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# Prime divisors of the Lagarias sequence

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Résumé. Nous donnons une solution à un problème posé par Lagarias [5] en 1985, en déterminant sous GRH la densité de l'ensemble des nombres premiers qui sont des diviseurs de termes de la suite $\{x_n\}_{n=1}^{\infty}$ définie par $x_0 = 3$, $x_1 = 1$ et la relation de récurrence $x_{n+1} = x_n + x_{n-1}$. Cela donne le premier exemple d'une suite de récurrence d'ordre 2 qui n'est pas 'à torsion' pour laquelle on sait déterminer la densité associée des diviseurs premiers.

Abstract. We solve a 1985 challenge problem posed by Lagarias [5] by determining, under GRH, the density of the set of prime numbers that occur as divisor of some term of the sequence $\{x_n\}_{n=1}^{\infty}$ defined by the linear recurrence $x_{n+1} = x_n + x_{n-1}$ and the initial values $x_0 = 3$ and $x_1 = 1$. This is the first example of a 'non-torsion' second order recurrent sequence with irreducible recurrence relation for which we can determine the associated density of prime divisors.

## 1. Introduction

In 1985, Lagarias [5] showed that the set of prime numbers that divide some Lucas number has a natural density 2/3 inside the set of all prime numbers. Here the Lucas numbers are the terms of the second order recurrent sequence $\{x_n\}_{n=1}^{\infty}$ defined by the linear recurrence $x_{n+1} = x_n + x_{n-1}$ and the initial values $x_0 = 2$ and $x_1 = 1$. Lagarias's method is a quadratic analogue of the approach used by Hasse [2, 3] in determining, for a given non-zero integer $a$, the density of the set of the prime divisors of the numbers of the form $a^n + 1$. Note that the sequence $\{a^n + 1\}_{n=1}^{\infty}$ also satisfies a second order recurrence.

Hasse and Lagarias apply the Chebotarev density theorem to a suitable tower of Kummer fields. Their method of 'Chebotarev partitioning' can be adapted to deal with the class of second order recurrent sequences that are now known as 'torsion sequences' [1, 8, 11]. For second order recurrent integer sequences that do not enjoy the rather special condition of being 'torsion', it can no longer be applied. In the case of the the Lucas numbers,

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changing the initial values into $x_0 = 3$ and $x_1 = 1$ (while leaving the recurrence $x_{n+1} = x_n + x_{n-1}$ unchanged) leads to a sequence for which Lagarias remarks that his method fails, and he wonders whether some modification of it can be made to work.

We will explain how non-torsion sequences lead to a question that is reminiscent of the Artin primitive root conjecture. In particular, we will see that for a non-torsion sequence, there is no number field $F$ (of finite degree) with the property that all primes having a given splitting behavior in $F$ divide some term of the sequence. It follows that Chebotarev partitioning can not be applied directly. However, it is possible to combine the technique of Chebotarev partitioning with the analytic techniques employed by Hooley [4] in his proof (under assumption of the generalized Riemann hypothesis) of Artin's primitive root conjecture. In the case of the modified Lucas sequence proposed by Lagarias, we give a full analysis of the situation and prove the following theorem.

**Theorem**. *Let $\{x_n\}_{n=0}^{\infty}$ be the integer sequence defined by $x_0 = 3$, $x_1 = 1$ and the linear recurrence $x_{n+1} = x_n + x_{n-1}$. If the generalized Riemann hypothesis holds, then the set of prime numbers that divide some term of this sequence has a natural density. It equals*

$$\frac{1573727}{1569610} \cdot \prod_{p \text{ prime}} \left(1 - \frac{p}{p^3 - 1}\right) \approx .577470679956\,.$$

Numerically, one finds that 45198 out of the first 78498 primes below $10^6$ divide the sequence: a fraction close to .5758.

## 2. Second order recurrences

Let $X = \{x_n\}_{n=0}^{\infty}$ be a second order recurrent sequence. It is our aim to determine, whenever it exists, the density (inside the set of all primes) of the set of prime numbers $p$ that divide some term of $X$.

We let $x_{n+2} = a_1 x_{n+1} + a_0 x_n$ be the recurrence satisfied by $X$, and denote by $f = T^2 - a_1 T - a_0 \in \mathbf{Z}[T]$ the corresponding characteristic polynomial. We factor $f$ over an algebraic closure of $\mathbf{Q}$ as $f = (T - \alpha)(T - \widetilde{\alpha})$.

In order to avoid trivialities, we will assume that $X$ does not satisfy a first order recurrence, so that $\alpha\widetilde{\alpha} = a_0$ does not vanish. The *root quotient* $r = r(f)$ of the recurrence, which is only determined up to inversion, is then defined as $r = \alpha/\widetilde{\alpha}$. It is either a rational number or a quadratic irrationality of norm 1. In the *separable case* $r \neq 1$ we have

$$x_n = c\alpha^n + \widetilde{c}\widetilde{\alpha}^n \qquad \text{with} \qquad c = \frac{x_1 - \widetilde{\alpha}x_0}{\alpha - \widetilde{\alpha}} \quad \text{and} \quad \widetilde{c} = \frac{x_1 - \alpha x_0}{\widetilde{\alpha} - \alpha}.$$

As our sequence is by assumption not of order smaller than 2, we have $c\widetilde{c} \neq 0$. Denote by

$$q = \frac{x_1 - \alpha x_0}{x_1 - \widetilde{\alpha} x_0} = -\widetilde{c}/c \in \mathbf{Q}(\alpha)^*$$

the *initial quotient* $q = q(X)$ of $X$. Just as the root quotient, this is a number determined up to inversion that is either rational or quadratic of norm 1. The elementary but fundamental observation for second order recurrences is that for almost all primes $p$, we have the fundamental equivalence

$$p \text{ divides } x_n \iff -\widetilde{c}/c = (\alpha/\widetilde{\alpha})^n \in \mathcal{O}/p\mathcal{O} \iff q = r^n \in (\mathcal{O}/p\mathcal{O})^*.$$

Here $\mathcal{O}$ is the ring of integers in the field generated by the roots of $f$. This is the ring $\mathbf{Z}$ if $f$ has rational roots, and the ring of integers of the quadratic field $\mathbf{Q}[X]/(f) = \mathbf{Q}(\sqrt{a_1^2 - 4a_0})$ otherwise. The equivalence above does not make sense for the finitely many primes $p$ for which either $r$ or $q$ is not invertible modulo $p$, but this is irrelevant for density purposes.

In the degenerate case where the root quotient $r$ is a root of unity, it is easily seen that the set of primes dividing some term of $X$ is either finite or cofinite in the set of all primes. We will further exclude this case, which includes the *inseparable case* $r = 1$, for which $q$ is not defined.

As we are essentially interested in the set of primes $p$ for which $q$ is in the subgroup generated by $r$ in the finite group $(\mathcal{O}/p\mathcal{O})^*$ of invertible residue classes modulo $p$, we can formulate the problem we are trying to solve without any reference to recurrent sequences. Depending on whether the root quotient $r$ is rational or quadratic, this leads to the following.

**Problem 1**. *Given two non-zero rational numbers $q$ and $r \neq \pm 1$, compute, whenever it exists, the density of the set of primes $p$ for which we have*

$$q \bmod p \in \langle r \bmod p \rangle \subset \mathbf{F}_p^*.$$

**Problem 2**. *Let $r$ be a quadratic irrationality of norm 1 and $\mathcal{O}$ the ring of integers of $\mathbf{Q}(r)$. Given an element $q \in \mathbf{Q}(r)$ of norm 1, compute, whenever it exists, the density of the set of rational primes $p$ for which we have*

$$q \bmod p \in \langle r \bmod p \rangle \subset (\mathcal{O}/p\mathcal{O})^*.$$

The instances of the two problems above where $(q \bmod r)$ is a torsion element in the group $\mathbf{Q}(r)^*/\langle r \rangle$ are referred to as torsion cases of the problem, and the sequences that give rise to them are known as *torsion sequences*. The sequences $\{a^n + 1\}_{n=0}^{\infty}$ studied by Hasse, the Lucas sequence $\{\varepsilon^n + \varepsilon^{-n}\}_{n=0}^{\infty}$ with $\varepsilon = \frac{1+\sqrt{5}}{2}$ treated by Lagarias and the Lucas-type sequences in [8] are torsion; in fact, they all have $q = -1$. The main theorem for torsion sequences, for which we refer to [11], is the following.

**Theorem**. *Let $X$ be a second order torsion sequence. Then the set $\pi_X$ of prime divisors of $X$ has a positive rational density.*

### 3. Non-torsion sequences

Problem 1 in the previous section is reminiscent of Artin's famous question on primitive roots: given a non-zero rational number $r \neq \pm 1$, for how many primes $p$ does $r$ generate the group $\mathbf{F}_p^*$ of units modulo $p$? (One naturally excludes the finitely many primes $p$ dividing the numerator or denominator of $r$ from consideration.) Artin's conjectural answer to this question is based on the observation that the index $[\mathbf{F}_p^* : \langle r \rangle]$ is divisible by $j$ if and only if $p$ splits completely in the splitting field $F_j = \mathbf{Q}(\zeta_j, r^{1/j})$ of the polynomial $X^j - r$ over $\mathbf{Q}$. Thus, $r$ is a primitive root modulo $p$ if and only $p$ does not split completely in *any* of the fields $F_j$ with $j > 1$. For fixed $j$, the set $S_j$ of primes that do split completely in $F_j$ has natural density $1/[F_j : \mathbf{Q}]$ by the Chebotarev density theorem. Applying an inclusion-exclusion argument to the sets $S_j$, one expects the set $S = S_1 \setminus \cup_{j \geq 1} S_j$ of primes for which $r$ is a primitive root to have natural density

$$\delta(r) = \sum_{j=1}^{\infty} \frac{\mu(j)}{[F_j : \mathbf{Q}]}. \tag{3.1}$$

Note that the right hand side of (3.1) converges for all $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ as $[F_j : \mathbf{Q}]$ is a divisor of $\varphi(j) \cdot j$ with cofactor bounded by a constant depending only on $r$.

A 'multiplicative version' of the 'additive formula' (3.1) for $\delta(r)$ is obtained if one starts from the observation that $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ is a primitive root if and only if $p$ does not split completely in any field $F_\ell$ with $\ell$ *prime*. The fields $F_\ell$ are of degree $\ell(\ell-1)$ for almost all primes $\ell$, and using the fact that they are almost 'independent', one can successively eliminate the primes that split completely in some $F_\ell$ to arrive at a heuristic density

$$\delta(r) = c_r \cdot \prod_{\ell \text{ prime}} (1 - \frac{1}{[F_\ell : \mathbf{Q}]}) = \widetilde{c}_r \cdot \prod_{\ell \text{ prime}} (1 - \frac{1}{\ell(\ell-1)}). \tag{3.2}$$

The correction factor $c_r$ for the 'dependency' between the fields $F_\ell$ is equal to 1 if the family of fields $\{F_\ell\}_\ell$ is linearly disjoint over $\mathbf{Q}$, i.e., if each field $F_{\ell_0}$ is linearly disjoint over $\mathbf{Q}$ from the compositum of the fields $F_\ell$ with $\ell \neq \ell_0$. If $r$ is not a perfect power in $\mathbf{Q}^*$, we have $\widetilde{c}_r = c_r$.

It turns out that the only possible obstruction to the linear disjointness of the fields $F_\ell$ occurs when $F_2 = \mathbf{Q}(\sqrt{r})$ is quadratic of odd discriminant. In this case, $F_2$ is contained in the compositum of the fields $F_\ell$ with $\ell$ dividing its discriminant. The value of $c_r$ is a rational number, and one can derive a closed formula for it as in [4, p. 220].

For example, taking $r=5$, one has $F_2 \subset F_5$ and the superfluous 'Euler factor' $1-[F_5:\mathbf{Q}]^{-1}=\frac{19}{20}$ at $\ell=5$ in the product $\prod_\ell(1-[F_\ell:\mathbf{Q}]^{-1})$ is 'removed' by the correction factor $c_5=\widetilde{c}_5=\frac{20}{19}$.

It is non-trivial to make the heuristics above into a proof. As Hooley [4] showed, it can be done if one is willing to assume estimates for the remainder term in the prime number theorem for the fields $F_j$ that are currently only known to hold under assumption of the generalized Riemann hypothesis. One should realize that only when we consider finitely many $\ell$ (or $j$) at a time, the Chebotarev density theorem gives us the densities we want. After taking a 'limit' over all $\ell$, we only know that the right hand side of (3.1) or (3.2) is an *upper density* for the set of primes $p$ for which $r$ is a primitive root. We have however no guarantee that we are left with a non-empty set of such $p$. Put somewhat differently, we can not obtain primes $p$ for which $(r \bmod p)$ is a primitive root by imposing a splitting condition on $p$ in a number field $F$ of finite degree; clearly, there is always some field $F_\ell$ that is linearly disjoint from $F$, and no splitting condition in $F$ will yield the 'correct' splitting behavior in $F_\ell$. A similar phenomenon occurs in the analysis of non-torsion cases of the Problems 1 and 2. This is exactly what makes non-torsion sequences so much harder to analyze than torsion sequences.

If $(q \bmod r)$ is not a torsion element in $\mathbf{Q}^*/\langle r\rangle$, then Problem 1 can be treated by a generalization of the arguments used by Artin. For each integer $i \geq 1$, one considers the set of primes $p$ (not dividing the numerator or denominator of either $q$ or $r$) for which the index $[\mathbf{F}_p^*:\langle r\rangle]$ is *equal* to $i$ and the index $[\mathbf{F}_p^*:\langle q\rangle]$ is *divisible* by $i$. These are the primes that split completely in the field $F_{i,1}=\mathbf{Q}(\zeta_i, r^{1/i}, q^{1/i})$, but not in any of the fields $F_{i,j}=\mathbf{Q}(\zeta_{ij}, r^{1/ij}, q^{1/i})$ with $j>1$. As before, inclusion-exclusion yields a conjectural value for the density $\delta_i(r,q)$ of this set of primes, and summing over $i$ we get

$$\delta(r,q)=\sum_{i=1}^{\infty}\delta_i(r,q)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mu(j)}{[F_{i,j}:\mathbf{Q}]} \tag{3.3}$$

as a conjectural value for the density in Problem 1. Note that $\delta_1(r,q)$ is nothing but the primitive root density $\delta(r)$ from (3.1).

The condition that $(q \bmod r)$ is not a torsion element in $\mathbf{Q}^*/\langle r\rangle$ means that $q$ and $r$ are multiplicatively independent in $\mathbf{Q}^*$. In this case $[F_{i,j}:\mathbf{Q}]$ is a divisor of $i^2 j\cdot\varphi(ij)$ with cofactor bounded by a constant depending only on $q$ and $r$. Thus the double sum in (3.3) converges, and under GRH one can show [9, 10] that its value is indeed the density one is asked to determine in Problem 1.

As in Artin's case, one can obtain a multiplicative version of (3.3) by a 'prime-wise' approach. One notes that the inclusion of subgroups $\langle q$ mod

$p\rangle \subset \langle r \bmod p\rangle$ in $\mathbf{F}_p^*$ means that for all primes $\ell$, we have an inclusion

$$\langle q \bmod p\rangle_\ell \subset \langle r \bmod p\rangle_\ell \tag{3.4}$$

of the $\ell$-primary parts of these subgroups. If we fix both $\ell$ and the number $k = \mathrm{ord}_\ell(p-1)$ of factors $\ell$ in the order of $\mathbf{F}_p^*$, this condition can be rephrased in terms of the splitting behavior of $p$ in the number field

$$\Omega_\ell^{(k)} = \mathbf{Q}(\zeta_{\ell^{k+1}}, r^{1/\ell^k}, q^{1/\ell^k}). \tag{3.5}$$

More precisely, we have $\mathrm{ord}_\ell(p-1) = k$ if and only if $p$ splits completely in $\mathbf{Q}(\zeta_{\ell^k})$ but not in $\mathbf{Q}(\zeta_{\ell^{k+1}})$; of the primes $p$ that meet this condition, we want those $p$ for which the order of the Frobenius elements over $p$ in $\mathrm{Gal}(\mathbf{Q}(\zeta_{\ell^k}, q^{1/\ell^k})/\mathbf{Q}(\zeta_{\ell^k}))$ divides the order of the Frobenius elements over $p$ in $\mathrm{Gal}(\mathbf{Q}(\zeta_{\ell^k}, r^{1/\ell^k})/\mathbf{Q}(\zeta_{\ell^k}))$. By the Chebotarev density theorem, one finds that the set of primes $p$ with $\mathrm{ord}_\ell(p-1) = k$, which has density $\ell^{-k}$ for $k \geq 1$, is a union of two sets that each have a density: the set of primes $p$ for which the inclusion (3.4) holds and the set of $p$ for which it does not. This 'Chebotarev partitioning' allows us to compute, for each $\ell$, the density of the primes $p$ for which we have the inclusion (3.4): summing over $k$ in the previous argument yields a lower density, and this is the required density as we can apply the same argument to the complementary set of primes.

For all but finitely many $\ell$, the fields $\mathbf{Q}(\zeta_{\ell^{k+1}}, q^{1/\ell^k})$ and $\mathbf{Q}(\zeta_{\ell^{k+1}}, r^{1/\ell^k})$ are linearly disjoint extensions of $\mathbf{Q}(\zeta_{\ell^{k+1}})$ with Galois group $\mathbf{Z}/\ell^k\mathbf{Z}$ for all $k \geq 0$. In this case the set of primes $p$ with $\mathrm{ord}_\ell(p-1) = k$ violating (3.4) has density

$$\ell^{-k}\sum_{i=1}^{k} \ell^{-i}(\ell^{-(i-1)} - \ell^{-i}) = (\ell^{-k} - \ell^{-3k})/(\ell+1).$$

Summing over $k$, we find that (3.4) does not hold for a set of primes of density $\ell/(\ell^3-1)$. As the fields

$$\Omega_\ell = \textstyle\bigcup_k \Omega_\ell^{(k)} = \mathbf{Q}(\zeta_{\ell^\infty}, r^{1/\ell^\infty}, q^{1/\ell^\infty})$$

for prime values of $\ell$ form a linearly disjoint family if we exclude finitely many 'bad' primes $\ell$, the multiplicative analogue of (3.3) reads

$$\delta(r,q) = c_{q,r} \cdot \prod_{\ell \text{ prime}} \left(1 - \frac{\ell}{\ell^3 - 1}\right). \tag{3.6}$$

As is shown in [9], the 'correction factor' $c_{q,r}$ is a rational number that admits a somewhat involved description in terms of $q$ and $r$. In practice, one finds its value most easily by starting from the additive formula (3.3).

In the situation of Problem 2, the arguments just given can be taken over without substantial changes from the rational case when one restricts to those rational primes $p$ that split completely in $\mathcal{O}$. Writing $K = \mathbf{Q}(r)$ and

$$F_{i,j} = K(\zeta_{ij}, r^{1/ij}, q^{1/i}),$$

we find that the density of the rational primes $p$ that are split in $\mathcal{O}$ and for which we have $q \bmod p \in \langle r \bmod p\rangle \subset (\mathcal{O}/p\mathcal{O})^*$ equals

$$\delta_{\text{split}}(r,q) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\mu(j)}{[F_{i,j} : \mathbf{Q}]}; \tag{3.7}$$

as in the case of Problem 1, one needs to assume the validity of the generalized Riemann hypothesis for this result. By (3.7), the computation of $\delta_{\text{split}}$ amounts to a degree computation for the family of fields $\{F_{i,j}\}_{i,j}$. In fact, because of the numerator $\mu(j)$ in (3.7) one may restrict to the case where $j$ is squarefree. As in the case of Problem 1, one finds (under GRH) that the split density equals

$$\delta_{\text{split}}(r,q) = c^+_{q,r} \cdot \prod_{\ell \text{ prime}} \left(1 - \frac{\ell}{\ell^3 - 1}\right) \tag{3.8}$$

for some rational number $c^+_{q,r}$. The next section provides a typical example of such a computation. It shows that the value of $c^+_{q,r}$ is not as simple a fraction as the analogous factor $c_r$ in (3.2).

For the rational primes $p$ that are inert in $\mathcal{O}$, the determination of the corresponding density $\delta_{\text{inert}}$ is more involved than in the split case. The group $(\mathcal{O}/p\mathcal{O})^*$ in Problem 2 is now cyclic of order $p^2 - 1$, and $(q \bmod p)$ and $(r \bmod p)$ are elements of the kernel

$$\kappa_p = \ker[N : (\mathcal{O}/p\mathcal{O})^* \longrightarrow \mathbf{F}_p] \tag{3.9}$$

of the norm map, which is cyclic of order $p+1$. In order to have the inclusion (3.4) of subgroups of $\kappa_p$ for all primes $\ell$, we fix $\ell$ and $k = \operatorname{ord}_\ell(p+1) \geq 1$ and rephrase (3.4) in terms of the splitting behavior of $p$ in the quadratic counterpart

$$\Omega_\ell^{(k)} = K(\zeta_{\ell^{k+1}}, r^{1/\ell^k}, q^{1/\ell^k})$$

of (3.5). Let us assume for simplicity that $\ell$ is an odd prime, and that $K$ is not the quadratic subfield of $\mathbf{Q}(\zeta_\ell)$. Then the requirement that $p$ be inert in $K$ and satisfy $\operatorname{ord}_\ell(p+1) = k \geq 1$ means that the Frobenius element of $p$ in $\operatorname{Gal}(K(\zeta_{\ell^{k+1}})/\mathbf{Q})$ is non-trivial on $K$ and has order $2\ell$ when restricted to $\mathbf{Q}(\zeta_{\ell^{k+1}})$. Let $B_k \subset K(\zeta_{\ell^{k+1}})$ be the fixed field of the subgroup generated by such a Frobenius element. Then $B_k$ does not contain $K$ or $\mathbf{Q}(\zeta_\ell)$, and $B_k \subset K(\zeta_{\ell^k})$ is a quadratic extension. Let $\sigma_k$ be the non-trivial automorphism of this extension. Then $\sigma_k$ acts by inversion on $\zeta_{\ell^k}$, and the norm-1-condition on $q$ and $r$ means that $\sigma_k$ also acts by inversion on $q$ and $r$. The Galois equivariancy of the Kummer pairing

$$\operatorname{Gal}(K(\zeta_{\ell^k}, r^{1/\ell^k}, q^{1/\ell^k})/K(\zeta_{\ell^k})) \times \langle q, r\rangle \longrightarrow \langle \zeta_{\ell^k}\rangle$$

shows that the natural action of $\sigma_k$ on $\mathrm{Gal}(K(\zeta_{\ell^k}, r^{1/\ell^k}, q^{1/\ell^k})/K(\zeta_{\ell^k}))$ is trivial, so $\Omega_\ell^{(k)}$ is abelian over $B_k$. It is the linearly disjoint compositum of the cyclotomic extension $B_k \subset B_k(\zeta_{\ell^{k+1}})$ and the abelian extension

$$B_k \subset B_k(r^{1/\ell^k} + r^{-1/\ell^k}, q^{1/\ell^k} + q^{-1/\ell^k}).$$

For almost all $\ell$, the group $\mathrm{Gal}(\Omega_\ell^{(k)}/B_k)$ is isomorphic to $\mathbf{Z}/2\ell\mathbf{Z} \times (\mathbf{Z}/\ell^k\mathbf{Z})^2$.

Just like in the rational case, we want those primes $p$ that have splitting field $B_k$ inside $K(\zeta_{\ell^{k+1}})$ and for which the order of the Frobenius elements over $p$ in $\mathrm{Gal}(B_k(q^{1/\ell^k} + q^{-1/\ell^k})/B_k)$ divides the order of the Frobenius elements over $p$ in $\mathrm{Gal}(B_k(r^{1/\ell^k} + r^{-1/\ell^k})/B_k)$. By the Chebotarev partition argument, we find again that for a 'generic' prime $\ell$, a fraction $\ell/(\ell^3-1)$ of the primes $p$ that are inert in $\mathcal{O}$ violates (3.4). Here 'generic' means that $\ell$ is odd and that $\Omega_\ell^{(k)}$ has degree $2\ell^{3k}(\ell-1)$ for $k \geq 1$. Under GRH, one can again deduce that the inert density $\delta_{\mathrm{inert}}(q, r)$ equals a rational constant $c_{q,r}^-$ times the infinite Euler product occurring in (3.6) and (3.8).

In general, there are various subtleties that need to be taken care of in the analysis above for $\ell = 2$, when 2-power roots of $q$ and $r$ are adjoined to $K$ or $K(\zeta_{2^n})$. We do not go into them in this paper. In the example in the next section, we deal with these complications by combining a simple ad hoc argument for a few 'bad' $\ell$ with the standard treatment for the 'good' $\ell$.

## 4. The Lagarias example

We now treat the explicit example of the modified Lucas sequence which is the subject of the theorem stated in the introduction. The roots of the characteristic polynomial $X^2 - X - 1$ of the recurrence are $\varepsilon = \frac{1+\sqrt{5}}{2}$ and its conjugate $\widetilde{\varepsilon} = \frac{1-\sqrt{5}}{2}$. The initial values $x_0 = 3$ and $x_1 = 1$ yield an initial quotient $q = \frac{1-3\varepsilon}{1-3\widetilde{\varepsilon}}$ of the sequence. As $\pi_{11} = 1 - 3\varepsilon \in \mathcal{O} = \mathbf{Z}[\varepsilon]$ has norm $-11$, we find that we have to solve Problem 2 for

$$q = \frac{\pi_{11}}{\widetilde{\pi}_{11}} = \frac{\pi_{11}^2}{-11} \qquad \text{and} \qquad r = \frac{\varepsilon}{\widetilde{\varepsilon}} = -\varepsilon^2.$$

We set $K = \mathbf{Q}(\varepsilon) = \mathbf{Q}(\sqrt{5})$ and $F_{i,j} = K(\zeta_{ij}, r^{1/ij}, q^{1/i})$ as in the previous section.

**4.1. Lemma**. *For $i, j \in \mathbf{Z}_{\geq 1}$ we have $[F_{i,j} : \mathbf{Q}] = 2^{1-t} i^2 j \varphi(ij)$, with*

$$t = t_{i,j} = \begin{cases} \#\{d \in \{4,5,11\} : d|ij\} & \text{if } i \text{ is even;} \\ \#\{d \in \{4,5\} : d|ij\} & \text{if } i \text{ is odd and } j \text{ is even;} \\ \#\{d \in \{5\} : d|ij\} & \text{if } ij \text{ is odd.} \end{cases}$$

*Proof.* As $K = \mathbf{Q}(\sqrt{5})$ is the quadratic subfield of $\mathbf{Q}(\zeta_5)$, the field $K(\zeta_{ij})$ has degree $2\varphi(ij)$ over $\mathbf{Q}$ if 5 does not divide $ij$ and degree $\varphi(ij)$ if it does.

As $q = \frac{\pi_{11}}{\bar\pi_{11}}$ is a quotient of two non-associate prime elements in $\mathcal{O}$ and $\varepsilon$ is a fundamental unit in $\mathcal{O}$, the polynomials $X^i - q$ and $X^{ij} - r$ are irreducible in $K[X]$ for all $i, j \in \mathbf{Z}_{\geq 1}$ by a standard result as [6, Theorem VI.9.1]. Moreover, the extension $K \subset K(q^{1/i})$ generated by a zero of $X^i - q$ is totally ramified at the primes of $K$ lying over 11, whereas the extension $K \subset K(r^{1/ij})$ generated by a zero of $X^{ij} - r$ is unramified above 11. It follows that $K \subset K(q^{1/i}, r^{1/ij})$ is of degree $i^2 j$ for all $i, j \in \mathbf{Z}_{\geq 1}$.

The intersection $K(q^{1/i}, r^{1/ij}) \cap K(\zeta_{ij})$ is contained in the maximal abelian subfield $K_0$ of $K(q^{1/i}, r^{1/ij})$, which equals

$$K_0 = \begin{cases} K(\sqrt{q}, \sqrt{r}) = K(\sqrt{-11}, \zeta_4) & \text{if } i \text{ is even;} \\ K(\sqrt{r}) = K(\zeta_4) & \text{if } i \text{ is odd and } j \text{ is even;} \\ K & \text{if } ij \text{ is odd.} \end{cases}$$

One trivially computes $K_0 \cap K(\zeta_{ij})$, and the lemma follows. □

We will need the preceding lemma only for squarefree $j$. In this case, we simply have $t = \#\{d \in \{5\} : d | ij\}$ for odd $i$.

If we substitute the explicit degrees from Lemma 4.1 in (3.7), we find that the split density for our example equals

$$\delta_{\mathrm{split}} = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2^{t_{i,j}} \frac{\mu(j)}{i^2 j \varphi(ij)},$$

where $t_{i,j}$ is as in Lemma 4.1. If we set

$$S_{m,n} = \sum_{\substack{i=1 \\ m|i}}^{\infty} \sum_{\substack{j=1 \\ mn|ij}}^{\infty} \frac{\mu(j)}{i^2 j \varphi(ij)},$$

then the expression above may be rewritten as

$$2\delta_{\mathrm{split}} = S_{1,1} + S_{2,2} + S_{2,5} + S_{2,11} + S_{2,10} + S_{2,22} + S_{2,55} + S_{2,110} + (S_{1,5} - S_{2,5}).$$

It is elementary to show [9, Theorem 4.2] that $S_{m,n}$ is the rational multiple

$$S_{m,n} = \frac{S}{m^3 n^3} \prod_{p|n} \frac{-p^4}{p^3 - p - 1} \prod_{\substack{p|m \\ p\nmid n}} \frac{p^3 + p^2}{p^3 - p - 1}$$

of the universal constant $S = \prod_{p \text{ prime}} (1 - \frac{p}{p^3 - 1})$ for $m, n \in \mathbf{Z}_{\geq 1}$. Simple arithmetic now yields the value

$$\delta_{\mathrm{split}} = \frac{712671}{1569610} \cdot S \approx .26151 \tag{4.2}$$

for the density (under GRH) of the primes $p \equiv \pm 1 \bmod 5$ dividing the Lagarias sequence. Numerically, one finds that 20416 primes out of the

78498 primes below $10^6$ are split in $K$ and divide our sequence: a fraction close to .2601.

For the inert primes of $K$, which satisfy $p \equiv \pm 2 \bmod 5$, we have a closer look at the 'bad' primes 2, 5 and 11. As for the rational problem, we define the extensions

$$\Omega_\ell = K(\zeta_{\ell^\infty}, r^{1/\ell^\infty}, q^{1/\ell^\infty})$$

for primes $\ell$ and note that, by Lemma 4.1, the family consisting of the extensions $\Omega_2\Omega_5\Omega_{11}$ and $\{\Omega_\ell\}_{\ell\neq 2,5,11}$ of $K$ is linearly independent over $K$.

Our first observation is that for the inert primes $p$, the order $p+1$ of the group $\kappa_p$ in (3.9) is never divisible by 5. Condition (3.4) is therefore automatic for the prime $\ell = 5$, and we can disregard the splitting behavior of $p$ in $\Omega_5$.

We next observe that for inert $p$, the element $r = -\varepsilon^2$ satisfies

$$r^{(p+1)/2} \equiv (-1)^{(p-1)/2} \bmod p. \tag{4.3}$$

For primes $p \equiv 3 \bmod 4$, this shows that $\langle r \bmod p\rangle_2$ is the 2-Sylow subgroup of $\kappa_p$, so that (3.4) is again automatic for $\ell = 2$. When we now impose that the inert primes congruent to 3 mod 4, which form a set of primes of density 1/4, have the correct splitting behavior in the extensions $\Omega_\ell$ for $\ell \neq 2, 5$, we are dealing with a linearly disjoint family and find (under GRH) that the set of these primes has density

$$\frac{1}{4}\cdot\prod_{\ell\neq 2,5}\left(1-\frac{\ell}{\ell^3-1}\right) = \frac{1}{4}\cdot\frac{7}{5}\cdot\frac{124}{119}\cdot S.$$

We next consider the inert primes $p \equiv 1 \bmod 4$. For these $p$, the congruence (4.3) shows that $(r \bmod p)$ has odd order in $\kappa_p$, so (3.4) is satisfied for $\ell = 2$ if and only if the order of $q = -\pi_{11}^2/11$ in $\kappa_p$ is also odd. As $\kappa_p$ is a cyclic group of order $p+1 \equiv 2 \bmod 4$, the order of $\bar{q} = (q \bmod p)$ is odd if and only if $\bar{q}$ is a square in $\kappa_p$. Let $x \in \mathcal{O}/p\mathcal{O}$ be a square root of $\bar{q}$. If $x$ is in $\kappa_p$, *i.e.*, if $x$ has norm 1 in $\mathbf{F}_p$, then its trace $x + 1/x$ is in $\mathbf{F}_p$, and we find that

$$(x+\frac{1}{x})^2 = 2+\bar{q}+\frac{1}{\bar{q}} = \frac{1}{-11} \bmod p$$

is a square modulo $p$. If $x$ is not in $\kappa_p$, then $x$ has norm $-1$ in $\mathbf{F}_p$ and

$$(x-\frac{1}{x})^2 = 2-\bar{q}-\frac{1}{\bar{q}} = \frac{3^2\cdot 5}{-11} \bmod p$$

is a square modulo $p$. As 5 is not a square modulo our inert prime $p$, we deduce

$$(q \bmod p) \text{ has odd order in } \kappa_p \quad\Longleftrightarrow\quad -11 \text{ is a square modulo } p. \tag{4.4}$$

If $p$ satisfies the equivalent conditions of (4.4), then $p$ is a square modulo 11 by quadratic reciprocity, and we have $11 \nmid p+1$. It follows that in this case, (3.4) is satisfied for $\ell = 2, 5$ and 11. Thus, the set of inert primes $p \equiv 1 \bmod 4$ satisfying the quadratic condition (4.4) is a set of primes of density 1/8, and the subset of those $p$ that have the correct splitting behavior in the extensions $\Omega_\ell$ for $\ell \neq 2, 5, 11$ has (under GRH) density

$$\frac{1}{8} \cdot \prod_{\ell \neq 2,5,11} \left(1 - \frac{\ell}{\ell^3 - 1}\right) = \frac{1}{4} \cdot \frac{7}{5} \cdot \frac{124}{119} \cdot \frac{1330}{1319} \cdot S.$$

Adding the fractions obtained for the inert primes congruent to $3 \bmod 4$ and to $1 \bmod 4$, we obtain

$$\delta_{\text{inert}} = \frac{61504}{112115} \cdot S \approx 0.3159598798268.$$

Numerically, one finds that 24781 primes out of the 78498 primes below $10^6$ are inert in $K$ and divide our sequence: a fraction close to .3157.

The sum $\delta_{\text{split}} + \delta_{\text{inert}}$ is the value $(\frac{712671}{1569610} + \frac{61504}{112115}) \cdot S = \frac{1573727}{1569610} \cdot S$ mentioned in the theorem in the introduction.

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