic* extension of G by a discrete Abelian group A is a topological central extension:

$$1 \to A \to \widetilde{G}(\mathbb{A}) \to G(\mathbb{A}) \to 1,$$

which splits over the group $G(k)$ of k-rational points. This means that the group $G(k)$ lifts to a subgroup of $\widetilde{G}(\mathbb{A})$. The covering group $\widetilde{G}(\mathbb{A})$ is called a metaplectic group, or metaplectic cover of G. Metaplectic groups are important since, just as the usual automorphic forms on G are functions on $G(\mathbb{A})$ which are invariant under translations by $G(k)$, forms of non-integral weight on G can be regarded as functions on $\widetilde{G}(\mathbb{A})$ which are invariant

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under translations by a lift of $G(k)$. Metaplectic extensions of G by A are classified by elements of $H^2(G(\mathbb{A}), A)$ which split when restricted to $G(k)$. The cohomology groups here are based on Borel-measurable cochains and A is regarded as a trivial, discrete $G(\mathbb{A})$-module. In this paper the algebraic group G will always be the general linear group GL_n/k and A will be a group of roots of unity in k.

Let μ_m be a group of roots of unity in k with $m = \#\mu_m$. There is a canonical metaplectic extension of SL_n/k by μ_m. This extension was constructed by T. Kubota [7], [8] in the case $n = 2$ and by H. Matsumoto [11] for general n (see also [12] or [5]). By embedding GL_n in SL_{n+1} one obtains a metaplectic extension on GL_n. Of course the extension of SL_n can be recovered from that of GL_n in the same way.

In this article we shall give a different construction of the metaplectic cover of SL_n. Our construction will be explicit in the sense that we are able to write down a cocycle corresponding to the extension. We shall show how the cocycle is related to the Gauss–Schering Lemma in the case $n = 1$, and as a corollary we obtain Weil's reciprocity law. The results described here are contained in a more general form in [4]. However we shall give more elementary proofs and limit ourselves to a minimum of notation. We shall also emphasize the connection between the cocycle and the Gauss–Schering Lemma.

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Notation. Throughout, k will denote a global field containing a primitive m-th root of unity. We shall write μ_m for the group of all m-th roots of unity in k. For a place v of k we shall write $\mathfrak{o}_v$ for the ring of integers in k_v and $(\cdot,\cdot)_{v,m}$ for the m-th power Hilbert symbol on k_v. We shall write π_v for a local uniformizer in k_v. Given a finite set S of places of k containing all the Archimedean places, we shall write $\mathbb{A}(S)$ for the restricted topological product of the k_v for $v \notin S$ with respect to the subrings $\mathfrak{o}_v$. Given an adele α we shall write α_v for the component of α in k_v. For an idele $\alpha \in \mathbb{A}(S)^\times$ we shall use the notation

$$|\alpha|_{\mathbb{A}(S)} = \prod_{v \notin S} |\alpha_v|_v.$$

We shall write $\mathfrak{o}^S$ for the ring of S-integers. The m-th power Legendre symbol on $\mathfrak{o}^S$ is defined by

$$\left(\frac{\alpha}{\mathfrak{b}}\right)_m = \prod_{v \mid \mathfrak{b},\ v \notin S} (\alpha, \pi_v)_{v,m}^{v_v(\mathfrak{b})},$$

where α is an S-integer and $\mathfrak{b}$ is an ideal of $\mathfrak{o}^S$ which is coprime to $m\alpha$.

Suppose T is an Abelian group, and regard μ_m as a trivial T-module. If σ is an inhomogeneous 2-cocycle on T with values in μ_m then the *commutator* of σ is the function $T \times T \to \mu_m$ given by:

$$[\alpha, \beta]_\sigma = \frac{\sigma(\alpha, \beta)}{\sigma(\beta, \alpha)}.$$

This map is bimultiplicative, skew-symmetric and depends only on the cohomology class of σ. We shall write $H^2_{sym}(T, \mu_m)$ for the subgroup of classes whose commutators are trivial. Cocycles in H^2_{sym} are called symmetric cocycles. It is known (see [6]) that the restriction map gives an isomorphism

$$H^2_{sym}(T, \mu_m) \to H^2_{sym}(T[m], \mu_m),$$

where $T[m]$ is the m-torsion subgroup of T.

2. The cocycle

We begin by describing the cocycle. Let S be the finite set of places v of k for which $|m|_v \neq 1$. Note that if k is a function field then S is empty and $\mathbb{A}(S) = \mathbb{A}$.

The group μ_m acts on the Cartesian product $\mathbb{A}(S)^n$ of n copies of $\mathbb{A}(S)$ by scalar multiplication. Let U be a μ_m-invariant compact, open neighbourhood of 0 in $\mathbb{A}(S)^n$ and let F be an open fundamental domain for the action of μ_m on $\mathbb{A}(S)^n \setminus \{0\}$. We shall write $f : \mathbb{A}(S)^n \to \mathbb{Z}$ for the characteristic function of F. Define for $\alpha, \beta \in \mathrm{GL}_n(\mathbb{A}(S))$,

$$(A) \qquad \sigma_{U,F}(\alpha, \beta) = \prod_{\zeta \in \mu_m} \zeta^{\left\{\int_U - \int_{\beta U}\right\} f(\alpha x) f(\zeta x) dx},$$

where the Haar measure dx on $\mathbb{A}(S)^n$ is normalized so that U has measure 1. The powers in this product are all rational numbers whose denominators are coprime to m. It therefore makes sense to raise ζ to such a power.

Theorem ([4]). *The function $\sigma_{U,F}$ is a continuous, inhomogeneous 2-cocycle on $\mathrm{GL}_n(\mathbb{A}(S))$. The cohomology class of $\sigma_{U,F}$ is independent of U and F. If k is a function field then $\sigma_{U,F}$ is metaplectic. Its restriction to SL_n corresponds to the canonical metaplectic extension of SL_n. Furthermore $\sigma_{U,F}$ splits on every compact subgroup of $\mathrm{GL}_n(\mathbb{A})$.*

Remark. In fact σ is not quite the standard metaplectic class on the whole group GL_n. In particular if m is even then σ cannot be obtained by pulling back a cocycle on a larger group SL_{n+r} to GL_r (see [4]). There is a sense in which σ is a nicer cocycle on GL_n than Matsumoto's, since it has the following functorial property. Let l be a finite extension of k of degree d. Then $\mathrm{Rest}^l_k(\mathrm{GL}_n/l)$ may be regarded as a subgroup of GL_{nd}/k. The cocycle $\sigma^{(l)}$ which we obtain on GL_n/l is the restriction of the cocycle $\sigma^{(k)}$ on GL_{nd}/k. This compatibility does not hold for Matsumoto's cocycles.

The theorem will be proved in sections 4, 5 and 6. At first sight the cocycle σ seems completely unrelated to any other number theoretical objects. In section 3 we shall show that it is in fact closely related to the Gauss–Schering Lemma.

3. The Gauss–Schering Lemma

One of Gauss' proofs of the quadratic reciprocity law is based on the following Lemma

Gauss' Lemma. *Let β be an odd prime number and α a natural number not divisible by β. Then*

$$\left(\frac{\alpha}{\beta}\right)_2 = (-1)^r,$$

where r is the number of residue classes $i \in \{1, 2, \dots, \frac{\beta-1}{2}\}$ modulo β such that αi is congruent to one of the numbers $\{-1, \dots, -\frac{\beta-1}{2}\}$ modulo β.

In work unpublished during his lifetime, Gauss [1] generalized this lemma to include composite β and 4th power residue symbols in the Gaussian integers. He went on to use the lemma to prove the biquadratic reciprocity law for the Gaussian integers. However his proof using the lemma is much harder than his later proof using Gauss sums. Some time later, Schering [13] published a proof that the lemma holds for composite β and the generalized lemma came to be known as the Gauss–Schering Lemma. More recently [2], [9], [10], [3] the Gauss–Schering lemma has been used to prove the general power reciprocity law in a number field.

We shall now recall a general form of the Gauss–Schering Lemma. Let k and $\mu_m \subset k$ be as above and let S be a finite set of places of k containing all Archimedian places. Then for an S-integer α and an ideal $\mathfrak{b} \subseteq \mathfrak{o}^S$ with $\mathfrak{b}$ coprime to $m\alpha$, the m-th power Legendre symbol $(\alpha/\mathfrak{b})_m$ is defined. By a $\frac{1}{m}$-set modulo $\mathfrak{b}$ we shall mean a subset $F \subset \mathfrak{o}^S/\mathfrak{b}$ such that

$$(\mathfrak{o}^S/\mathfrak{b}) \setminus \{0\} = \mathop{\dot{\bigcup}}_{\zeta \in \mu_m} \zeta F,$$

where the union here is disjoint.

Generalized Gauss–Schering Lemma. *Let $\alpha \in \mathfrak{o}^S$ and let $\mathfrak{b} \subseteq \mathfrak{o}^S$ be coprime to $m\alpha$. Then for any $\frac{1}{m}$-set F modulo $\mathfrak{b}$, one has*

$$\left(\frac{\alpha}{\mathfrak{b}}\right)_m = \prod_{\zeta \in \mu_m} \zeta^{r(\zeta)},$$

where $r(\zeta)$ is the number of elements $i \in F$ such that $\alpha i \in \zeta F$.

A proof of this may be found for example in [9] or [3].

We shall now reformulate the lemma slightly. Let $f : \mathfrak{o}^S/\mathfrak{b} \to \{0,1\}$ be the characteristic function of F. The power $r(\zeta)$ in the Gauss–Schering Lemma can now be expressed in the form

$$r(\zeta) = \sum_{i\in(\mathfrak{o}^S/\beta)\setminus\{0\}} f(\alpha i) f(\zeta i).$$

We therefore have

$$\left(\frac{\alpha}{\mathfrak{b}}\right)_m = \prod_{\zeta\in\mu_m} \zeta^{\sum_{i\in(\mathfrak{o}^S/\mathfrak{b})\setminus\{0\}} f(\alpha i) f(\zeta i)}.$$

Replacing the sum by an integral over $\mathbb{A}(S)$ we obtain a formula for $(\alpha/\mathfrak{b})_m$ of exactly the same form as (A). More precisely, if we regard $\mathfrak{o}^S \setminus \{0\}$ as being embedded diagonally in $\mathbb{A}(S)^\times = \mathrm{GL}_1(\mathbb{A}(S))$ then we have

$$\left(\frac{\alpha}{\beta}\right)_m = \sigma_{U,F}(\alpha,\beta),$$

where $U = \prod_{v\notin S} \mathfrak{o}_v$ and F coincides on $U \setminus \beta U$ with a $\frac{1}{m}$-set modulo β.

The above relation between Legendre symbols and the cocycle $\sigma_{U,F}$ is still rather unsatisfactory since our choice of F depends on β. We shall now give a relation between the commutator of σ and Hilbert symbols.

Proposition 1. *For $\alpha, \beta \in \mathbb{A}(S)^\times$ the following holds:*

$$[\alpha,\beta]_\sigma = (-1)^{\frac{(m-1)(|\alpha|_{\mathbb{A}(S)}-1)(|\beta|_{\mathbb{A}(S)}-1)}{m^2}} \prod_{v\notin S} (\alpha_v,\beta_v)_{v,m}.$$

Remark. Hilbert symbols are partially skew symmetric in the sense that $(\alpha,\beta)_{v,m} = (\beta,\alpha)_{v,m}^{-1}$. However if m is even then it is not always true that $(\alpha,\alpha)_{v,m} = 1$. If the Hilbert symbol fails to be skew symmetric in this way then it clearly cannot be the commutator of a cocycle. The factor $(-1)^{\frac{(m-1)(|\alpha|_{\mathbb{A}(S)}-1)(|\beta|_{\mathbb{A}(S)}-1)}{m^2}}$ in the above formula compensates for the lack of skew symmetry in the Hilbert symbols.

Proof. Note that $\mathbb{A}(S)^\times$ is generated by the following set:

$$\prod_{v\notin S} \mathfrak{o}_v^\times \cup \{\pi_v : v \notin S\}.$$

Here by abuse of notation π_v denotes an idele whose w-component is 1 for $w \neq v$ and whose v-component is a local uniformizer of k_v. Since both sides of the equation in the proposition are bilinear and skew symmetric, it suffices to prove the equality for α and β in the generating set.

First suppose $\alpha, \beta \in \prod_{v\notin S} \mathfrak{o}_v^\times$. Since $\alpha U = \beta U = U$, we have

$$\sigma_{U,F}(\alpha,\beta) = \sigma_{U,F}(\beta,\alpha) = 1.$$

Therefore $[\alpha,\beta]_\sigma = 1$.

We next treat the case $\alpha \in \prod_{v\notin S} \mathfrak{o}_v^\times$ and $\beta = \pi_w$. For any choice of F it follows as before that $\sigma_{U,F}(\beta,\alpha) = 1$. We therefore have

$$[\alpha,\beta]_\sigma = \sigma_{U,F}(\alpha,\beta).$$

We must show that $\sigma_{U,F}(\alpha,\beta) = (\alpha_w, \pi_w)_{w,m}$. Note that we are still free to choose F in such a way that we can calculate $\sigma_{U,F}(\alpha,\beta)$. First let F_1 be a lift to $\mathfrak{o}_w^\times$ in a $\frac{1}{m}$-set F_0 modulo π_w. Define

$$F_2 = F_1 \times U',$$

where $U' = \prod_{v\notin S\cup\{w\}} \mathfrak{o}_v$. Then F_2 is an open fundamental domain for μ_m in $U \setminus \beta U$. This may be extended to an open fundamental domain F_3 for μ_m in $\mathbb{A}(S) \setminus \{0\}$. We shall write f_i for the characteristic function of F_i. We have

$$\begin{aligned}
\int_{U\setminus\beta U} f_3(\zeta x) f_3(\alpha x) dx &= \int_{U\setminus\beta U} f_2(\zeta x) f_2(\alpha x) dx \\
&= \int_{U'} \int_{\mathfrak{o}_w^\times} f_1(\zeta x) f_1(\alpha x) dx \\
&\equiv \sum_{x\in(\mathfrak{o}_w/\pi_w \mathfrak{o}_w)^\times} f_0(\zeta x) f_0(\alpha x) dx \bmod m.
\end{aligned}$$

The result now follows from the Gauss–Schering Lemma taking $F = F_3$.

Finally suppose $\alpha = \pi_v$ and $\beta = \pi_w$ with $v \neq w$. We must show that

$$[\alpha,\beta]_\sigma = (-1)^{\frac{(m-1)(|\alpha|_{\mathbb{A}(S)}-1)(|\beta|_{\mathbb{A}(S)}-1)}{m^2}}.$$

This is a routine but long calculation and is left to the reader. $\square$

As a corollary to this and the theorem we obtain:

Weil's Reciprocity Law. *Let k be a global field of positive characteristic. For $\alpha,\beta \in k^\times$ we have*

$$\prod_v (\alpha,\beta)_{v,m} = 1.$$

4. The Cocycle Relation

The Legendre symbol $(\alpha/\beta)_m$ is defined for composite β by multiplicativity. Thus in order to prove that the Gauss–Schering Lemma holds for composite β one must show that the right hand side of the formula in the lemma is also multiplicative in β. If one follows the proof of this fact through without assuming that α and β commute or are coprime then instead of obtaining a multiplicativity relation, one obtains the cocycle relation for $\sigma_{U,F}$:

$$\sigma(\alpha\beta,\gamma)\sigma(\alpha,\beta)=\sigma(\alpha,\beta\gamma)\sigma(\beta,\gamma).$$

We shall go through this proof now.

We simply calculate $\sigma(\alpha,\beta\gamma)$ as follows:

$$\begin{aligned}\sigma(\alpha,\beta\gamma) &= \prod_{\zeta\in\mu_m}\zeta^{\left\{\int_U-\int_{\beta U}+\int_{\beta U}-\int_{\beta\gamma U}\right\}f(\alpha x)f(\zeta x)dx}\\ &= \sigma(\alpha,\beta)\prod_{\zeta\in\mu_m}\zeta^{\left\{\int_{\beta U}-\int_{\beta\gamma U}\right\}f(\alpha x)f(\zeta x)dx}\\ &= \sigma(\alpha,\beta)\prod_{\zeta\in\mu_m}\zeta^{|\det\beta|_{\mathbb{A}(S)}\left\{\int_U-\int_{\gamma U}\right\}f(\alpha\beta x)f(\zeta\beta x)dx}.\end{aligned}$$

Since $|\det\beta|_{\mathbb{A}(S)}$ is congruent to 1 modulo m, we have

$$\begin{aligned}&\sigma(\alpha,\beta\gamma)=\sigma(\alpha,\beta)\\ &\quad\times\prod_{\zeta\in\mu_m}\zeta^{\left\{\int_U-\int_{\gamma U}\right\}\Big(f(\alpha\beta x)f(\zeta x)+f(\alpha\beta x)\big(f(\zeta\beta x)-f(\zeta x)\big)\Big)dx}\\ &\qquad=\sigma(\alpha,\beta)\sigma(\alpha\beta,\gamma)\prod_{\zeta\in\mu_m}\zeta^{\left\{\int_U-\int_{\gamma U}\right\}f(\alpha\beta x)(f(\zeta\beta x)-f(\zeta x))dx}.\end{aligned}$$

It remains to show that

$$\sigma(\beta,\gamma)=\prod_{\zeta\in\mu_m}\zeta^{\left\{\int_U-\int_{\gamma U}\right\}f(\alpha\beta x)(f(\zeta x)-f(\zeta\beta x))dx}. \tag{B}$$

This follows from the following lemma.

Lemma 1.

$$\prod_{\zeta\in\mu_m}\zeta^{\left\{\int_U-\int_{\gamma U}\right\}(f(\alpha\beta x)-f(x))(f(\zeta x)-f(\zeta\beta x))dx}=1.$$

Proof. Since U and γU are μ_m-invariant, it is sufficient to show that for any non-zero vector $y\in\mathbb{A}(S)^n\setminus\{0\}$ we have

$$\prod_{\zeta\in\mu_m}\zeta^{\sum_{x\in\mu_m y}(f(\alpha\beta x)-f(x))(f(\zeta x)-f(\zeta\beta x))}=1,$$

where $\mu_m y$ denotes the μ_m orbit of y. This follows easily from the relation

$$\sum_{\zeta\in\mu_m} f(\zeta x) = 1.$$

□

From the lemma we have

$$\prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\gamma U}\right\} f(\alpha\beta x)(f(\zeta x) - f(\zeta\beta x))dx}$$
$$= \prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\gamma U}\right\} f(x)(f(\zeta x) - f(\zeta\beta x))dx}.$$

Note that $f(x)f(\zeta x) = 0$ unless $\zeta = 1$. We therefore have

$$\prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\gamma U}\right\} f(\alpha\beta x)(f(\zeta x) - f(\zeta\beta x))dx}$$
$$= \prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\gamma U}\right\} - f(x)f(\zeta\beta x)dx}.$$

Replacing x by $\zeta^{-1}x$ we obtain:

$$\prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\gamma U}\right\} f(\alpha\beta x)(f(\zeta x) - f(\zeta\beta x))dx}$$
$$= \prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\gamma U}\right\} - f(\zeta^{-1}x)f(\beta x)dx}.$$

Finally replacing ζ by ζ^{-1} we obtain (B).

5. Independence of U and F

We shall now show that the cohomology class of $\sigma_{U,F}$ is independent of U and F. We first fix F and vary U. Let U' be another μ_m-invariant, compact, open neighbourhood of 0 in $\mathbb{A}(S)$ and define

$$\tau(\alpha) = \prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{U'}\right\} f(\alpha x)f(\zeta x)dx}.$$

We shall show that

$$\frac{\sigma_{U,F}(\alpha,\beta)}{\sigma_{U',F}(\alpha,\beta)} = \frac{\tau(\alpha)\tau(\beta)}{\tau(\alpha\beta)}.$$

First note that

$$\begin{aligned}\frac{\sigma_{U,F}(\alpha,\beta)}{\sigma_{U',F}(\alpha,\beta)} &= \prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{U'}\right\}(f(\alpha x)f(\zeta x) - f(\alpha\beta x)f(\beta\zeta x))dx} \\ &= \tau(\alpha)\prod_{\zeta\in\mu_m} \zeta^{\left\{\int_{U'} - \int_{U}\right\} f(\alpha\beta x)f(\beta\zeta x))dx} \\ &= \frac{\tau(\alpha)}{\tau(\alpha\beta)}\prod_{\zeta\in\mu_m} \zeta^{\left\{\int_{U'} - \int_{U}\right\} f(\alpha\beta x)(f(\beta\zeta x)-f(\zeta x))dx}.\end{aligned}$$

Now using Lemma 1 we obtain

$$\frac{\sigma_{U,F}(\alpha,\beta)}{\sigma_{U',F}(\alpha,\beta)} = \frac{\tau(\alpha)}{\tau(\alpha\beta)}\prod_{\zeta\in\mu_m} \zeta^{\left\{\int_{U'} - \int_{U}\right\} f(x)(f(\beta\zeta x)-f(\zeta x))dx}.$$

Again since $f(x)f(\zeta x) = 0$ unless $\zeta = 1$, we obtain the result.

A similar argument shows that if F' is another open fundamental domain for the action of μ_m on $\mathbb{A}(S)\setminus\{0\}$ then we have

$$\frac{\sigma_{U,F}(\alpha,\beta)}{\sigma_{U,F'}(\alpha,\beta)} = \frac{\epsilon(\alpha)\epsilon(\beta)}{\epsilon(\alpha\beta)},$$

where

$$\epsilon(\alpha) = \prod_{\zeta\in\mu_m} \zeta^{\left\{\int_U - \int_{\alpha U}\right\} f'(x)f(\zeta x)dx}.$$

Here f' denotes the characteristic function of F'.

6. The rest of the proof

The other properties of the cocycle are almost immediate from its definition with suitable choices of F and U. We begin by showing that σ splits on any compact subgroup of $\mathrm{GL}_n(\mathbb{A}(S))$. Let K be a compact subgroup of $\mathrm{GL}_n(\mathbb{A}(S))$. Then one may choose the neighbourhood U of 0 in $\mathbb{A}(S)^n$ to be K-invariant. One then has immediately for $\alpha,\beta\in K$,

$$\sigma_{F,U}(\alpha,\beta) = 1.$$

We next show that σ splits on $\mathrm{GL}_n(k)$ in the case that k is a function field. To prove this we would like to take F to be invariant under translations by k^n and U to be a fundamental domain for k^n in $\mathbb{A}^n$. Then for $\alpha,\beta\in\mathrm{GL}_n(k)$ and $\zeta\in\mu_m$ the function

$$x \mapsto f(\alpha x)f(\zeta x)$$

is periodic modulo translations by k^n. On the other hand U and βU are two different fundamental domains for k^n. We therefore have

$$\left\{\int_U - \int_{\beta U}\right\} f(\alpha x)f(\zeta x) = 0,$$

which implies

$$\sigma_{F,U}(\alpha,\beta) = 1.$$

There is a problem with this method. Unfortunately there is no open fundamental domain F for μ_m in $\mathbb{A}^n \setminus \{0\}$ which is k^n-invariant. However if we are only interested in the restriction of σ to $\mathrm{GL}_n(k)$, we may instead take F to be an open fundamental domain for μ_m in $\mathbb{A}^n \setminus k^n$. Thus its characteristic function f will be discontinuous on k^n. Since U and βU have the same intersection with k^n, it follows that the integrals

$$\left\{ \int_U - \int_{\beta U} \right\} f(\alpha x) f(\zeta x) dx$$

will still be rationals whose denominators are coprime to m. We may therefore use such an F to define a cocycle. Now it is possible to take F to be translation invariant modulo k^n, so the above argument shows that σ splits on $\mathrm{GL}_n(k)$.

Finally we give a sketch proof that the restriction of σ to SL_n corresponds to the canonical metaplectic extension of SL_n. The canonical extension is determined by its restriction to the subgroup T of diagonal matrices with determinant 1. For a diagonal matrix α we shall write α_i for the i-th entry on the diagonal of α.

On $T(\mathbb{A}(S))$ the canonical extension is given by the cocycle

$$c(\alpha,\beta) = \prod_{v \notin S} \prod_{1 \le i < j \le n} (\alpha_i, \beta_j)_{v,m}.$$

One calculates that the commutator of c is

$$[\alpha,\beta]_c = \prod_{v \notin S} \prod_{i=1}^{n} (\alpha_i, \beta_i)_{v,m}.$$

By generalizing Proposition 1 one may show that this is the same as the commutator of σ on T. Thus σ/c represents a symmetric class on $T(\mathbb{A}(S))$. To show that σ is cohomologous to c on $\mathrm{SL}_n(\mathbb{A}(S))$ it remains only to show that σ/c splits on $T(\mathbb{A}(S))[m]$. Since $T(\mathbb{A}(S))[m]$ is relatively compact in $\mathrm{GL}_n(\mathbb{A}(S))$ it follows that σ splits on $T(\mathbb{A}(S))[m]$. On the other hand if $\alpha, \beta \in T(\mathbb{A}(S))[m]$ then since the Hilbert symbols for $v \notin S$ are unramified, it follows that $c(\alpha,\beta) = 1$. This finishes the proof.

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