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Lower powers of elliptic units

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Résumé. Dans un article antérieur [Sch2] il est démontré que les corps de classes de rayon d'un corps quadratique imaginaire possèdent comme générateurs des produits simples de valeurs singulières de la forme de Klein defini plus bas. Dans l'article présent le deuxième auteur a généralisé les résultats de [Sch2] et en même temps corrigé une erreur dans le Théorème 2 de [Sch2]. Le premier auteur a implémenté le calcul de ces produits dans un progamme de KASH et ainsi effectué le calcul des exemples en fin d'article. Ces exemples, ainsi que les cas particuliers traités dans [Sch2], démontrent, que les nombres définis par le Théorème 1 ont des polynômes minimaux à très petits coefficients. En plus, à part des exceptions triviales, ces produits de valeurs singulières constituent même des générateurs, ce qui mène à la conjecture énoncée après le Théorème 1.

Abstract. In the previous paper [Sch2] it has been shown that ray class fields over quadratic imaginary number fields can be generated by simple products of singular values of the Klein form defined below. In the present article the second named author has constructed more general products that are contained in ray class fields thereby correcting Theorem 2 of [Sch2]. An algorithm for the computation of the algebraic equations of the numbers in Theorem 1 of this paper has been implemented in a KASH program by the first named author, who also calculated the list of examples at the end of this article. As in the special cases treated in [Sch2] these examples exhibit again that the coefficients of the algebraic equations are rather small. Moreover, apart from trivial exceptions, all numbers computed so far turn out to be generators of the corresponding ray class field, thereby suggesting the conjecture formulated more precisely after Theorem 1.

Introduction and results

We let Γ be a lattice in $\mathbb{C}$ and ω_1, ω_2 a $\mathbb{Z}$-basis of Γ with $\Im(\frac{\omega_1}{\omega_2}) > 0$. The normalized Klein form is then defined by

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$$\varphi\left(z \,\middle|\, \begin{matrix}\omega_1\\ \omega_2\end{matrix}\right) = 2\pi i\ e^{-\frac{zz^*}{2}}\sigma(z|\Gamma)\eta(\frac{\omega_1}{\omega_2})^2\omega_2^{-1}.$$

Here σ denotes the σ-function of Γ and η the Dedekind η-function. The number z^* is defined by

$$z^* = z_1\eta_1 + z_2\eta_2 \tag{1}$$

with the real coordinates z_1, z_2 of $z = z_1\omega_1 + z_2\omega_2$ and the quasi-periods η_1, η_2 of the elliptic Weierstrass ζ-function of Γ belonging to ω_1, ω_2.

In what follows let K be a quadratic imaginary number field of discriminant d, $\mathfrak{O}$ the ring of integers in K and $\mathfrak{f}$ an integral ideal of $\mathfrak{O}$. We denote by $K_{\mathfrak{f}}$ the ray class field modulo $\mathfrak{f}$ over K. Using this notation we now state our main result.

Theorem 1. *Let ξ be an element of K^*, $\mathfrak{f}$ the denominator of the ideal (ξ) and $f := \min(\mathbb{N} \cap (\mathfrak{f} \setminus \{0\}))$. We use the notation $[\omega_1, \omega_2] := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and choose an element $\alpha \in K$, $\Im(\alpha) > 0$, so that $\mathfrak{O} = [\alpha, 1]$ and*

$$\mathrm{tr}(\alpha) \equiv 0 \ mod\ 3, \qquad \mathrm{tr}(\alpha) \equiv \begin{cases} 0 \ mod\ 4 & if\ 2|d, \\ 1 \ mod\ 4 & if\ 2 \nmid d, \end{cases}$$

which is always possible.
Let s be some integer ≥ 1 and define ideals $\mathfrak{b}_1, ..., \mathfrak{b}_s$ and $\mathfrak{c}$ of $\mathfrak{O}$ of norm $b_1, .., b_s, c$ prime to $6f$ by

$$\overline{\mathfrak{b}_i} = [\alpha, b_i], \qquad \overline{\mathfrak{b}_i\mathfrak{c}} = [\alpha, b_i c], i = 1, ..., s,$$

where the bar denotes complex conjugation. Note that the above second set of equations imposes further constraints on α. Let $\lambda_1, ..., \lambda_s,\ \lambda$ be numbers from $\mathfrak{O} \setminus \mathfrak{f}$ and $n_1, ..., n_s \in \mathbb{Z}$. We suppose that the ideals $\gcd((2), (\lambda_i))$ are equal for all i with $2 \nmid n_i$ and that $\gcd(N(\lambda), 6f) = 1$, where $N(\cdot)$ denotes the norm.
We define

$$\Theta := \prod_{i=1}^{s} \varphi\left(\xi\lambda_i \,\middle|\, \begin{matrix}\frac{\alpha}{b_i}\\ 1\end{matrix}\right)^{n_i}$$

and decompose

$$\mathfrak{f} = f_1\mathfrak{f}_2$$

with $f_1 \in \mathbb{N}$ and a primitive ideal $\mathfrak{f}_2$ of norm f_2. By f_2^* we denote the non split part of f_2 and by f_2^{**} the split part of f_2. Now we make the following assumptions:
1) $n_1 + .. + n_s \equiv 0 \bmod 2$,
2) $n_1 b_1 + ... + n_s b_s \equiv 0 \bmod 4$, *if* $2|d$ *and* $2 \nmid \mathfrak{f}$,
3) $n_1 b_1 + ... + n_s b_s \equiv 0 \bmod 3$, *if* $3|d$ *and* $3 \nmid \mathfrak{f}$,
4) $n_1 b_1 N(\lambda_1) + ... + n_s b_s N(\lambda_s) \equiv 0 \bmod \frac{2f}{f_2^{**}\gcd(2, f_2^{**})}$.

We choose a canonical basis $\mathfrak{f}_2 = [\tilde{\alpha}, f_2]$ *with some other integral element* $\tilde{\alpha} \in \mathfrak{O}$, $\Im(\tilde{\alpha}) > 0$. *Here* $\mathrm{tr}(\tilde{\alpha})$ *is prime to* f_2^{**} *and so there is a solution* a *of the congruence*

$$\mu a \ \mathrm{tr}(f_1\tilde{\alpha}) - N(\mathfrak{f}\xi) \sum_{i=1}^{s} n_i b_i N(\lambda_i) \equiv 0 \ mod\ 2f,$$

where $\mu = 1$ *if* f_2^{**} *is even and* $\mu = 2$ *if* f_2^{**} *is odd. We set* $\zeta := \exp(\frac{2\pi i\mu a}{2f})$. *Then*

$$\zeta\Theta \in K_{\mathfrak{f}},$$

and the action of the Frobenius map $\sigma(\mathfrak{c}\lambda)$ *of* $K_{\mathfrak{f}}/K$ *belonging to the ideal* $\mathfrak{c}\lambda$ *is given by*

$$(\zeta\Theta)^{\sigma(\mathfrak{c}\lambda)} = \zeta^{N(\mathfrak{c}\lambda)}\epsilon(\lambda,\alpha)^{(n_1b_1+\ldots+n_sb_s)c} \prod_{i=1}^{s} \varphi\left(\xi\lambda_i\lambda \,\middle|\, \begin{matrix}\frac{\alpha}{b_i c}\\ 1\end{matrix}\right)^{n_i},$$

where $\epsilon(\lambda,\alpha)$ *is the 12-th root of unity defined in (11).*

Numerical experience shows that apart from trivial exceptions the numbers defined in Theorem 1 are even generators of $K_{\mathfrak{f}}$ over K and in the case of Θ being a power of a quotient of two φ-values this has been proved in [Sch2] under certain assumptions about the λ_i and $\mathfrak{f}$. Moreover the computations done so far make us believe in the following

Conjecture. *Let the* λ_i *in Theorem 1 all be prime to the conductor* $\mathfrak{f}$ *so that raised to the power* $12f$ *the factors in the definition of* Θ *are conjugate numbers in* $K_{\mathfrak{f}}$. *We assume further that the product of these* $12f$*-th powers is not a norm to a proper subfield (i.e. the formal sum* $\sum_i n_i[\lambda_i\mathfrak{b}_i]$ *of ray classes* $[\lambda_i\mathfrak{b}_i]$ *modulo* $\mathfrak{f}$ *of the ideals* $\lambda_i\mathfrak{b}_i$ *with the* n_i*'s as coefficients is not equal to a multiple of the formal sum over the elements of a proper subgroup of the ray class group modulo* $\mathfrak{f}$*). Then* $\zeta\Theta$ *is a generator of* $K_{\mathfrak{f}}/K$.

Theorem 1 is a generalization of Theorem 2 in [Sch2], where, as H. Cohen has pointed out to the authors, the hypothesis "$\gcd(\mathfrak{f},\bar{\mathfrak{f}}) = 1$" has to be added. So in order to correct that theorem of [Sch2] we write down a special case of Theorem 1 of this paper. Herein the number $\left(\frac{\varphi\left(\xi\lambda\middle|\begin{smallmatrix}\frac{\alpha}{b}\\ 1\end{smallmatrix}\right)}{\varphi\left(\xi\middle|\begin{smallmatrix}\alpha\\ 1\end{smallmatrix}\right)}\right)^n$ is a conjugate of the number $\left(\frac{\varphi(\lambda|\mathfrak{f}\mathfrak{b}^{-1})}{\varphi(1|\mathfrak{f})}\right)^n$ defined in [Sch2].

Theorem 2. *Let* ξ *be an element of* K^*, $\mathfrak{f}$ *the denominator of the ideal* (ξ) *and* $f := \min(\mathbb{N}\cap(\mathfrak{f}\setminus\{0\}))$. *We use the notation* $[\omega_1,\omega_2] := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *and choose an element* $\alpha \in K$, $\Im(\alpha) > 0$, *so that* $\mathfrak{O} = [\alpha, 1]$ *and*

$$\mathrm{tr}(\alpha) \equiv 0 \ mod\ 3, \qquad \mathrm{tr}(\alpha) \equiv \begin{cases} 0 \ mod\ 4 & if\ 2|d, \\ 1 \ mod\ 4 & if\ 2\nmid d, \end{cases}$$

which is always possible.
We define ideals $\mathfrak{b}$ *and* $\mathfrak{c}$ *of* $\mathfrak{O}$ *of norm* b, c *prime to* $6f$ *by*

$$\overline{\mathfrak{b}} = [\alpha, b], \quad \overline{\mathfrak{b}\mathfrak{c}} = [\alpha, bc],$$

where the bar denotes complex conjugation. Note that the above second set of equations imposes further constraints on α. *Let* $\lambda_1, \ \lambda$ *be numbers from* $\mathfrak{O} \setminus \mathfrak{f}$. *We suppose* $\gcd(N(\lambda_1), 2) = \gcd(N(\lambda), 6f) = 1$, *where* $N(\cdot)$ *denotes the norm.*
We define

$$\Theta := \frac{\varphi\left(\xi\lambda_1 \,\middle|\, \begin{smallmatrix}\frac{\alpha}{b}\\ 1\end{smallmatrix}\right)}{\varphi\left(\xi \,\middle|\, \begin{smallmatrix}\alpha\\ 1\end{smallmatrix}\right)}$$

and decompose $\mathfrak{f} = f_1\mathfrak{f}_2$ *with* $f_1 \in \mathbb{N}$ *and a primitive ideal* $\mathfrak{f}_2$ *of norm* f_2.
By f_2^* *we denote the non split part of* f_2 *and by* f_2^{**} *the split part of* f_2.
Now we make the following assumptions:
1) $b \equiv 1 \bmod 4$, *if* $2|d$ *and* $2 \nmid \mathfrak{f}$,
2) $b \equiv 1 \bmod 3$, *if* $3|d$ *and* $3 \nmid \mathfrak{f}$,
3) $bN(\lambda_1) \equiv 1 \bmod \frac{2f}{f_2^{**}\gcd(2, f_2^{**})}$.
We choose a canonical basis $\mathfrak{f}_2 = [\tilde{\alpha}, f_2]$ *with some other integral element* $\tilde{\alpha} \in \mathfrak{O}$, $\Im(\tilde{\alpha}) > 0$. *Here* $\mathrm{tr}(\tilde{\alpha})$ *is prime to* f_2^{**} *and so there is a solution* a *of the congruence*

$$\mu a \ \mathrm{tr}(f_1\tilde{\alpha}) - N(\mathfrak{f}\xi)(N(\lambda_1\mathfrak{b}) - 1) \equiv 0 \ mod \ 2f,$$

where $\mu = 1$ *if* f_2^{**} *is even and* $\mu = 2$ *if* f_2^{**} *is odd. We set* $\zeta := \exp(\frac{2\pi i\mu a}{2f})$.
Then

$$\zeta\Theta \in K_{\mathfrak{f}}.$$

The action of the Frobenius map $\sigma(\mathfrak{c}\lambda)$ *of* $K_{\mathfrak{f}}/K$ *belonging to the ideal* $\mathfrak{c}\lambda$ *is given by*

$$(\zeta\Theta)^{\sigma(\mathfrak{c}\lambda)} = \zeta^{N(\mathfrak{c}\lambda)}\epsilon(\lambda,\alpha)^{(b-1)c}\frac{\varphi\left(\xi\lambda_1\lambda \,\middle|\, \begin{smallmatrix}\frac{\alpha}{bc}\\ 1\end{smallmatrix}\right)}{\varphi\left(\xi\lambda \,\middle|\, \begin{smallmatrix}\frac{\alpha}{c}\\ 1\end{smallmatrix}\right)},$$

where $\epsilon(\lambda, \alpha)$ *is the 12-th root of unity defined in (11).*

According to Theorem 3 in [Sch2] all powers of the numbers defined in Theorem 2 of this paper are generators for $K_{\mathfrak{f}}/K$ under certain conditions. For these conditions to be satisfied it is necessary that the ideal $\lambda_1\mathfrak{b}$ is not in the principle ray class modulo $\mathfrak{f}$, a condition which, according to our Conjecture is also believed to be sufficient. However there are cases, where such a pair $\lambda_1, \mathfrak{b}$ cannot be found. Generalizing an idea that the authors have been told by H. Cohen it then follows from class field theory that $K_{\mathfrak{f}} \subseteq K(\zeta_{12f})$. Moreover the above inclusion implies that the Hilbert class field of K is Abelian over $\mathbb{Q}$. Thus K is one of the finite number of quadratic imaginary fields with only one class per genus. Further we can conclude that $\bar{\mathfrak{f}} = \mathfrak{f}$, and it follows more precisely $K(\zeta_f) \subseteq K_{\mathfrak{f}} \subseteq K(\zeta_{12f})$. In particular this implies that $K_{\mathfrak{f}}$ can be constructed using roots of unity, when no pair $\lambda_1, \mathfrak{b}$ satisfying the hypothesis of Theorem 2 exists.

Proof of Theorem 1

The proof of Theorem 1 requires the reciprocity law of complex multiplication which we are now going to explain (see [La, St]). For a natural number N let F_N be the field of modular functions belonging to the group

$$\Gamma(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) \mid M \equiv \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\},$$

that have at every cusp q-expansion coefficients in the N-th cyclotomic field. F_N is generated over F_1 by a primitive N-th roots unity ζ_N and the functions

$$\tau_x(\omega) := \tau\left((x_1, x_2) \left(\begin{smallmatrix} \omega \\ 1 \end{smallmatrix}\right) \mid \left(\begin{smallmatrix} \omega \\ 1 \end{smallmatrix}\right)\right), \quad x = (x_1, x_2) \in \tfrac{1}{N}(\mathbb{Z} \times \mathbb{Z}) \setminus (\mathbb{Z} \times \mathbb{Z}),$$

where τ denotes Weber's τ-function. The extension F_N/F_1 is Galois and its Galois group is isomorphic to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\left\{ \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$. For an integral matrix A of determinant a prime to N the action of the corresponding automorphism of F_N/F_1 is given by

$$\zeta_N \circ A = \zeta_N^a \text{ and } \tau_x \circ A = \tau_{xA}.$$

To compute the action of A on an arbitrary function f on F_N we observe that

$$[\tau_x \circ M](\omega) = \tau_x(M(\omega)) \text{ for } M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),$$

where in abuse of notation we write $M(\omega)$ instead of $\frac{a\omega+b}{c\omega+d}$. Looking at the q-expansion of τ_x (see [De]) we see that $\tau_x \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ is obtained by applying the automorphism $\sigma_a = (\zeta_N \mapsto \zeta_N^a)$ of $\mathbb{Q}_N$ to the q-coefficients of τ_x. So for an arbitrary function $f \in F_N$ with q-expansion

$$f = \sum_n a_n q^{\frac{n}{N}}, \quad q^{\frac{1}{N}} = e^{\frac{2\pi i \omega}{N}}.$$

we also have

$$[f \circ M](\omega) = f(M(\omega)) \text{ for } M \in \mathrm{SL}_2(\mathbb{Z})$$

and

$$f \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) = \sum_n a_n^{\sigma_a} q^{\frac{n}{N}}.$$

For an arbitrary integral matrix A of determinant a prime to N this action can then be computed via a decomposition

$$A \equiv M_1 \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) M_2 \text{ mod } N \text{ with } M_1, M_2 \in \mathrm{SL}_2(\mathbb{Z}),$$

which always exists.

With these definitions and notations we then have the

Reciprocity law. *Let $\mathfrak{a} = [\alpha_1, \alpha_2]$ be an ideal in K and $\alpha := \frac{\alpha_1}{\alpha_2}$ with $\Im(\alpha) > 0$. Then for $f \in F_N$ with $f(\alpha) \neq \infty$ we have*

$$f(\alpha) \in K_N,$$

and the action of the Frobenius map $\sigma(\mathfrak{c})$ belonging to an integral ideal $\mathfrak{c}$ prime to N is given by

$$f(\alpha)^{\sigma(\mathfrak{c})} = [f \circ cC^{-1}](C(\alpha)),$$

where C is an integral matrix of determinant $c > 0$ such that $C\binom{\alpha_1}{\alpha_2}$ is a basis of $\mathfrak{a}\bar{\mathfrak{c}}$.

We remark that the reciprocity law is proved for primitive prime ideals $\mathfrak{c}$ first. By multiplicativity it is then generalized to primitive ideals $\mathfrak{c}$ and it is in fact also valid for integral ideals $\mathfrak{c}$. This can easily be derived observing on the one hand that $f \circ A$ only depends on A modulo N and that on the other hand given an integral ideal $\mathfrak{c}$ prime to N there is a relation

$$\lambda_1 \mathfrak{c} = \lambda_2 \mathfrak{c}_0$$

with integral numbers $\lambda_i \equiv 1 \bmod N$ and a primitive ideal $\mathfrak{c}_0$ from the ray class modulo N of $\mathfrak{c}$.

To apply the reciprocity law to the functions of Theorem 1 we start by collecting some transformation formulas and q-expansions. For a complex lattice $\Gamma = [\omega_1, \omega_2], \Im(\frac{\omega_1}{\omega_2}) > 0$, and $\omega \in \Gamma$ we obtain from the transformation formula of the σ-function

$$\varphi\left(z + \omega \,\middle|\, \begin{matrix}\omega_1 \\ \omega_2\end{matrix}\right) = \psi(\omega) e^{\frac{1}{2} l(z,\omega)} \varphi\left(z \,\middle|\, \begin{matrix}\omega_1 \\ \omega_2\end{matrix}\right) \tag{2}$$

$$\text{with} \quad l(z_1, z_2) := z_1 z_2^* - z_1^* z_2 \text{ and } \psi(\omega) = \begin{cases} 1 & \text{if } \omega \in 2\Gamma, \\ -1 & \text{if } \omega \in \Gamma \setminus 2\Gamma, \end{cases}$$

where the z_i^* have been defined in (1). The $\mathbb{R}$-linear function l is alternate. For any basis ω_1, ω_2 of Γ such that $\Im(\frac{\omega_1}{\omega_2}) > 0$, one has

$$l(\omega_1, \omega_2) = 2\pi i, \tag{3}$$

and for the calculation of l the rule

$$l(z_1 \xi, z_2) = l(z_1, \bar{\xi} z_2) \tag{4}$$

for complex conjugate numbers $\xi, \bar{\xi}$ (see [Ro]) is very useful.

For $x = (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q}$ we consider the function

$$g_x(z) = \varphi\left((x_1, x_2)\binom{z}{1} \,\middle|\, \begin{matrix} z \\ 1 \end{matrix}\right). \tag{5}$$

Then besides (2) we have a second transformation formula

$$g_x(M(z)) = \epsilon(M)^2 g_{xM}(z) \tag{6}$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$. The factor $\epsilon(M)^2$ is a 12-th root of unity that comes from the transformation formula of the η-function:

$$\eta(M(z))^2 = \epsilon(M)^2(cz+d)\eta(z)^2.$$

If $c \geq 0$ and $d > 0$ if $c = 0$ we have the explicit formula

$$\epsilon(M)^2 = \zeta_{12}^{ab+c(d(1-a^2)-a)+3(a-1)c_1}. \tag{6}$$

Here $c_1 = c$ if c is odd and $c_1 = 1$ if c is even, and we use the notation $\zeta_{12} = \exp(\frac{2\pi i}{12})$. This formula is easily derived from [Me]. In fact there are two formulas in [Me], one in the case $2 \nmid c$ and another in the case $2 \nmid a$. The above formula is obtained by applying the quadratic reciprocity law to the Legendre symbol $(\frac{c}{a})$ in front of the second formula in [Me], which then in the case $2 \nmid c$ coincides with the first formula. To evaluate $\epsilon(M)^2$ for arbitrary $M \in \mathrm{SL}_2(\mathbb{Z})$ we have to use the relation

$$\epsilon(-M)^2 = -\epsilon(M)^2. \tag{7}$$

g_x has the q-expansion (see [La])

$$g_x(z) = -q^{\frac{1}{2}B_2(x_1)}e^{\pi i x_2(x_1-1)}(1-Q)\prod_{n=1}^{\infty}(1-q^nQ)(1-q^nQ^{-1}) \tag{8}$$

with

$$B_2(X) = X^2 - X + \frac{1}{6}, \quad q = e^{2\pi i z}, \quad Q = e^{2\pi i(x_1 z + x_2)}.$$

Now let $\mathfrak{a} = [\alpha_1, \alpha_2]$ be an ideal of $\mathfrak{O}$ with $\alpha = \frac{\alpha_1}{\alpha_2}$ in the upper half plane and $\delta \in \frac{1}{f}\mathfrak{a}$, where f is defined as in Theorem 1. Then we can write

$$\delta = (x_1, x_2)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \text{ with } x = (x_1, x_1) \in \tfrac{1}{f}(\mathbb{Z}\times\mathbb{Z})$$

and

$$\varphi\left(\delta \,\middle|\, \begin{matrix}\alpha_1\\ \alpha_2\end{matrix}\right) = g_x(\alpha).$$

(5) and (8) imply that g_x is in F_{12f^2} and the reciprocity law tells us that

$$\varphi\left(\delta \,\middle|\, \begin{matrix}\alpha_1\\ \alpha_2\end{matrix}\right) = g_x(\alpha) \in K_{12f^2}.$$

To compute the Galois action let $\mathfrak{c}$ be a primitive ideal of $\mathfrak{O}$ prime to $6f$ and let $\sigma(\mathfrak{c})$ denote the Frobenius automorphism of K_{12f^2}/K belonging to $\mathfrak{c}$. Then according to the reciprocity law the action of $\sigma(\mathfrak{c})$ is given by

$$g_x(\alpha)^{\sigma(\mathfrak{c})} = [g_x \circ cC^{-1}](C(\alpha)),$$

where C is a rational matrix of determinant $c = N(\mathfrak{c})$ that transforms the basis of $\mathfrak{a}$ into a basis of $\mathfrak{a}\bar{\mathfrak{c}}$. With a decomposition of the matrix cC^{-1},

$$cC^{-1} = M_1 \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} M_2, \qquad M_1, M_2 \in \mathrm{SL}_2(\mathbb{Z}),$$

we then compute using the transformation formula (5)

$$\begin{aligned} g_x \circ cC^{-1} &= [[g_x \circ M_1] \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)] \circ M_2 \\ &= [[\epsilon(M_1)^2 g_{xM_1}] \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)] \circ M_2 \\ &= [\epsilon(M_1)^{2c} g_{xM_1\left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)}] \circ M_2 \\ &= \epsilon(M_1)^{2c}\epsilon(M_2)^2 g_{xM_1\left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)M_2} \\ &= \epsilon(M_1)^{2c}\epsilon(M_2)^2 g_{xcC^{-1}}. \end{aligned}$$

By homogeneity of φ we find

$$g_{xcC^{-1}}(C(\alpha)) = \varphi\left(\delta \,\middle|\, \tfrac{1}{c} C \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\right).$$

So the Galois action becomes as already shown in [Ro, Sch1]

$$\varphi\left(\delta \,\middle|\, \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}\right)^{\sigma(\mathfrak{c})} = \epsilon(M_1)^{2c}\epsilon(M_2)^2 \varphi\left(\delta \,\middle|\, \frac{1}{c} C \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\right) \tag{9}$$

and herein $\frac{1}{c} C \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right)$ is a basis of $\mathfrak{a}\mathfrak{c}^{-1}$.

Now we consider two special cases:

(i) First we assume that $\mathfrak{a} = [\alpha, 1]$ and $\mathfrak{a}\bar{\mathfrak{c}} = [\alpha, c]$. Then $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)$, whence $cC^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and by the formulas (6),(7) we get

$$\varphi\left(\delta \,\middle|\, \begin{matrix} \alpha \\ 1 \end{matrix}\right)^{\sigma(\mathfrak{c})} = (-1)^{\frac{c-1}{2}} \varphi\left(\delta \,\middle|\, \begin{matrix} \frac{\alpha}{c} \\ 1 \end{matrix}\right). \tag{10}$$

(ii) Next we assume $\mathfrak{a} = \mathfrak{O} = [\alpha, 1]$ and $\mathfrak{c} = (\lambda)$ to be a principal ideal prime to $12f$. We write

$$\lambda = u + v\alpha \text{ with } \quad u, v \in \mathbb{Z}.$$

As $\sigma(\lambda)$ depends on λ only mod $12f^2$ we can change λ modulo $12f^2$ thereby achieving that

$$v \neq 0 \text{ and } \gcd(u, v) = 1.$$

Then

$$C = \begin{pmatrix} u & vN \\ -v & u + vS \end{pmatrix},$$

where S and N denote trace and norm of α, and we find the decomposition

$$cC^{-1} = M_1 \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} M_2,$$

$$M_1 = \begin{pmatrix} avN + b(u+vS) & -1 \\ 1 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} v & u \\ a & b \end{pmatrix}$$

with $a, b \in \mathbb{Z}$ such that $\det(M_2) = 1$. Moreover as $v \neq 0$ we can choose $a > 0$. So using the explicit formula (6) for $\epsilon(M_i)$ we can now evaluate the factor

$$\epsilon(\lambda, \alpha) := \epsilon(M_1)^{2c}\epsilon(M_2)^2. \tag{11}$$

By the formula (9), it does not depend on the auxiliary choice of a and b. We omit the boring case by case calculation using the fact that $c = u^2 + uvS + v^2N$ is prime to 6. The result is

$$\epsilon(\lambda,\alpha)^3 = \begin{cases} \zeta_4^{(-uvN+1)c-uv-u} & \text{if } 2\nmid u, \\ \zeta_4^{Sc+c-v+1} & \text{if } 2|u, \end{cases} \quad \epsilon(\lambda,\alpha)^4 = \begin{cases} \zeta_3^{-uv(Nc+1)} & \text{if } 3\nmid u, \\ \zeta_3^{Sc} & \text{if } 3|u. \end{cases}$$

Moreover, in case (ii), assume that $S = \operatorname{tr}(\alpha)$ satisfies the congruences stated in Theorem 1. Then it can be worked out that

$$\begin{aligned} &\epsilon(\lambda,\alpha)^6 = 1, \ \text{ if } 2 \nmid d \text{ or } 2|(\lambda - 1), \\ &\epsilon(\lambda,\alpha)^4 = 1, \ \text{ if } 3 \nmid d \text{ or } 3|(\lambda - 1). \end{aligned} \tag{12}$$

Observing further that for $\mathfrak{c} = (\lambda)$ we have $\frac{1}{c}C\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix}\alpha\\ 1\end{pmatrix}$, the above formula (9) becomes

$$\varphi\left(\delta \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right)^{\sigma(\lambda)} = \epsilon(\lambda,\alpha)\varphi\left(\delta\lambda \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right) \tag{13}$$

where by homogeneity of φ the factor $\frac{1}{\lambda}$ in the second argument is replaced by the factor λ in the first argument. Now we use (10) and (13) to prove the assertions of Theorem 1. The relative automorphisms of $K_{12f^2}/K_{\mathfrak{f}}$ are of the form $\sigma(\lambda)$, $\lambda = 1 + \omega$, $\omega \in \mathfrak{f}$. By (13) we compute using the transformation formula (2) and the properties (3) and (4) of l

$$\begin{aligned} \varphi\left(\xi\lambda_i \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right)^{\sigma(\lambda)} &= \epsilon(\lambda,\alpha)\varphi\left(\xi\lambda_i\lambda \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right) \\ &= \epsilon(\lambda,\alpha)\psi(\xi\lambda_i\omega)e^{\frac{1}{2}l(\xi\lambda_i,\xi\lambda_i\omega)}\varphi\left(\xi\lambda_i \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right) \\ &= \epsilon(\lambda,\alpha)\psi(\xi\lambda_i\omega)e^{\frac{1}{2}N(\xi\lambda_i)l(1,\omega)}\varphi\left(\xi\lambda_i \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right). \end{aligned} \tag{14}$$

To evaluate $l(1,\omega)$ we choose in $\mathfrak{f}$ a canonical basis, $\mathfrak{f} = f_1[\tilde{\alpha}, f_2]$, $\Im(\tilde{\alpha}) > 0$, and write

$$\omega = wf + yf_1\tilde{\alpha} \text{ with } w, y \in \mathbb{Z}.$$

Then, keeping in mind that $\tilde{\alpha} \equiv \alpha \bmod \mathbb{Z}$, we get $l(1,\omega) = -2\pi i f_1 y$. Now we apply $\sigma(\mathfrak{b}_j)$ to both sides in of (14) and obtain using (10)

$$\varphi\left(\xi\lambda_j \,\middle|\, \begin{matrix}\frac{\alpha}{b_j}\\ 1\end{matrix}\right)^{n_j(\sigma(\lambda)-1)} = \epsilon(\lambda,\alpha)^{n_jb_j}\psi(\xi\lambda_j\omega)^{n_jb_j}e^{-\frac{2\pi i}{2f}n_jb_jN(\mathfrak{f}\xi\lambda_j)y}.$$

The action of $\sigma(\lambda)$ on Θ is then given by

$$\Theta^{\sigma(\lambda)-1} = \epsilon(\lambda,\alpha)^{\sum_{j=1}^{s} n_j b_j} \left(\prod_{j=1}^{s} \psi(\xi\lambda_j\omega)^{n_j b_j}\right) e^{-\frac{2\pi i}{2f}\sum_{j=1}^{s} n_j b_j N(\mathfrak{f}\xi\lambda_j)y}.$$

Further we find

$$\zeta_{2f}^{\mu(\sigma(\lambda)-1)} = \zeta_{2f}^{\mu(N(\lambda)-1)} = \zeta_{2f}^{\mu\ \mathrm{tr}(f_1\tilde{\alpha})y}.$$

The assumptions about the b_j, λ_j and n_j together with the property (12) of $\epsilon(\lambda,\alpha)$ now imply that $\zeta\Theta$ is invariant under the action of the Galois group of $K_{12f^2}/K_{\mathfrak{f}}$ and so is contained in $K_{\mathfrak{f}}$. The computation of the conjugates as described in the Theorem follows from (10) and (13).

Examples

By the following examples it is shown on the one hand, that the minimal polynomials of the numbers constructed in Theorem 1 mostly have rather small coefficients. On the other hand all numbers considered in the following are generators for $K_{\mathfrak{f}}/K$ thereby confirming our conjecture.

Using Theorem 1 we determine Θ and its conjugates as well as the root of unity ζ by which we get the minimal polynomial of $\zeta\Theta$ over K. In all cases $\zeta\Theta$ turns out to be a generator of $K_{\mathfrak{f}}/K$, though in the examples 5.-7. the conditions of our conjecture are not satisfied.

In the following $m_{\theta,K}$ denotes the minimal polynomial of θ over K. Further we denote by $\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_5, \mathfrak{p}_7$ prime ideals dividing $2, 3, 5, 7$ and the number α is assumed to be of the form $\alpha = \frac{u+\sqrt{d}}{2}$ with the discriminant d of K and $u = \mathrm{tr}(\alpha) \in \mathbb{Z}$.

1. Let $d = -7$, $\mathfrak{f} = (9)$ and

$$\Theta = \varphi\left(\xi \,\middle|\, \begin{matrix}\frac{\alpha}{11}\\ 1\end{matrix}\right)\varphi\left(\xi \,\middle|\, \begin{matrix}\frac{\alpha}{7}\\ 1\end{matrix}\right)$$

where $\mathrm{tr}(\alpha) = 189$, $\xi = \frac{1}{9}$. Then $\zeta = 1$ and

$$\begin{aligned} m_{\Theta,K} = {} & X^{36} - 6X^{33} + 15X^{32} + 27X^{30} - 45X^{29} + 63X^{28} - 81X^{27} \\ & + 54X^{26} - 54X^{25} + 414X^{24} + 117X^{23} + 135X^{22} - 747X^{21} \\ & + 162X^{20} + 351X^{19} + 1269X^{18} + 108X^{17} - 387X^{16} - 576X^{15} \\ & + 450X^{14} + 1638X^{13} + 1683X^{12} + 891X^{11} - 171X^{10} - 1002X^{9} \\ & - 1044X^{8} - 351X^{7} + 531X^{6} + 729X^{5} + 531X^{4} + 297X^{3} \\ & + 117X^{2} + 27X + 3. \end{aligned}$$

2. Let $d=-24$, $\mathfrak{f}=(2)\mathfrak{p}_3\mathfrak{p}_5$ and

$$\Theta = \varphi\left(\xi \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right)\varphi\left(\xi \,\middle|\, \begin{matrix}\frac{\alpha}{59}\\ 1\end{matrix}\right)$$

where $\mathrm{tr}(\alpha)=624$, $\xi=\frac{3+16\sqrt{-24}}{30}$. Then $\zeta=1$ and

$$\begin{aligned} m_{\Theta,K} = {} & X^{16}+(2-2\sqrt{-6})X^{15}+(-17-\sqrt{-6})X^{14}+(-6+22\sqrt{-6})X^{13}\\ & +(98+4\sqrt{-6})X^{12}+(-16-57\sqrt{-6})X^{11}+(-159+24\sqrt{-6})X^{10}\\ & +(62+56\sqrt{-6})X^{9}+(41-37\sqrt{-6})X^{8}+(-112-26\sqrt{-6})X^{7}\\ & +(-87+15\sqrt{-6})X^{6}+(2+27\sqrt{-6})X^{5}+(35-5\sqrt{-6})X^{4}\\ & +(-24-17\sqrt{-6})X^{3}+(-26+2\sqrt{-6})X^{2}+(2+3\sqrt{-6})X+1. \end{aligned}$$

3. Let $d=-11$, $\mathfrak{f}=(9)$ and

$$\Theta = \frac{\varphi\left(\xi \,\middle|\, \begin{matrix}\frac{\alpha}{47}\\ 1\end{matrix}\right)\varphi\left(2\xi \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right)^2}{\varphi\left(\xi \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right)}$$

where $\mathrm{tr}(\alpha)=417$, $\xi=\frac{1}{9}$. Then $\zeta=1$ and

$$\begin{aligned} m_{\Theta,K} = {} & X^{18}+9X^{17}+36X^{16}+(95-4\sqrt{-11})X^{15}+(189-39\sqrt{-11})X^{14}\\ & +(192-147\sqrt{-11})X^{13}+(-233-246\sqrt{-11})X^{12}\\ & +(-960-60\sqrt{-11})X^{11}+(-735+408\sqrt{-11})X^{10}\\ & +(935+534\sqrt{-11})X^{9}+(1716-9\sqrt{-11})X^{8}+(84-441\sqrt{-11})X^{7}\\ & +(-1379-144\sqrt{-11})X^{6}+(-603+258\sqrt{-11})X^{5}\\ & +(345+195\sqrt{-11})X^{4}+(218+\sqrt{-11})X^{3}+(-12-3\sqrt{-11})X^{2}\\ & +(12+3\sqrt{-11})X+1. \end{aligned}$$

4. Let $d=-31$, $\mathfrak{f}=\mathfrak{p}_2^3$ and

$$\Theta = \frac{\varphi\left(\xi \,\middle|\, \begin{matrix}\frac{\alpha}{31}\\ 1\end{matrix}\right)}{\varphi\left(\xi \,\middle|\, \begin{matrix}\alpha\\ 1\end{matrix}\right)^3}$$

where $\mathrm{tr}(\alpha)=93$, $\xi=\frac{15+\sqrt{-31}}{16}$. Then $\zeta=1$ and

$$\begin{aligned} m_{2\Theta,K} = {} & X^{6}+8X^{5}+(18-2\sqrt{-31})X^{4}+24X^{3}+(66-2\sqrt{-31})X^{2}\\ & +(40-8\sqrt{-31})X+(4-4\sqrt{-31}). \end{aligned}$$

The denominator of Θ can be obtained by the known factorisation of the singular values of φ.

5. Let $d=-20$, $\mathfrak{f}=(10)$ and

$$\Theta=\varphi\left(\sqrt{-5}\xi\,\middle|\,\begin{matrix}\alpha\\1\end{matrix}\right)\varphi\left((10+\sqrt{-5})\xi\,\middle|\,\begin{matrix}\frac{\alpha}{7}\\1\end{matrix}\right)$$

where $\mathrm{tr}(\alpha)=48$, $\xi=\frac{1}{10}$. Then $\zeta=1$ and

$$\begin{aligned}m_{\Theta,K}= &X^{40}+(-30-10\sqrt{-5})X^{35}+(378+140\sqrt{-5})X^{30}\\ &+(-1596-710\sqrt{-5})X^{25}+(1684+1290\sqrt{-5})X^{20}\\ &+(366-3770\sqrt{-5})X^{15}+(816+1490\sqrt{-5})X^{10}\\ &+(-24+50\sqrt{-5})X^{5}+1.\end{aligned}$$

The special form of $m_{\Theta,K}$ shows, that Θ^5 does not generate $K_{\mathfrak{f}}/K$. This is not in contrast to our conjecture, because $\sqrt{-5}$ is not prime to $\mathfrak{f}$. Rather one can write Θ^5 with $\tilde{\xi}=\frac{\sqrt{-5}}{10}$ as

$$\Theta^5=\left(\varphi\left(\tilde{\xi}\,\middle|\,\begin{matrix}\alpha\\1\end{matrix}\right)\varphi\left((1-2\sqrt{-5})\tilde{\xi}\,\middle|\,\begin{matrix}\frac{\alpha}{7}\\1\end{matrix}\right)\right)^5$$

and Theorem 1 supplies $\Theta^5\in K_{\tilde{\mathfrak{f}}}$ with $\tilde{\mathfrak{f}}=(2)\mathfrak{p}_5$. So this example also shows, that in general the exponents of the numbers given in Theorem 1 can not be reduced.

6. Let $d=-52$, $\mathfrak{f}=\mathfrak{p}_7$ and

$$\Theta=\varphi\left(\xi\,\middle|\,\begin{matrix}\alpha\\1\end{matrix}\right)\varphi\left(\xi\,\middle|\,\begin{matrix}\frac{\alpha}{11}\\1\end{matrix}\right)$$

with $\mathrm{tr}(\alpha)=72$, $\xi=\frac{1-\sqrt{-13}}{7}$. Then $\zeta=\zeta_7^3$ and

$$\begin{aligned}m_{\zeta\Theta,K}=X^6+(1+\sqrt{-13})X^5+(-6+\sqrt{-13})X^4+(1+\sqrt{-13})X^3\\ +(-6+\sqrt{-13}).\end{aligned}$$

Here $\zeta\Theta$ is a generator of $K_{\mathfrak{f}}/K$, though it is a relative norm in the sense of our conjecture. Also $(\zeta\Theta)^2$ generates $K_{\mathfrak{f}}/K$ with

$$m_{(\zeta\Theta)^2,K}=X^6+(47-16\sqrt{-13})X^4+(46-24\sqrt{-13})X^2+(23-12\sqrt{-13}).$$

Considering $m_{(\zeta\Theta)^2,K}$ we see, that $(\zeta\Theta)^4$ generates only a proper subfield of $K_{\mathfrak{f}}$ over K. $K_{\mathfrak{f}}$ itself is generated by $(\zeta\Theta)^4$ over $K_{(1)}$ at least.

7. With "small" denominators $\mathfrak{f}$ the Hilbert classfield $K_{(1)}$ can often be constructed by the products in Theorem 1. This does not follow from our conjecture as $K_{(1)}$ has the conductor $\mathfrak{f}=(1)$.

Let $d=-119$, $\mathfrak{f}=\mathfrak{p}_2$ and

$$\Theta=\frac{\varphi\left(\xi\,\middle|\,\begin{matrix}\alpha\\1\end{matrix}\right)}{\varphi\left(\xi\,\middle|\,\begin{matrix}\frac{\alpha}{17}\\1\end{matrix}\right)}$$

where $\mathrm{tr}(\alpha) = 357$, $\xi = \frac{3+\sqrt{-119}}{4}$. Then $\zeta = 1$ and

$$m_{\Theta,K} = X^{10} - 5X^9 + 8X^8 + 7X^7 + 5X^6 - 25X^5 + 5X^4 + 7X^3 + 8X^2 - 5X + 1.$$

Let $d = -71$, $\mathfrak{f} = \mathfrak{p}_3$ and

$$\Theta = \varphi\left(\xi \,\middle|\, \begin{matrix}\alpha \\ 1\end{matrix}\right)^2$$

with $\mathrm{tr}(\alpha) = 9$, $\xi = \frac{5+\sqrt{-71}}{6}$. Then $\zeta = \bar{\zeta}_3$ and

$$\begin{aligned} m_{\zeta\Theta,K} = \ & X^7 + (1+\sqrt{-71})X^5 + \tfrac{57+3\sqrt{-71}}{2}X^4 + \tfrac{105-3\sqrt{-71}}{2}X^3 \\ & + (13 - 5\sqrt{-71})X^2 + (46 + \sqrt{-71}). \end{aligned}$$

References

[De] M. Deuring, *Die Klassenkörper der komplexen Multiplikation.* Enzykl. d. math. Wiss. I/2, 2. Aufl., Heft 10, Stuttgart, 1958.

[La] S. Lang, *Elliptic functions.* Addison Wesley, 1973.

[Me] C. Meyer, *Über einige Anwendungen Dedekindscher Summen.* Journal Reine Angew. Math. **198** (1957), 143–203.

[Ro] G. Robert, *La racine 12-ième canonique de* $\frac{\Delta(L)^{[\underline{L}:L]}}{\Delta(\underline{L})}$. Sém. de th. des nombres Paris, 1989-90.

[Sch1] R. Schertz, *Niedere Potenzen elliptischer Einheiten.* Proc. of the International Conference on Class Numbers and Fundamental Units of Algebraic Number Fields, Katata, Japan (1986), 67–87.

[Sch2] R. Schertz, *Construction of Ray Class Fields by Elliptic Units.* J. Théor. Nombres Bordeaux **9** (1997), 383–394.

[St] H. Stark, *L-functions at* $s = 1$*, IV.* Adv. Math. **35** (1980), 197–235.

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