Journal de Théorie des Nombres
de Bordeaux **13** (2001), 227–240

The representation of almost all numbers as sums of unlike powers

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Résumé. Nous prouvons dans cet article que presque tout entier s'écrit comme la somme d'un cube, d'un bicarré, ..., et d'une puissance dixième.

Abstract. We prove in this article that almost all large integers have a representation as the sum of a cube, a biquadrate, ..., and a tenth power.

1. Introduction

Although the somewhat esoteric appearance of additive problems involving sums of mixed powers attracts a thinner audience than the more conventional versions of Waring's problem, these mixed problems have provided useful specimens for the testing and development of new technology since the earliest days of the Hardy-Littlewood method (see, for example, [4, 7, 15]). Following early investigations of Roth [8, 9], particular attention has focused on sums of ascending powers. When r is a natural number, let $H(r)$ denote the least number s such that all sufficiently large integers n are represented in the form

$$n = x_1^r + x_2^{r+1} + \cdots + x_s^{r+s-1},$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$. Also, let $H^+(r)$ denote the corresponding number s, where we instead merely seek to represent almost all integers n, in the sense of natural density. Then Roth [8] established that $H^+(2) \leqslant 3$, a conclusion that is transparently best possible, and subsequently (see [9]) provided the upper bound $H(2) \leqslant 50$. Improving on previous work of Vaughan [13, 14], Thanigasalam [10, 11, 12], and Brüdern [1, 2], it has recently been shown by Ford [5, 6] that $H(2) \leqslant 14$, and moreover Ford also supplies the bounds $H(3) \leqslant 72$ and, for large r, gives $H(r) \ll r^2 \log r$. Our purpose in this paper is to bound $H^+(r)$ for the smallest value of r as yet unresolved.

Manuscrit reçu le 15 septembre 1999.

[1]Written while the author visited the University of Michigan at Ann Arbor, and enjoyed the benefits of a Fellowship from the Consiglio Nazionale delle Ricerche (Italy).

[2]Packard Fellow, and supported in part by NSF grant DMS-9970440.

Theorem 1. *One has $H^+(3) \leqslant 8$.*

The lower bound $H^+(3) \geqslant 5$ is immediate from the observation that $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}<1$, and so the conclusion of Theorem 1 is not astronomically far from the truth. Cursory computations indicate that the methods of this paper yield a bound at least as strong as $H^+(4) \leqslant 20$. We note also that when r is large, the methods of [6, §5] are easily adapted to give $H^+(r) \ll r^{3/2}(\log r)^{1/2}$. We establish Theorem 1 by means of the Hardy-Littlewood method, exploiting recent new estimates for mean values of smooth Weyl sums (see, in particular, [3, 18, 21, 22, 24, 26]. These methods yield a lower bound for the number of representations in the prescribed form of the expected size predicted by a formal application of the circle method. When n is a natural number, let $\nu(n)$ denote the number of representations of n in the shape

$$n = x_1^3 + x_2^4 + \cdots + x_8^{10}, \tag{1}$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 8)$. Then we obtain the following theorem.

Theorem 2. *There is a positive number τ satisfying the property that, for all but $O(N(\log N)^{-\tau})$ of the natural numbers n with $1 \leqslant n \leqslant N$, one has $\nu(n) \gg n^{\frac{1081}{2520}}$.*

We offer an outline of the proof of Theorem 2 in §2 below, wherein we also negotiate certain preliminaries. Plainly, Theorem 1 is an immediate consequence of Theorem 2. Throughout, ε will denote a sufficiently small positive number, and k will denote a positive integer, usually in the range $3 \leqslant k \leqslant 10$. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, with implicit constants depending at most on ε and k, unless otherwise indicated. In an effort to simplify our analysis, we adopt the following convention concerning the number ε. Whenever ε appears in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, when y is a real number we write $[y]$ for the greatest integer not exceeding y.

The first author is grateful to the Department of Mathematics at the University of Michigan, Ann Arbor, for its generous hospitality and excellent working conditions during the period in which this paper was written.

2. Preliminaries to the main argument

In this section we describe the strategy underlying our application of the circle method, and at the same time record auxiliary estimates for later use. As mentioned in the introduction, we make fundamental use of smooth Weyl sums. In this context, when X and Y are real numbers with $2 \leqslant Y \leqslant X$, define the set of Y-smooth numbers up to X by

$$\mathcal{A}(X,Y) = \{n \in [1,X] \cap \mathbb{Z} : p|n \text{ and } p \text{ prime implies that } p \leqslant Y\}.$$

As usual, we write $e(z)$ for $e^{2\pi iz}$, and when k is a natural number, we define

$$(2) \quad f_k(\alpha; X,Y) = \sum_{x \in \mathcal{A}(X,Y)} e(\alpha x^k) \quad \text{and} \quad F_k(\alpha; X) = \sum_{X/2 < x \leqslant X} e(\alpha x^k).$$

Finally, when s is a positive real number, we write

$$U_s^{(k)}(X,Y) = \int_0^1 |f_k(\alpha; X,Y)|^s d\alpha.$$

We say that an exponent $\mu_s^{(k)}$ is *permissible* whenever the exponent has the property that, for each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon, s, k)$ such that whenever $Y \leqslant X^\eta$, then one has

$$U_s^{(k)}(X,Y) \ll_{\varepsilon,s,k} X^{\mu_s^{(k)}+\varepsilon}.$$

In the argument used to establish Theorem 2, we make use of the permissible exponents listed in the table below. For $k = 4$, these exponents follow from the table in §2 of [3]. When $k = 5, 6, 7$, these exponents are provided in the appendix of [21] (but see [22, §9] for $k = 7$ and $s = 36$). Finally, when $k = 8, 9, 10$, these exponents are recorded in [22, §§10,11,12], on noting the remarks concerning process D^s concluding §8 of that paper. Note that the exponent λ_s in these sources corresponds here to our μ_{2s}. We take $\delta = 10^{-10}$, and fix η to be a positive number, small enough so that for each s and k listed in the table, whenever X is sufficiently large and $Y \leqslant X^\eta$, one has

$$U_s^{(k)}(X,Y) \ll X^{\mu_s^{(k)}+\delta}.$$

Table of permissible exponents.

k	s	$\mu_s^{(k)}$	k	s	$\mu_s^{(k)}$	k	s	$\mu_s^{(k)}$
4	7.7	4.358530	7	14	8.541090	9	20	12.746344
4	12	8.000000	7	16	10.152633	9	22	14.410584
5	10	5.925080	7	36	29.000000	10	30	20.930371
5	18	13.000000	8	18	11.452911	10	32	22.753746
6	12	7.231564	8	20	13.128307	10	60	50.000000
6	14	8.850572	8	72	64.000000			
6	24	18.000000						

Consider next a positive number N sufficiently large in terms of η, and define

$$P_3 = (N/4)^{1/3} \quad \text{and} \quad P_k = N^{1/k} \quad (4 \leqslant k \leqslant 10). \tag{3}$$

When n is an integer with $N/2 < n \leqslant N$, we consider the number $\nu^*(n)$ of representations of n in the form (1) with $P_3/2 < x_1 \leqslant P_3$ and $x_{k-2} \in \mathcal{A}(P_k, P_k^\eta)$ $(4 \leqslant k \leqslant 10)$. Plainly, one has $\nu(n) \geqslant \nu^*(n)$ for each such integer n. For the sake of concision, we modify the notation introduced in (2) by writing

$$F_3(\alpha) = F_3(\alpha; P_3) \quad \text{and} \quad f_k(\alpha) = f_k(\alpha; P_k, P_k^\eta) \quad (4 \leqslant k \leqslant 10).$$

Also, we write

$$\mathcal{F}(\alpha) = F_3(\alpha) f_4(\alpha) \dots f_{10}(\alpha), \tag{4}$$

and when $\mathfrak{B} \subseteq [0,1)$, we define

$$\nu^*(n; \mathfrak{B}) = \int_{\mathfrak{B}} \mathcal{F}(\alpha) e(-n\alpha) d\alpha. \tag{5}$$

Then by orthogonality, one has $\nu^*(n) = \nu^*(n; [0,1))$.

We estimate the integral (5) by means of the circle method. Our primary Hardy-Littlewood dissection is defined as follows. Write $L = (\log N)^\delta$, and denote by $\mathfrak{P}$ the union of the major arcs

$$\mathfrak{P}(q,a) = \{\alpha \in [0,1) \; : \; |\alpha - a/q| \leqslant LN^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant L$ and $(a,q) = 1$. Also, define $\mathfrak{p} = [0,1) \setminus \mathfrak{P}$. The object of the first phase of our analysis is to show that, for a suitable positive number τ,

$$\int_{\mathfrak{p}} |\mathcal{F}(\alpha)|^2 d\alpha \ll \mathcal{F}(0)^2 N^{-1} L^{-\tau}, \tag{6}$$

whence, as a consequence of Bessel's inequality,

$$\sum_{N/2 < n \leqslant N} |\nu^*(n; \mathfrak{p})|^2 \leqslant \int_{\mathfrak{p}} |\mathcal{F}(\alpha)|^2 d\alpha \ll \mathcal{F}(0)^2 N^{-1} L^{-\tau}. \tag{7}$$

This first objective we achieve in three steps. Define $\mathfrak{M}$ to be the union of the arcs

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) \; : \; |q\alpha - a| \leqslant P_3^{3/4} N^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant P_3^{3/4}$ and $(a,q) = 1$, and write $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. In §3, we estimate the minor arc contribution $\nu^*(n; \mathfrak{m})$ in mean square. Let $\mathfrak{N}$ denote the union of the arcs

$$\mathfrak{N}(q,a) = \{\alpha \in [0,1) \; : \; |q\alpha - a| \leqslant N^{\delta - 1}\},$$

with $0 \leqslant a \leqslant q \leqslant N^\delta$ and $(a,q) = 1$. Then in §5 we prune the set $\mathfrak{M}$ down to $\mathfrak{N}$, establishing that $\nu^*(n; \mathfrak{M} \setminus \mathfrak{N})$ makes a negligible contribution to (7) in mean square. We prune down to the thin set $\mathfrak{P}$ in §6, thereby completing the proof of (7). Experts will recognise that the primary difficulty in our analysis lies with the small number of classical Weyl sums present in (5). Thus, while the work in §§3 and 5 is essentially routine, the pruning process of §6 requires a technical lemma not available in the literature. Fortunately, recent work of Brüdern and Wooley [3] (see also [22, Lemma 5.4]) provides the inspiration to surmount the latter difficulty in §4.

In the second phase of our analysis, in §7, we employ major arc technology familiar to aficionados of the new iterative methods in order to establish a lower bound of the shape

$$\nu^*(n;\mathfrak{P}) \gg \mathcal{F}(0)N^{-1}, \tag{8}$$

uniformly for $N/2 < n \leqslant N$. In combination with (7), this lower bound shows that

$$\nu^*(n) = \nu^*(n;\mathfrak{P}) + \nu^*(n;\mathfrak{p}) \gg \mathcal{F}(0)N^{-1}(1 + O(L^{-\tau/3}))$$

for all but $O(NL^{-\tau/3})$ of the integers n with $N/2 < n \leqslant N$. The conclusion of Theorem 2 follows immediately, whence also Theorem 1.

3. The minor arc contribution

Our goal in this section is to estimate $\nu^*(n;\mathfrak{m})$ in mean square, and this we achieve with a swift application of Hölder's inequality. We remark that mixed mean values incorporating efficient differencing processes of the type used by Ford [6] are not worthwhile in the present context. Although we have developed more efficient processes that do improve the quality of our bounds here, it transpires that such improvements leave no visible trace in our final analysis.

k	4	5	6	7	8	9
s_k	3.850	5.000	6.608	7.964	9.369	10.734

Define the numbers s_k $(4 \leqslant k \leqslant 9)$ as in the table above, and define s_{10} by means of $s_4^{-1} + \cdots + s_{10}^{-1} = 1$. Then recalling (4) and applying Hölder's inequality, we obtain

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^2 d\alpha \leqslant \Big(\sup_{\alpha\in\mathfrak{m}} |F_3(\alpha)|\Big)^2 \prod_{k=4}^{10} \Big(\int_0^1 |f_k(\alpha)|^{2s_k} d\alpha\Big)^{1/s_k}.$$

Suppose that $5 \leqslant k \leqslant 10$. Then on writing $t_k = [s_k + 1]$ and $\theta_k = t_k - s_k$, we find that a second application of Hölder's inequality yields

$$\int_0^1 |f_k(\alpha)|^{2s_k} d\alpha \leqslant \Big(\int_0^1 |f_k(\alpha)|^{2t_k-2} d\alpha\Big)^{\theta_k} \Big(\int_0^1 |f_k(\alpha)|^{2t_k} d\alpha\Big)^{1-\theta_k}.$$

Also, in view of the definition of $\mathfrak{m}$, it follows from [16, Lemma 1] that

$$\sup_{\alpha\in\mathfrak{m}} |F_3(\alpha)| \ll P_3^{3/4+\varepsilon}. \tag{9}$$

Thus, if we write $\nu_k = (\theta_k\mu_{2t_k-2}^{(k)} + (1-\theta_k)\mu_{2t_k}^{(k)})/s_k$ for $5 \leqslant k \leqslant 10$, and suppose that each $\mu_s^{(k)}$ is a permissible exponent, then we obtain

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^2 d\alpha \ll N^\varepsilon P_3^{3/2} P_4^{\mu_{7.7}^{(4)}/3.85} P_5^{\nu_5} \ldots P_{10}^{\nu_{10}}.$$

On recalling (3) and the table of exponents from §2, therefore, a modicum of computation reveals that with a real number ϕ exceeding 0.0023,

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^2 d\alpha \ll \mathcal{F}(0)^2 N^{-1-\phi}. \tag{10}$$

4. Preparations for pruning

Before initiating the first pruning process, we record some notation and recall certain auxiliary estimates. Write

$$v_3(\beta) = \int_{P_3/2}^{P_3} e(\beta\gamma^3)d\gamma \quad \text{and} \quad v_k(\beta) = \int_0^{P_k} e(\beta\gamma^k)d\gamma \quad (k \geqslant 4). \tag{11}$$

Also, when $k \geqslant 2$, define

$$S_k(q,a) = \sum_{r=1}^{q} e(ar^k/q),$$

and define the multiplicative function $\kappa_k(q)$ on prime powers π^l by taking

$$\kappa_k(\pi^{uk+v}) = \begin{cases} k\pi^{-u-1/2}, & \text{when } u \geqslant 0 \text{ and } v = 1, \\ \pi^{-u-1}, & \text{when } u \geqslant 0 \text{ and } 2 \leqslant v \leqslant k. \end{cases} \tag{12}$$

Then by [17, Lemma 3], whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$, one has

$$q^{-1}S_k(q,a) \ll \kappa_k(q) \quad \text{and} \quad q^{-1/2} \leqslant \kappa_k(q) \ll q^{-1/k}. \tag{13}$$

Next define $F_3^*(\alpha)$ for $\alpha \in [0,1)$ by taking

$$F_3^*(\alpha) = q^{-1}S_3(q,a)v_3(\alpha - a/q) \tag{14}$$

when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, and by taking this function to be zero otherwise. Then by [19, Theorem 4.1],

$$\sup_{\alpha\in\mathfrak{M}} |F_3(\alpha) - F_3^*(\alpha)| \ll P_3^{3/8+\varepsilon}. \tag{15}$$

We note also that by applying partial integration to (11), it follows from (13) and (14) that whenever $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$ and $(a,q) = 1$, one has

$$F_3^*(\alpha) \ll \kappa_3(q)P_3(1 + N|\alpha - a/q|)^{-1}. \tag{16}$$

Before describing our technical pruning lemma, we define an auxiliary set of major arcs. When $1 \leqslant X \leqslant P_3$, let $\mathfrak{W}(X)$ denote the union of the intervals

$$\mathfrak{W}(q,a;X) = \{\alpha \in [0,1) \ : \ |q\alpha - a| \leqslant XN^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant X$ and $(a,q) = 1$.

Lemma 1. *Suppose that* $k \geqslant 4$, $1 \leqslant X \leqslant P_k$ *and* $A > 1$. *Write* $t = [k/2]$, *and take* $\mathcal{A}$ *to be a subset of* $[1, P_k] \cap \mathbb{Z}$. *Define the function* $\Upsilon(\alpha)$ *for* $\alpha \in \mathfrak{W}(X)$ *by taking*

$$\Upsilon(\alpha) = \kappa_3(q)^2(1 + N|\alpha - a/q|)^{-A}, \tag{17}$$

when $\alpha \in \mathfrak{W}(q,a;X) \subseteq \mathfrak{W}(X)$. *Then for each* $\varepsilon > 0$,

$$\int_{\mathfrak{W}(X)} \Upsilon(\alpha)\Big|\sum_{x\in\mathcal{A}} e(\alpha x^k)\Big|^{2t} d\alpha \ll X^\varepsilon P_k^{2t} N^{-1}.$$

Proof. We follow closely the argument of the proof of [22, Lemma 5.4], noting initially that the argument leading to inequality (5.8) of that paper shows that

$$\int_{\mathfrak{W}(X)} \Upsilon(\alpha)\Big|\sum_{x\in\mathcal{A}} e(\alpha x^k)\Big|^{2t} d\alpha \ll P_k^{2t} N^{-1} \sum_{1\leqslant q\leqslant X} \kappa_3(q)^2 \sigma(q), \tag{18}$$

where $\sigma(q) = \sum_{r|q} r\kappa_k(r)^{2t}$. The function $\kappa_k(r)$ is multiplicative with respect to r, and thus $\sigma(q)$ is likewise a multiplicative function of q. Further, the argument of the proof of [22, Lemma 5.4] provides for each prime number p the upper bounds $\sigma(p) \leqslant 1 + k^{2t}p^{-1}$ and $\sigma(p^h) \ll p^{h/k}$ $(h \geqslant 2)$. When $k \geqslant 4$, therefore, we deduce from (12) that

$$\kappa_3(p)^2\sigma(p) \ll p^{-1},$$

$$\kappa_3(p^{3u+1})^2\sigma(p^{3u+1}) \ll p^{-2u-1+(3u+1)/k} \ll p^{-u-1+1/k} \quad (u \geqslant 1),$$

$$\kappa_3(p^{3u+2})^2\sigma(p^{3u+2}) \ll p^{-2u-2+(3u+2)/k} \ll p^{-u-1} \quad (u \geqslant 0).$$

The multiplicative properties of $\sigma(q)$ and $\kappa_3(q)$ thus ensure that for a suitable constant B depending at most on k,

$$\sum_{1\leqslant q\leqslant X} \kappa_3(q)^2\sigma(q) \leqslant \prod_{p\leqslant X}\Big(1 + \sum_{h=1}^{\infty} \kappa_3(p^h)^2\sigma(p^h)\Big) \leqslant \prod_{p\leqslant X}(1 + Bp^{-1}) \ll X^\varepsilon.$$

The conclusion of the lemma now follows immediately from (18). $\square$

We also require a weak estimate of Weyl type for the generating function $f_{10}(\alpha)$.

Lemma 2. *For each* $\alpha \in \mathfrak{M} \setminus \mathfrak{N}$, *one has* $|f_{10}(\alpha)| \ll P_{10}^{1-\delta/200}$.

Proof. Suppose that $\alpha \in \mathfrak{M} \setminus \mathfrak{N}$. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$, $1 \leqslant q \leqslant N^{1-\delta}$ and $|q\alpha - a| \leqslant N^{\delta-1}$. But since $\alpha \notin \mathfrak{N}$, one necessarily has $q > N^{\delta}$. Thus, on applying [25, Lemma 3.1] with $k=10$, $M = P_{10}^{3/4}$ and $t = w = 30$, we deduce that whenever $\mu_{60} = 50 + \Delta$ is a permissible exponent, one has

$$\begin{aligned} f_{10}(\alpha) &\ll q^{\varepsilon} P_{10}^{1+\varepsilon} \left(P_{10}^{\Delta} (q^{-1} + 2N^{-1/4} + qN^{-1}) \right)^{1/(2t^2)} + P_{10}^{3/4} \\ &\ll P_{10}^{1+\varepsilon} (P_{10}^{\Delta} N^{-\delta})^{1/1800}. \end{aligned}$$

But we find from the table in §2 that $\Delta = 0$ is admissible, and so $|f_{10}(\alpha)| \ll P_{10}^{1-\delta/180+\varepsilon}$. The conclusion of the lemma follows immediately. □

5. A wide set of major arcs

In this section we prune the major arcs $\mathfrak{M}$ down to the set $\mathfrak{N}$, in preparation for further pruning in the next section. We begin by replacing the generating function $F_3(\alpha)$, implicit in $\nu^*(n;\mathfrak{M})$, by its approximation $F_3^*(\alpha)$. In this context, define

$$\mathcal{F}_1(\alpha) = F_3^*(\alpha) f_4(\alpha) \dots f_{10}(\alpha). \tag{19}$$

Lemma 3. *One has*

$$\int_{\mathfrak{M}} |\mathcal{F}(\alpha) - \mathcal{F}_1(\alpha)|^2 d\alpha \ll \mathcal{F}(0)^2 N^{-1-\delta}.$$

Proof. On recalling (4) and the values of s_k from §3, it follows from Hölder's inequality that

$$\int_{\mathfrak{M}} |\mathcal{F}(\alpha) - \mathcal{F}_1(\alpha)|^2 d\alpha \leqslant \Big(\sup_{\alpha \in \mathfrak{M}} |F_3(\alpha) - F_3^*(\alpha)| \Big)^2 \prod_{k=4}^{10} \Big(\int_0^1 |f_k(\alpha)|^{2s_k} d\alpha \Big)^{1/s_k}.$$

A comparison of (9) and (15) thus reveals that the argument of §3 leading to (10) again applies, and the conclusion of the lemma follows. □

We next dispose of the contribution of the set of arcs $\mathfrak{M} \setminus \mathfrak{N}$.

Lemma 4. *One has*

$$\int_{\mathfrak{M} \setminus \mathfrak{N}} |\mathcal{F}_1(\alpha)|^2 d\alpha \ll \mathcal{F}(0)^2 N^{-1-\delta/2000}.$$

Proof. Recalling (19) and applying Hölder's inequality, a trivial estimate for $f_9(\alpha)$ yields

$$\int_{\mathfrak{M} \setminus \mathfrak{N}} |\mathcal{F}_1(\alpha)|^2 d\alpha \leqslant \Big(\sup_{\alpha \in \mathfrak{M} \setminus \mathfrak{N}} |f_{10}(\alpha)| \Big)^2 P_9^2 J_1^{2/3} J_2^{1/18} \prod_{k=5}^{8} I_k^{1/t_k}, \tag{20}$$

where

$$J_1 = \int_{\mathfrak{M}} |F_3^*(\alpha)^3 f_4(\alpha)^2| d\alpha, \quad J_2 = \int_0^1 |f_4(\alpha)|^{12} d\alpha,$$

$$I_k = \int_0^1 |f_k(\alpha)|^{2t_k} d\alpha \quad (5 \leqslant k \leqslant 8),$$

and here we take $t_5 = 9$, $t_6 = 12$, $t_7 = 18$, $t_8 = 36$. On recalling the permissible exponents from the table in §2, moreover, one has

(21) $$J_2 \ll P_4^{8+\varepsilon} \quad \text{and} \quad I_k \ll P_k^{2t_k-k+\varepsilon} \quad (5 \leqslant k \leqslant 8).$$

In order to estimate J_1, we note that by (13) and (16), whenever $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, one has $|F_3^*(\alpha)| \ll P_3 \Delta(\alpha)^{1/3}$, where $\Delta(\alpha)$ is the function defined for $\alpha \in \mathfrak{M}$ by taking $\Delta(\alpha) = (q + N|q\alpha - a|)^{-1}$, when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$. Observe also that $|f_4(\alpha)|^2 = \sum_{l \in \mathbb{Z}} \psi(l) e(l\alpha)$, where $\psi(l)$ denotes the number of solutions of the equation $z_1^4 - z_2^4 = l$, with $z_i \in \mathcal{A}(P_4, P_4^{\eta})$ $(i = 1, 2)$. Plainly, one has $\psi(0) \ll P_4$ and $\sum_{l \in \mathbb{Z}} \psi(l) = f_4(0)^2 \ll P_4^2$, and so by [2, Lemma 2], it follows that

$$\int_{\mathfrak{M}} |F_3^*(\alpha)^3 f_4(\alpha)^2| d\alpha \ll N \int_{\mathfrak{M}} \Delta(\alpha) |f_4(\alpha)|^2 d\alpha \ll N^{\varepsilon} \left(P_3^{3/4} P_4 + P_4^2 \right).$$

On recalling Lemma 2, we therefore deduce from (20) and (21) that

$$\int_{\mathfrak{M} \setminus \mathfrak{N}} |\mathcal{F}_1(\alpha)|^2 d\alpha \ll N^{\varepsilon - 1} \mathcal{F}(0)^2 P_{10}^{-\delta/100},$$

and this suffices to establish the conclusion of the lemma. □

6. Pruning

By wielding the technical pruning lemma prepared in §4, we are able to prune the set of arcs $\mathfrak{N}$ down to the thin set $\mathfrak{P}$ in a single stroke.

Lemma 5. *One has*

$$\int_{\mathfrak{N} \setminus \mathfrak{P}} |\mathcal{F}_1(\alpha)|^2 d\alpha \ll \mathcal{F}(0)^2 N^{-1} L^{-1/2}.$$

Proof. Suppose that $L \leqslant X \leqslant N^{\delta}$, and define $\mathfrak{V}(X) = \mathfrak{W}(2X) \setminus \mathfrak{W}(X)$, where $\mathfrak{W}(X)$ is as defined in §4. Then [20, Lemmata 7.2 and 8.5] show that

(22) $$\sup_{\alpha \in \mathfrak{V}(X)} |f_k(\alpha)| \ll P_k X^{-1/(2k+1)} \quad (4 \leqslant k \leqslant 10),$$

the former lemma applying in the interval $(\log N)^{10000} \leqslant X \leqslant N^{\delta}$, and the latter for $(\log N)^{\delta} \leqslant X \leqslant (\log N)^{10000}$. Recalling (19) and applying Hölder's inequality, we obtain

(23) $$\int_{\mathfrak{V}(X)} |\mathcal{F}_1(\alpha)|^2 d\alpha \leqslant \sup_{\alpha \in \mathfrak{V}(X)} |f_6(\alpha) \ldots f_{10}(\alpha)|^2 J_4^{1/2} J_5^{1/2},$$

where

$$J_k=\int_{\mathfrak{W}(2X)}|F_3^*(\alpha)^2f_k(\alpha)^4|d\alpha\quad (k=4,5).$$

But by (16), one has $F_3^*(\alpha)^2\ll P_3^2\Upsilon(\alpha)$, where $\Upsilon(\alpha)$ is the function defined for $\alpha\in\mathfrak{W}(X)$ as in (17). Thus it follows from Lemma 1 that whenever $L\leqslant X\leqslant N^\delta$, one has $J_k\ll X^\varepsilon P_3^2P_k^4N^{-1}$ $(k=4,5)$. On substituting the latter estimates into (23), and making use also of (22), we deduce that

$$\int_{\mathfrak{V}(X)}|\mathcal{F}_1(\alpha)|^2d\alpha\ll\mathcal{F}(0)^2N^{-1}X^{-1/2}. \tag{24}$$

In order to complete the proof of the lemma, we have merely to note that $\mathfrak{N}\setminus\mathfrak{P}$ is contained in the union of the sets $\mathfrak{V}(X)$ as X runs over the values 2^lL with $l\geqslant 0$ and $2^lL\leqslant N^\delta$. On summing over the latter values of X, it follows from (24) that the desired conclusion does indeed hold. $\square$

Collecting together (10) with the conclusions of Lemmata 3, 4 and 5, we find that

$$\begin{aligned}\int_{\mathfrak{p}}|\mathcal{F}(\alpha)|^2d\alpha\ll&\int_{\mathfrak{m}}|\mathcal{F}(\alpha)|^2d\alpha+\int_{\mathfrak{M}}|\mathcal{F}(\alpha)-\mathcal{F}_1(\alpha)|^2d\alpha\\ &+\int_{\mathfrak{M}\setminus\mathfrak{N}}|\mathcal{F}_1(\alpha)|^2d\alpha+\int_{\mathfrak{N}\setminus\mathfrak{P}}|\mathcal{F}_1(\alpha)|^2d\alpha\\ \ll&\mathcal{F}(0)^2N^{-1}L^{-1/2},\end{aligned}$$

whence the desired estimate (6) follows immediately.

7. The main term

Before establishing the lower bound (8), we introduce some further notation. Write c_η for $\rho(\eta^{-1})$, where $\rho(t)$ is the Dickman function (see, for example, [19, §12.1]). For our purposes here it suffices to note only that when $\eta>0$ one has $c_\eta>0$. Next, when $4\leqslant k\leqslant 10$, define $f_k^*(\alpha)$ for $\alpha\in\mathfrak{P}$ by taking

$$f_k^*(\alpha)=c_\eta q^{-1}S_k(q,a)v_k(\alpha-a/q), \tag{25}$$

when $\alpha\in\mathfrak{P}(q,a)\subseteq\mathfrak{P}$. As a consequence of [23, Lemma 8.5], one has

$$\sup_{\alpha\in\mathfrak{P}}|f_k(\alpha)-f_k^*(\alpha)|\ll P_k(\log N)^{-1/4}. \tag{26}$$

Also, from [19, Theorem 4.1], it follows that

$$\sup_{\alpha\in\mathfrak{P}}|F_3(\alpha)-F_3^*(\alpha)|\ll L^{1+\varepsilon}. \tag{27}$$

Lemma 6. *One has*

$$\int_{\mathfrak{P}} \mathcal{F}(\alpha)e(-n\alpha)d\alpha \gg \mathcal{F}(0)N^{-1}.$$

Proof. We begin by replacing the exponential sums $F_3(\alpha)$ and $f_k(\alpha)$ by their approximations $F_3^*(\alpha)$ and $f_k^*(\alpha)$. Write

$$\mathcal{F}^*(\alpha) = F_3^*(\alpha)f_4^*(\alpha)\dots f_{10}^*(\alpha).$$

Then since the measure of $\mathfrak{P}$ is $O(L^3N^{-1})$, on making liberal use of trivial estimates for generating functions, one finds from (26) and (27) that

$$\begin{aligned}(28)\quad \int_{\mathfrak{P}} \mathcal{F}(\alpha)e(-n\alpha)d\alpha - \int_{\mathfrak{P}} \mathcal{F}^*(\alpha)e(-n\alpha)d\alpha &\ll L^3(\log N)^{-1/4}\mathcal{F}(0)N^{-1}\\ &\ll \mathcal{F}(0)(NL)^{-1}.\end{aligned}$$

But on recalling (25) and (14), we have

$$(29)\qquad \int_{\mathfrak{P}} \mathcal{F}^*(\alpha)e(-n\alpha)d\alpha = c_\eta^7 J_0(n) \sum_{1\leqslant q\leqslant L} A(q,n),$$

where

$$(30)\qquad J_0(n) = \int_{-L/N}^{L/N} \Big(\prod_{k=3}^{10} v_k(\beta)\Big)e(-n\beta)d\beta$$

and

$$(31)\qquad A(q,n) = q^{-8}\sum^{q} a = 1(a,q) = 1\Big(\prod_{k=3}^{10} S_k(q,a)\Big)e(-na/q).$$

We complete the singular integral $J_0(n)$ to obtain the new integral

$$(32)\qquad J(n) = \int_{-\infty}^{\infty} \Big(\prod_{k=3}^{10} v_k(\beta)\Big)e(-n\beta)d\beta.$$

On recalling (11), a partial integration yields the bounds

$$v_3(\beta) \ll P_3(1+N|\beta|)^{-1} \quad \text{and} \quad v_k(\beta) \ll P_k(1+N|\beta|)^{-1/k} \quad (4\leqslant k\leqslant 10).$$

On substituting these bounds into (30) and (32), we deduce that

$$(33)\qquad J(n) - J_0(n) \ll \mathcal{F}(0)\int_{L/N}^{\infty} (1+N\beta)^{-2}d\beta \ll \mathcal{F}(0)N^{-1}L^{-1}.$$

We may rewrite (32) in the form

$$J(n) = \int_{-\infty}^{\infty}\int_{\mathcal{B}} e(\beta(\gamma_1^3+\gamma_2^4+\dots+\gamma_8^{10}-n))d\boldsymbol{\gamma}d\beta,$$

where

$$\mathcal{B} = [P_3/2, P_3]\times[0,P_4]\times\dots\times[0,P_{10}].$$

When $N/2 < n \leqslant N$, therefore, one certainly has that

$$[(n/8)^{1/3}, (n/4)^{1/3}] \times [0, n^{1/4}] \times \cdots \times [0, n^{1/10}] \subseteq \mathcal{B},$$

and hence an application of Fourier's integral formula rapidly establishes that

$$J(n) \gg n^{\frac{1}{3}+\cdots+\frac{1}{10}-1} \gg \mathcal{F}(0)N^{-1}. \tag{34}$$

Next we turn our attention to the singular series, which we complete to obtain

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(q,n).$$

Recalling first (13) and (31), we have $A(q,n) \ll q\kappa_3(q)\kappa_4(q)\dots\kappa_{10}(q)$, and in particular, by virtue of (13), one has the upper bound $A(q,n) \ll q^{-3/7}$. When π is a prime number and $h = 1$ or 2, moreover, the formulae (12) ensure that $A(\pi^h, n) \ll \pi^{-3}$. But the standard theory of exponential sums shows that $A(q,n)$ is a multiplicative function of q (see, for example, [19, §2.6]). Thus it follows that whenever π is a prime number and $0 < \theta \leqslant 1/35$,

$$\sum_{h=1}^{\infty} (\pi^h)^{\theta} |A(\pi^h, n)| \ll \pi^{2\theta-3} + \sum_{h=3}^{\infty} \pi^{h\theta - 3h/7} \ll \pi^{-6/5}. \tag{35}$$

Consequently, there is a fixed positive number B with the property that

$$\sum_{1\leqslant q\leqslant Q} q^{\theta}|A(q,n)| \leqslant \prod_{p\leqslant Q} (1 + Bp^{-6/5}) \ll 1,$$

whence

$$\sum_{q>L} |A(q,n)| \leqslant \sum_{q>L} (q/L)^{1/35} |A(q,n)| \ll L^{-1/35}.$$

Thus we arrive at the conclusion

$$\mathfrak{S}(n) - \sum_{1\leqslant q\leqslant L} A(q,n) \ll L^{-1/35}. \tag{36}$$

Next write

$$\omega_{\pi}(n) = \sum_{h=0}^{\infty} A(\pi^h, n),$$

and observe that by (35), one has for each prime π that

$$\omega_{\pi}(n) - 1 \ll \pi^{-6/5}. \tag{37}$$

The multiplicative property of $A(q,n)$ together with the latter estimate shows that we may rewrite $\mathfrak{S}(n)$ as an absolutely convergent product $\mathfrak{S}(n) =$

$\prod_\pi \omega_\pi(n)$. We aim now to show that $\mathfrak{S}(n) \gg 1$, uniformly in n. Assuming this inequality, it follows from (28), (29), (33), (34) and (36) that for $N/2 < n \leqslant N$, one has

$$\int_{\mathfrak{P}} \mathcal{F}(\alpha)e(-n\alpha)d\alpha \gg (\mathcal{F}(0)N^{-1} + O(\mathcal{F}(0)(NL)^{-1}))(1 + O(L^{-1/35})),$$

which yields the conclusion of the lemma.

In order to establish that $\mathfrak{S}(n) \gg 1$, we begin by noting that the proof of [19, Lemma 2.12] shows that when $h \geqslant 1$, one has

$$\sum_{l=0}^{h} A(\pi^l, n) = \pi^{-7h}\Omega(\pi^h, n),$$

where $\Omega(\pi^h, n)$ denotes the number of incongruent solutions of the congruence

$$x_1^3 + x_2^4 + \cdots + x_8^{10} \equiv n \pmod{\pi^h}. \tag{38}$$

When π is not equal to 3 and $h = 1$, it follows from the Cauchy-Davenport theorem (see [19, Lemma 2.14]) that the congruence (38) is soluble with $\pi \nmid x_1$. When $\pi^h = 9$, on the other hand, the latter conclusion is easily verified by hand. Then the methods of [19, §2.6], in combination with (37), therefore show that for a sufficiently large but fixed positive number C, one has

$$\mathfrak{S}(n) \gg \prod_{\pi > C} (1 - \pi^{-7/6}) \gg 1,$$

uniformly in n. This completes the proof of the lemma. □

In view of the discussion concluding §2, the proofs of Theorems 1 and 2 follow immediately from (6) and the conclusion of Lemma 6.

References

[1] J. Brüdern, *Sums of squares and higher powers. II.* J. London Math. Soc. (2) **35** (1987), 244–250.

[2] J. Brüdern, *A problem in additive number theory.* Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.

[3] J. Brüdern, T.D. Wooley, *On Waring's problem: two cubes and seven biquadrates.* Tsukuba Math. J. **24** (2000), 387–417.

[4] H. Davenport, H. Heilbronn, *On Waring's problem: two cubes and one square.* Proc. London Math. Soc. (2) **43** (1937), 73–104.

[5] K.B. Ford, *The representation of numbers as sums of unlike powers.* J. London Math. Soc. (2) **51** (1995), 14–26.

[6] K.B. Ford, *The representation of numbers as sums of unlike powers. II.* J. Amer. Math. Soc. **9** (1996), 919–940.

[7] C. Hooley, *On a new approach to various problems of Waring's type.* In: Recent progress in analytic number theory, vol. 1 (Durham, 1979), Academic Press, London (1981), 127–191.

[8] K.F. Roth, *Proof that almost all positive integers are sums of a square, a positive cube and a fourth power.* J. London Math. Soc. **24** (1949), 4–13.

[9] K.F. ROTH, *A problem in additive number theory.* Proc. London Math. Soc. (2) **53** (1951), 381–395.
[10] K. THANIGASALAM, *On additive number theory.* Acta Arith. **13** (1967/68), 237–258.
[11] K. THANIGASALAM, *On sums of powers and a related problem.* Acta Arith. **36** (1980), 125–141.
[12] K. THANIGASALAM, *On certain additive representations of integers.* Portugal. Math. **42** (1983/84), 447–465.
[13] R.C. VAUGHAN, *On the representation of numbers as sums of powers of natural numbers.* Proc. London Math. Soc. (3) **21** (1970), 160–180.
[14] R.C. VAUGHAN, *On sums of mixed powers.* J. London Math. Soc. (2) **3** (1971), 677–688.
[15] R.C. VAUGHAN, *A ternary additive problem.* Proc. London Math. Soc. (3) **41** (1980), 516–532.
[16] R.C. VAUGHAN, *On Waring's problem for cubes.* J. Reine Angew. Math. **365** (1986), 122–170.
[17] R.C. VAUGHAN, *On Waring's problem for smaller exponents* . Proc. London Math. Soc. (3) **52** (1986), 445–463.
[18] R.C. VAUGHAN, *A new iterative method in Waring's problem.* Acta Math. **162** (1989), 1–71.
[19] R.C. VAUGHAN, *The Hardy-Littlewood method.* Cambridge Tract No. 125, 2nd Edition, Cambridge University Press, 1997.
[20] R.C. VAUGHAN, T.D. WOOLEY, *On Waring's problem: some refinements.* Proc. London Math. Soc. (3) **63** (1991), 35–68.
[21] R.C. VAUGHAN, T.D. WOOLEY, *Further improvements in Waring's problem.* Acta Math. **174** (1995), 147–240.
[22] R.C. VAUGHAN, T.D. WOOLEY, *Further improvements in Waring's problem, IV: higher powers.* Acta Arith. **94** (2000), 203–285.
[23] T.D. WOOLEY, *On simultaneous additive equations, II.* J. Reine Angew. Math. **419** (1991), 141–198.
[24] T.D. WOOLEY, *Large improvements in Waring's problem.* Ann. of Math. (2) **135** (1992), 131–164.
[25] T.D. WOOLEY, *New estimates for smooth Weyl sums.* J. London Math. Soc. (2) **51** (1995), 1–13.
[26] T.D. WOOLEY, *Breaking classical convexity in Waring's problem: sums of cubes and quasi-diagonal behaviour.* Invent. Math. **122** (1995), 421–451.

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