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Two theorems on meromorphic functions used as a principle for proofs on irrationality

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Résumé. Dans cet article, nous nous intéressons à deux théorèmes dûs à Nikishin et Chudnovsky se rapportant à des fonctions méromorphes. Notre propos est ici de déduire simplement de certaines propriétés de fonctions méromorphes, satisfaisant à des conditions arithmétiques, des résultats d'irrationalité bien que déjà connus mais non triviaux. L'intérêt de cette approche ne réside pas dans les résultats (obtenus comme corollaires de nos théorèmes) sur l'irrationalité de nombres, dont la transcendance a été établie depuis longtemps (cf. [4], [9] and [10]). Il réside plutôt dans l'intervention de théorèmes concernant les coefficients de Taylor de fonctions méromorphes qui ont une caractérisation arithmétique. De la même manière que Niven [6] utilise la méthode d'Hermite pour donner tous les résultats connus sur l'irrationalité de valeurs de fonctions trigonométriques, nous utilisons les résultats de Nikishin et Chudnovsky (cf. [2], [8]), pour déduire l'irrationalité de valeurs de fonctions non-élémentaires.

Abstract. In this paper we discuss two theorems on meromorphic functions of Nikishin and Chudnovsky. Our purpose is to show, how to derive some well-known but not obvious results on irrationality in a systematic and simple way from properties of meromorphic functions with arithmetic conditions. As far as it stands, we have no new results on irrationality, to the contrary some results on numbers of the corollaries are known already since a long time to be transcendental (cf. [4], [9] and [10]). Our main intention lies in theorems on meromorphic functions whose Taylor coefficients are arithmetically characterized. Like Niven [6] used Hermite's method to give all known results on irrationality of trigonometric functions, we use methods going back to Nikishin [5] and Chudnovsky (cf. [2] and [8]), to give results on irrationality of values of non-elementary functions.

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1. On a theorem of Nikishin

In [5] E.M. Nikishin gave a short proof of the following theorem on entire functions satisfying certain arithmetic conditions:

Theorem 1. *Let f be a transcendental entire function of strict order $\leq \sigma < 2$* [1]*. All derivatives of f at the points 0 and $\lambda \neq 0$, as well as λ shall be numbers of the Gauss field $\mathbb{Q}(i)$. Let $D_0(n)$ and $D_\lambda(n)$ be the least common multiple of the denominators of the numbers $f^{(k)}(0)$ and $f^{(k)}(\lambda)$, $k = 0, \dots, n$, respectively. If for a given n*

$$\int_0^1 t^n (1-t)^n f^{(2n+1)}(\lambda t)\, dt \neq 0, \tag{1.1}$$

for $D(n) := D_0(n) D_\lambda(n)$ the relation

$$D(n) \geq \alpha^n n^{[(2/\sigma)-1]n}$$

holds, where α does not depend on n.

Nikishin remarks that from this theorem the irrationality of the numbers $\exp(p/q)$ and π can be deduced. Obviously one can extract much more out of it; sometimes, however, the condition (1.1) causes some difficulty. That is the reason why we have tried to weaken condition (1.1). With a slight generalization of Nikishin's method we can prove the following result:

Theorem 1'. *Let f be a transcendental entire function of strict order $\leq \sigma < 3/2$. All derivatives of f at the points 0 and $\lambda \neq 0$, as well as λ shall be numbers of the Gauss field $\mathbb{Q}(i)$. Let $D_0(n)$ and $D_\lambda(n)$ be the least common multiple of the denominators of the numbers $f^{(k)}(0)$ and $f^{(k)}(\lambda)$, $k = 0, \dots, n$, respectively. Then for infinitely many $n \in \mathbb{N}$*

$$D(2n) := D_0(2n) D_\lambda(2n) \geq C^n n^{\delta(\sigma) n},$$

where

$$\delta(\sigma) := \begin{cases} (2/\sigma) - 1, & \text{if} \quad 0 < \sigma \leq 1 \\ (3/\sigma) - 2, & \text{if} \quad 1 \leq \sigma < 3/2 \end{cases}$$

and $C > 0$ does not depend on n.

Proof of Theorem 1'. We shall not give the whole proof of Theorem 1' but refer for the first part to the paper of Nikishin [5]. To avoid condition (1.1) we proceed in the following way:
We take the Padé approximation for e^z as mentioned in [5],

$$A_n(z) e^z + B_n(z) = R_n(z), \tag{1.2}$$

[1] *i.e.* there is a $C > 0$, such that $\log |f|_r \leq C r^\sigma$ for $r \geq r_0$

with

$$R_n(z) = \frac{z^{2n+1}}{n!} \int_0^1 t^n (1-t)^n e^{tz} \, dt.$$

$A_n(z)$ and $B_n(z)$ are polynomials of degree n with integer coefficients and the property $A_n(z) = -B_n(-z)$. If we substitute z in (1.2) by the operator $\lambda d/dz$ and then apply the operator $R_n(\lambda d/dz)$ to the entire function $f^{(l)}$ we obtain

$$\text{(1.3)} \quad A_n(\lambda d/dz) f^{(l)}(z+\lambda) + B_n(\lambda d/dz) f^{(l)}(z) = \frac{\lambda^{2n+1}}{n!} \int_0^1 t^n (1-t)^n f^{(2n+l+1)}(z+\lambda t) \, dt.$$

This holds for all $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$. For a verification of this identity compare [5]. Since (1.2) also holds for $n-1$, it immediately follows that (cf. Siegel [11], pp. 6)

$$\text{(1.4)} \qquad A_{n-1}(z) R_n(z) - A_n(z) R_{n-1}(z) = c_n z^{2n-1}, \ c_n \neq 0.$$

Substitution of z in (1.4) by the operator $\lambda d/dz$ and application of the resulting operators to f yields

$$\sum_{k=0}^{n-1} a_{k,n-1} \lambda^k F_n^{(k)}(z) - \sum_{k=0}^{n} a_{k,n} \lambda^k F_{n-1}^{(k)}(z) = c_n \lambda^{2n-1} f^{(2n-1)}(z),$$

where $F_n(z) := R_n(\lambda d/dz) f(z)$ and $A_n(z) := \sum_{k=0}^n a_{k,n} z^k \in \mathbb{Z}[z]$. Since by assumption f is a transcendental entire function, there are infinitely many $n \in \mathbb{N}$ with $f^{(2n-1)}(0) \neq 0$. Therefore there exists an infinite subset $M \subseteq \mathbb{N}$ with the property: for all $n \in M$ there is a $0 \leq k_n \leq n$, such that

$$\text{(1.5)} \qquad F_n^{(k_n)}(0) \neq 0 \quad \text{or} \quad F_{n-1}^{(k_n)}(0) \neq 0.$$

Without loss of generality it can be assumed that in (1.5) always the first or second case holds (take otherwise a subsequence of M).

Lemma. *For sufficiently large $n \in \mathbb{N}$ we have* $\max\{|F_n^{(k_n)}(0)|, |F_{n-1}^{(k_n)}(0)|\} \leq {C_1}^n n^{-\delta(\sigma)n}$, *where C_1 is a constant independent of n and $\delta(\sigma)$ as in Theorem 1'.*

Proof of the Lemma. Obviously

$$\text{(1.6)} \quad F_n^{(k_n)}(0) = R_n(\lambda d/dz) f^{(k_n)}(z)\Big|_{z=0} = \frac{\lambda^{2n+1}}{n!} \int_0^1 t^n (1-t)^n f^{(2n+k_n+1)}(\lambda t) \, dt$$

holds. By Cauchy's integral formula it follows that

$$f^{(2n+k_n+1)}(\lambda t) = \frac{(2n+k_n+1)!}{2\pi i} \int_{B_r(\lambda t)} \frac{f(\xi)}{(\xi - \lambda t)^{2n+k_n+2}}\, d\xi.$$

Choosing $r = n^{1/\sigma}$ and taking into account the estimation

$$\max_{|\xi|=r}\{|f(\xi)|\} \le \exp(C_2 r^\sigma),$$

where the constant C_2 does not depend on r, we obtain for sufficiently large values of n

$$|f^{(2n+k_n+1)}(\lambda t)| \le C_3^n \frac{(2n+k_n+1)!}{n^{(2n+k_n+1)/\sigma}}.$$

There the constant C_3 does not depend on n. A simple calculation shows that

$$\frac{n!}{(2n+1)!} \frac{(2n+k_n+1)!}{n^{(2n+k_n+1)/\sigma}} \le C_4^n n^{-\delta(\sigma)n}$$

and with

$$\int_0^1 t^n (1-t)^n\, dt = \frac{(n!)^2}{(2n+1)!}$$

we finally get the inequality $|F_n^{(k_n)}(0)| \le C_1^n n^{-\delta(\sigma)n}$. Obviously the same estimation holds for $F_{n-1}^{(k_n)}(0)$. The proof of the Lemma is completed.

Now we assume $F_n^{(k_n)}(0) \neq 0$ for all $n \in M$. If we take $l = k_n$ in (1.3) and use the identity $A_n(z) = -B_n(-z)$, we get at the point $z = 0$

$$(1.7) \quad \sum_{k=0}^{n} a_{k,n} \lambda^k [f^{(k_n+k)}(\lambda) - (-1)^k f^{(k_n+k)}(0)]$$
$$= \frac{\lambda^{2n+1}}{n!} \int_0^1 t^n (1-t)^n f^{(2n+k_n+1)}(\lambda t)\, dt = F_n^{(k_n)}(0).$$

If C_5 denotes the denominator of λ, we obtain by (1.7) and the assumptions about the values of the derivatives of f at the points 0 and λ

$$D(2n) C_5^n F_n^{(k_n)}(0) \in \mathbb{Z}[i] \backslash \{0\}$$

and therefore

$$|D(2n) C_5^n F_n^{(k_n)}(0)|^2 \ge 1.$$

Taking into account the estimation of the Lemma for $F_n^{(k_n)}(0)$ this leads for all $n \in M$ to

$$D(2n) \ge C^n n^{\delta(\sigma)n}$$

with a constant $C > 0$ not depending on n. Since the case $F_{n-1}^{(k_n)}(0) \neq 0$ yields the same estimation, the proof of Theorem 1' is completed .

Remark. Instead of $\mathbb{Q}(i)$ we can consider any imaginary quadratic number field.

Application. As we don't require condition (1.1), Theorem 1' can be applied more easily than Theorem 1 in order to gain results on the arithmetic nature of the values of non-elementary functions. As an example, we derive some results on irrationality for certain hypergeometric series and the incomplete gamma function.

Corollary 1. *For all $x \neq 0$ in an imaginary quadratic number field K and all $a \in \mathbb{Q}\backslash\{0, -1, -2, \dots\}$ the confluent hypergeometric function*

$$\Phi(1, 1 + a; x) := \sum_{n \geq 0} \frac{x^n}{(a + 1) \cdots (a + n)}$$

never has values in the same field K.
In particular, all real zeros different from 0 of the so-called incomplete gamma function $\gamma(a, x)$ for $a \in \mathbb{Q}\backslash\{0, -1, -2, \dots\}$ are irrational and the complex zeros different from 0 are in no imaginary quadratic extension of $\mathbb{Q}$. Here $\gamma(a, x)$ is defined by

$$\gamma(a, x) := \int_0^x e^{-t} t^{a-1} dt$$

for real $x > 0$ and for $\mathrm{Re}(a) > 0$ and by analytic continuation for complex values of a and x.

Proof of Corollary 1. The entire function $y(x) := \Phi(1, 1+a; x)$ has order of growth 1 and fulfills the differential equation

$$y'(x) + \left(\frac{a}{x} - 1\right) y(x) = \frac{a}{x} \,. \tag{1.8}$$

If we consider $y(x)$ at the point 0, then $d_k := y^{(k)}(0) = k!/(a+1) \cdots (a+k)$ for all $k \geq 0$. Following Siegel [11], pp. 54, there is a constant $C_0 > 0$, independent of k, such that we have for the denominators of the numbers $d_0, \dots, d_k$

$$\text{l.c.m.}\{\mathrm{den}(d_0), \dots, \mathrm{den}(d_k)\} \leq C_0^{k+1}, \quad k \geq 0.$$

Let us suppose that there is a number $w \in K\backslash\{0\}$ with $y(w) = q \in K$. With $A := \mathrm{den}(1/w)$, $D := \mathrm{den}(a)$ and

$$B := \begin{cases} \mathrm{den}(q), & \text{if} \quad q \neq 0 \\ \quad 1 \quad , & \text{if} \quad q = 0 \end{cases}$$

the differential equation (1.8) inductively yields for all $k \geq 0$

$$B(AD)^k y^{(k)}(w) \in \mathcal{O}_K,$$

$\mathcal{O}_K$ denoting the ring of integers of K. Setting $f = y$ in the notation of Theorem 1' we therefore get $D(2n) = D_0(2n)D_w(2n) \leq C^n$ for all sufficiently large n. As this limitation of the growth of the denominators contradicts Theorem 1', the assumption made above is wrong. Hereby the statement about the arithmetic nature of values of $\Phi(1, 1 + a; x)$ is proved.

For $a \in \mathbb{Q}\backslash\{0, -1, -2, \dots\}$

$$\gamma(a, x) = \frac{1}{a} x^a e^{-x} \Phi(1, 1 + a; x)$$

holds (cf. Erdélyi [3], p. 133). If $\gamma(a, x) = 0$ for an $x \in K\backslash\{0\}$, $\Phi(1, 1+a; x)$ would vanish. This contradicts the considerations above and so the claim about the zeros of the incomplete gamma function is proved.

2. On a theorem of Chudnovsky

In connection with the work of Nikishin we mention a far-reaching theorem on meromorphic functions with arithmetic conditions by G.V. Chudnovsky [2] (also cf. E. Reyssat [8]), which is proved within modern transcendence theory:

Theorem 2. *Let f be a transcendental meromorphic function of order $\leq \rho$ [2]. S denotes the set of all algebraic numbers $w \in \overline{\mathbb{Q}}$, such that all derivatives $f^{(k)}(w)$ are rational integers. Then the set S is of cardinality at most ρ.*

Like Niven [6] applied Hermite's proof of the transcendence of e to prove irrationality of values of elementary functions, we use a simple generalization of Theorem 2 to get results on the arithmetic nature of values of non-elementary functions, too. In our case $\overline{\mathbb{Q}}$ is replaced by a certain algebraic number field K of degree 4 and the values $f^{(k)}(w)$, $w \in K$, are certain algebraic numbers with a restricted growth of the denominators. With this modification one gets the following result:

Theorem 2'. *Let f be a transcendental meromorphic function of order $\rho < 2$ and K a quadratic extension of $\tilde{K}$, where $\tilde{K} = \mathbb{Q}$ or any imaginary quadratic field. Let there be a sequence of positive integers b_k and a constant*

[2] a meromorphic function is said to be of order $\leq \rho$, if it can be expressed as a quotient of two entire functions of order $\leq \rho$

C with

$$\text{l.c.m.}\{b_0, \dots, b_k\} \leq C^{k+1}, \tag{2.1}$$

such that for $w_1, w_2 \in K$ one has $b_k f^{(k)}(w_i) \in \mathcal{O}_{\tilde{K}}$ (ring of integers of $\tilde{K}$). Then $w_1 = w_2$.

Remark. Theorem 2' can be proven with some minor changes in the proof of Theorem 0.1 in Chudnovsky [2] because of the nature of the denominators b_k. Therefore we omit the proof. Furthermore, one can easily see (compare the way how Chudnovsky obtained the inequalities of Lemma 0.10 in [2], p. 391) that if a function of order < 1 is taken, then a stronger growth of the denominators b_k can be allowed:

Theorem 3. *Let f be of order $\rho < 1$ and K as in Theorem 2'. If at the point $w_1 \in K$ condition (2.1) of Theorem 2' holds and if there exist $C_0, C_1 \in \mathbb{N}$, such that for $w_2 \in K$*

$$[(k + C_1)!]^{C_0} f^{(k)}(w_2) \in \mathcal{O}_{\tilde{K}}, \ k \geq 0,$$

then $w_1 = w_2$.

In order to show that Theorem 3 can be applied in a much more general setting than Theorem 1, we give an application concerning the irrationality of certain continued fractions (compare e.g. Bertrand [1]):

Corollary 2. *For any imaginary quadratic $\alpha \in \tilde{K}$, $\alpha \neq 0$, which is not a zero of*

$$\Phi_\nu(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!(k+\nu)!}, \ \nu \in \mathbb{N}_0,$$

the value of the logarithmic derivation of Φ_ν is never in $\tilde{K}$. In particular, for all such α with $\alpha\Phi_\nu'(\alpha) \neq 0$ and $\nu \in \mathbb{N}_0$ the value of the continued fraction

$$\nu + 1 + \frac{\alpha|}{|\nu+2} + \frac{\alpha|}{|\nu+3} + \frac{\alpha|}{|\nu+4} + \cdots$$
$$= \nu + 1 + \cfrac{\alpha}{\nu + 2 + \cfrac{\alpha}{\nu + 3 + \cfrac{\alpha}{\nu + 4 + \dots}}}$$

is not in $\tilde{K}$.

Proof of Corollary 2. Let $\nu \in \mathbb{N}_0$ be fixed. The entire function Φ_ν has order 1/2. The function

$$f_\nu(z) := \frac{d}{dz}\left(\frac{z\Phi'_\nu(z)}{\Phi_\nu(z)}\right)$$

is meromorphic and of order $\leq 1/2$. As Φ_ν fulfills a linear differential equation of second order one obtains

$$f_\nu(z) = 1 - z\left(\frac{\Phi'_\nu(z)}{\Phi_\nu(z)}\right)^2 - \nu\frac{\Phi'_\nu(z)}{\Phi_\nu(z)}$$

and this equation inductively yields

$$f_\nu^{(k)}(z) = P_k\left(z, \frac{1}{z}, \frac{\Phi'_\nu}{\Phi_\nu}(z)\right), \text{ with } P_k \in \mathbb{Z}[X_1, X_2, X_3]. \tag{2.2}$$

Here $\deg P_{k,X_i} \leq k+2$ for $i = 1, 2, 3$. Furthermore, a simple calculation shows that

$$[(k+\nu+1)!]^{\nu+1} f_\nu^{(k)}(0) \in \mathbb{Z} \subseteq \mathcal{O}_{\tilde{K}}.$$

Let us assume that there is an $\alpha \in \tilde{K}$, $\alpha \neq 0$, with $\Phi_\nu(\alpha) \neq 0$ and $\Phi'_\nu(\alpha)/\Phi_\nu(\alpha) = \beta \in \tilde{K}$. With $C := \operatorname{den}(\alpha)\operatorname{den}(1/\alpha)\operatorname{den}(\beta)$ equation (2.2) yields

$$C^{k+2} f_\nu^{(k)}(\alpha) \in \mathcal{O}_{\tilde{K}}.$$

If we now apply Theorem 3 with $f = f_\nu$, $C_0 = C_1 = \nu + 1$, $w_1 = \alpha$ and $w_2 = 0$ then $\alpha = 0$ follows. Hence the assumption made above was wrong, which proves the statement about the values of the logarithmic derivation of Φ_ν.

In order to obtain the statement about the continued fractions we consider the entire function

$$K_\nu(z) := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(\nu+1)\cdots(\nu+n)}\left(\frac{z}{2}\right)^{2n}.$$

The identity

$$\frac{\Phi_\nu}{\Phi'_\nu}(\alpha) = -i\sqrt{\alpha}\frac{K_\nu}{K'_\nu}(2i\sqrt{\alpha}). \tag{2.3}$$

can easily be verified for all $\alpha \in \tilde{K}$ with $\alpha\Phi'_\nu(\alpha) \neq 0$. Following [7], p. 210, the right side of (2.3) can be expanded into the continued fraction

$$\nu + 1 + \cfrac{\alpha}{\nu + 2 + \cfrac{\alpha}{\nu + 3 + \cdots}}.$$

Thus Corollary 2 is proved.

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