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On sums of Hecke series in short intervals

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RÉSUMÉ. On a $\sum_{K-G\le\kappa_j\le K+G}\alpha_j H_j^3(\frac12) \ll_\varepsilon GK^{1+\varepsilon}$ pour $K^\varepsilon \le G \le K$, ou $\alpha_j = |\rho_j(1)|^2(\cosh\pi\kappa_j)^{-1}$, et $\rho_j(1)$ est le premier coefficient de Fourier de forme de Maass correspondant à la valeur propre $\lambda_j = \kappa_j^2 + \frac14$ à laquelle le série de Hecke $H_j(s)$ est attachée. Ce résultat fournit l'estimation nouvelle $H_j(\frac12) \ll_\varepsilon \kappa_j^{\frac13+\varepsilon}$.

ABSTRACT. We have $\sum_{K-G\le\kappa_j\le K+G}\alpha_j H_j^3(\frac12) \ll_\varepsilon GK^{1+\varepsilon}$ for $K^\varepsilon \le G \le K$, where $\alpha_j = |\rho_j(1)|^2(\cosh\pi\kappa_j)^{-1}$, and $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue $\lambda_j = \kappa_j^2 + \frac14$ to which the Hecke series $H_j(s)$ is attached. This result yields the new bound $H_j(\frac12) \ll_\varepsilon \kappa_j^{\frac13+\varepsilon}$.

1. Introduction and statement of results

The purpose of this paper is to obtain a bound for sums of Hecke series in short intervals which, as a by-product, gives a new bound for $H_j(\frac12)$. We begin by stating briefly the necessary notation and some results involving the spectral theory of the non-Euclidean Laplacian. For a competent and extensive account of spectral theory the reader is referred to Y. Motohashi's monograph [13].

Let $\{\lambda_j = \kappa_j^2 + \frac14\} \cup \{0\}$ be the eigenvalues (discrete spectrum) of the hyperbolic Laplacian

$$\Delta = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$$

acting over the Hilbert space composed of all Γ-automorphic functions which are square integrable with respect to the hyperbolic measure. Let $\{\psi_j\}$ be a maximal orthonormal system such that $\Delta\psi_j = \lambda_j\psi_j$ for each $j \ge 1$ and $T(n)\psi_j = t_j(n)\psi_j$ for each integer $n \in \mathbb{N}$, where

$$\big(T(n)f\big)(z) \;=\; \frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b=1}^{d} f\left(\frac{az+b}{d}\right)$$

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is the Hecke operator. We shall further assume that $\psi_j(-\bar{z}) = \varepsilon_j \psi_j(z)$ with $\varepsilon_j = \pm 1$. We then define ($s = \sigma + it$ will denote a complex variable)

$$H_j(s) \;=\; \sum_{n=1}^{\infty} t_j(n) n^{-s} \qquad (\sigma > 1),$$

which we call the Hecke series associated with the Maass wave form $\psi_j(z)$, and which can be continued to an entire function. The Hecke series satisfies the functional equation

$$\begin{aligned} H_j(s) = 2^{2s-1}\pi^{2s-2}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j) \\ \times \big(\varepsilon_j \cosh(\pi\kappa_j) - \cos(\pi s)\big)\big) H_j(1-s), \end{aligned}$$

which by the Phragmén–Lindelöf principle (convexity) implies the bound

$$H_j(\tfrac12) \;\ll_\varepsilon\; \kappa_j^{\frac12+\varepsilon}. \tag{1.1}$$

It is known that $H_j(\frac12) \ge 0$ (see Katok–Sarnak [8] and for the proofs of (1.2)–(1.4) see [11] or [13]), and

$$\sum_{\kappa_j \le K} \alpha_j H_j^2(\tfrac12) \;=\; (A \log K + B)K^2 + O(K \log^6 K) \qquad (A > 0). \tag{1.2}$$

Here as usual we put

$$\alpha_j = |\rho_j(1)|^2 (\cosh \pi\kappa_j)^{-1},$$

where $\rho_j(1)$ is the first Fourier coefficient of $\psi_j(z)$. Moreover we have

$$\sum_{\kappa_j \le K} \alpha_j H_j^4(\tfrac12) \ll K^2 \log^{15} K \tag{1.3}$$

and

$$\sum_{j=1}^{\infty} \alpha_j H_j^3(\tfrac12) h_0(\kappa_j) = \left(\frac83 + O\left(\frac{1}{\log K}\right)\right) \pi^{-3/2} K^3 G \log^3 K \tag{1.4}$$

with

$$K^{\frac12} \log^5 K \le G \le K^{1-\varepsilon}, \tag{1.5}$$

$$h_0(r) = (r^2 + \tfrac14)\left(e^{-\left(\frac{r-K}{G}\right)^2} + e^{-\left(\frac{r+K}{G}\right)^2}\right). \tag{1.6}$$

Apart from its intrinsic interest, the asymptotic formula (1.4) has an important application in the theory of the Riemann zeta-function. Namely it immediately implies that there are infinitely many κ such that

$$\sum_{\kappa_j = \kappa} \alpha_j H_j^3(\tfrac12) \;>\; 0,$$

which is essential in establishing Ω–results for the function $E_2(T)$, which represents the error term in the asymptotic formula for the fourth moment of $|\zeta(\frac{1}{2}+it)|$ (see [13, Chapter 5]). Instead of the sum in (1.4) we shall consider the sum $\sum_{K-G\le\kappa_j\le K+G}\alpha_j H_j^3(\frac{1}{2})$ and seek an upper bound for it, which is especially interesting when $G=K^\varepsilon$. In that case it follows from (1.1) and (1.2) (or from (1.3), or from (1.4)) that

$$\sum_{K-K^\varepsilon\le\kappa_j\le K+K^\varepsilon}\alpha_j H_j^3(\tfrac{1}{2}) \ll_\varepsilon K^{\frac{3}{2}+\varepsilon}, \tag{1.7}$$

where here and later $\varepsilon>0$ denotes arbitrarily small constants, not necessarily the same ones at each occurrence. We can suppose that

$$\sum_{K-K^\varepsilon\le\kappa_j\le K+K^\varepsilon}\alpha_j H_j^3(\tfrac{1}{2}) \ll_\varepsilon K^{1+\alpha+\varepsilon} \qquad (0\le\alpha\le\tfrac{1}{2}), \tag{1.8}$$

and it is reasonable to expect that (1.8) holds with $\alpha=0$. This is indeed so, and is the content of the following

Theorem. *We have*

$$\sum_{K-G\le\kappa_j\le K+G}\alpha_j H_j^3(\tfrac{1}{2}) \ll_\varepsilon GK^{1+\varepsilon} \tag{1.9}$$

for

$$K^\varepsilon \le G \le K. \tag{1.10}$$

In view of the convention made above on the use of ε's, the above result strictly speaking means that, for given ε sufficiently small, the bound (1.9) holds with $GK^{1+\varepsilon_1}$ and $\lim_{\varepsilon\to0}\varepsilon_1=0$, provided that (1.10) holds.

Corollary 1. *We have (1.8) with $\alpha=0$.*

From $H_j(\frac{1}{2})\ge0$ and the bound

$$\alpha_j = \frac{|\rho_j(1)|^2}{\cosh(\pi\kappa_j)} \gg_\varepsilon \kappa_j^{-\varepsilon}$$

of H. Iwaniec [6] we obtain

Corollary 2. *We have*

$$H_j(\tfrac{1}{2}) \ll_\varepsilon \kappa_j^{\frac{1}{3}+\varepsilon}. \tag{1.11}$$

This seems to be the first unconditional improvement over (1.1), and represents the limit of our method. Note that H. Iwaniec [7] obtained (1.11) assuming a certain hypothesis (the referee remarked that, using a trickier amplifier based on the equality $\lambda_f(p)^2-\lambda_f(p^2)=1$, Iwaniec observed that his method actually gives unconditionally $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^{\frac{5}{12}+\varepsilon}$, but this result sharper than (1.1) does not seem to have appeared in print). His paper contains several other interesting results, including a bound for sums of squares of $H_j(s)$ over κ_j's in short intervals.

We remark that W. Luo [10] proved the bound

$$H_j(\tfrac{1}{2} + i\kappa_j) \ll_\varepsilon \kappa_j^{\frac{1}{4}+\varepsilon}$$

by exploiting some special properties of the Hecke series at the points $s = \frac{1}{2} \pm i\kappa_j$, but our method certainly cannot give such a sharp bound for $H_j(\frac{1}{2})$, for which one expects the bound $H_j(\frac{1}{2}) \ll_\varepsilon \kappa_j^\varepsilon$, and more generally one conjectures that $H_j(\frac{1}{2} + it) \ll_\varepsilon (|t|\kappa_j)^\varepsilon$. This bound may be viewed as a sort of the "Lindelöf hypothesis" for $H_j(\frac{1}{2})$. Since $H_j(s)$ bears several analogies (i.e., the functional equation) to $\zeta^2(s)$, then the bound (1.11) represents the analogue of the classical estimate $\zeta(\frac{1}{2} + it) \ll |t|^{1/6}$.

Cubic moments of automorphic L-functions $L_f(s, \chi)$ have been recently investigated by J.B. Conrey and H. Iwaniec [1]. Although they also exploit the idea of the nonnegativity of cubes of central values of automorphic L–functions, their methods are quite different from ours. One of their main results is the bound

$$\sum_{f \in F^\star} L_f^3(\tfrac{1}{2}, \chi) \ll_\varepsilon q^{1+\varepsilon},$$

where $F^\star$ is the set of all primitive cusp forms of weight k (an even integer ≥ 12) and level dividing q, where $\chi(n) = (\frac{n}{q})$ for odd, squarefree q.

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2. Beginning of proof

Before we begin the proof, some further notation will be necessary. If one denotes the left-hand side of (1.4) by $\mathcal{C}(K, G)$, then with $\lambda = C \log K$ ($C > 0$) one has ([13, (3.4.18)], with the extraneous factor $(1-(\kappa_j/K)^2)^\nu$ omitted)

$$(2.1)\qquad \begin{aligned} \mathcal{C}(K,G) = &\sum_{f \leq 3K} f^{-\frac{1}{2}} \exp\Big(-\Big(\frac{f}{K}\Big)^\lambda\Big)\mathcal{H}(f; h_0) \\ &- \sum_{\nu=0}^{N_1} \sum_{f \leq 3K} f^{-\frac{1}{2}} U_\nu(fK)\mathcal{H}(f; h_\nu) + O(1), \end{aligned}$$

with ($h_0(r)$ is given by (1.6))

$$(2.2)\qquad h_\nu(r) \;=\; h_0(r)\left(1 - \Big(\frac{r}{K}\Big)^2\right)^\nu,$$

$$\begin{aligned}
\mathcal{H}(f;h) &= \sum_{\nu=1}^{7} \mathcal{H}_\nu(f;h),\\
\mathcal{H}_1(f;h) &= -2\pi^{-3} i \left\{ (\gamma - \log(2\pi\sqrt{f}))(\hat{h})'(\tfrac12) + \tfrac14 (\hat{h})''(\tfrac12) \right\} d(f) f^{-\frac12},\\
\mathcal{H}_2(f;h) &= \pi^{-3} \sum_{m=1}^{\infty} m^{-\frac12} d(m) d(m+f) \Psi^{+}\Big(\frac{m}{f};h\Big) \quad \Big(d(n) = \sum_{\delta|n} 1\Big),
\end{aligned}$$

$$\mathcal{H}_3(f;h) = \pi^{-3} \sum_{m=1}^{\infty} (m+f)^{-\frac12} d(m) d(m+f) \Psi^{-}\Big(1 + \frac{m}{f};h\Big), \tag{2.3}$$

$$\begin{aligned}
\mathcal{H}_4(f;h) &= \pi^{-3} \sum_{m=1}^{f-1} m^{-\frac12} d(m) d(f-m) \Psi^{-}\Big(\frac{m}{f};h\Big),\\
\mathcal{H}_5(f;h) &= -(2\pi^3)^{-1} f^{-\frac12} d(f) \Psi^{-}(1;h),\\
\mathcal{H}_6(f;h) &= -12\pi^{-2} i \sigma_{-1}(f) f^{\frac12} h'(-\tfrac12 i),\\
\mathcal{H}_7(f;h) &= -\pi^{-1} \int_{-\infty}^{\infty} \frac{|\zeta(\frac12 + ir)|^4}{|\zeta(1+2ir)|^2} \sigma_{2ir}(f) f^{-ir} h(r)\,\mathrm{d}r \quad \Big(\sigma_a(f) = \sum_{d|f} d^a\Big),
\end{aligned}$$

where

$$\begin{aligned}
\hat{h}(s) &= \int_{-\infty}^{\infty} r h(r) \frac{\Gamma(s+ir)}{\Gamma(1-s+ir)}\,\mathrm{d}r,\\
\Psi^{+}(x;h) &= \int_{(\beta)} \Gamma^2(\tfrac12 - s) \tan(\pi s) \hat{h}(s) x^s\,\mathrm{d}s,
\end{aligned}$$

and

$$\Psi^{-}(x;h) = \int_{(\beta)} \Gamma^2(\tfrac12 - s) \frac{\hat{h}(s)}{\cos(\pi s)} x^s\,\mathrm{d}s,$$

with $-\frac32 < \beta < \frac12$, N_1 is a sufficiently large integer,

$$U_\nu(x) = \frac{1}{2\pi i \lambda} \int_{(-\lambda^{-1})} (4\pi^2 K^{-2} x)^w u_\nu(w) \Gamma\big(\frac{w}{\lambda}\big)\,\mathrm{d}w \ll \Big(\frac{x}{K^2}\Big)^{-\frac{C}{\log K}} \log^2 K,$$

where $u_\nu(w)$ is a polynomial in w of degree $\le 2N_1$, whose coefficients are bounded. A prominent feature of Motohashi's explicit expression for $\mathcal{C}(K,G)$ is that it contains series and integrals with the classical divisor function $d(n)$ only, with no quantities from spectral theory. Therefore the problem of obtaining an upper bound for $\mathcal{C}(K,G)$ is a problem of classical analytic number theory.

Now we are ready to begin with the proof of our result. We shall start from the obvious bound

$$\sum_{K-G \le \kappa_j \le K+G} \alpha_j H_j^3(\tfrac12) \ll K^{-2} \mathcal{C}(K,G) \qquad (K^\varepsilon \le G \le K), \tag{2.4}$$

so that the proof of the Theorem reduces to showing that

$$\mathcal{C}(K,G) \ll_\varepsilon K^{3+\varepsilon}G \qquad (K^\varepsilon \le G \le K). \tag{2.5}$$

The delicate machinery of (2.1)–(2.3) was developed by Motohashi in order to establish the asymptotic formula (1.4), where special care must be taken in order to produce the (weak) error term $O(1/\log K)$. To achieve this, Motohashi assumed the bound $G \ge K^{\frac12}\log^5 K$ in (1.5), which immediately rendered several contributions in (2.1) negligibly small. However, in (2.5) we are not aiming at an asymptotic formula for $\mathcal{C}(K,G)$, but only at an upper bound. To obtain this we could start from first principles, but it seemed expedient to utilize the machinery of (2.1)–(2.3). First of all, by going through the proof of (1.4), it is seen that it is the term $\nu = 0$ in (2.1) whose contributions should be considered, because the bound for the ν-th term will be essentially the same as the bound for the term $\nu = 0$, only it will be multiplied by $(G/K)^\nu$. We note that the factors $\exp(-(f/K)^\lambda)$ and $U_\nu(fK)$ in (2.1) can be conveniently removed by partial summation. Next we follow the analysis carried out in [13, pp. 120 and 128-129] to show that the contribution of $\nu = 1,3,5,6,7$ in (2.3) to (2.1) will be $\ll_\varepsilon K^{3+\varepsilon}G$. Indeed we have

$$\mathcal{H}_1(f;h_0) \ll d(f)f^{-1/2}K^3G\log^2 K, \quad \mathcal{H}_3(f;h_0) \ll e^{-C\log^2 K} \quad (C>0)$$

by [13, (3.4.20)-(3.4.24)], and in view of [13, (3.3.44)]

$$\mathcal{H}_5(f;h_0) \ll d(f)f^{-1/2}, \quad \mathcal{H}_6(f;h_0) \ll \sigma_{-1}(f)f^{1/2}K.$$

Finally to deal with $\mathcal{H}_7(f;h_0)$ note that we have $1/\zeta(1+ir) \ll \log(|r|+1)$, $\zeta(\frac12+ir) \ll |r|^{1/6+\varepsilon}$ (see [4]) and

$$\sum_{n=1}^\infty \sigma_{2ir}(n)n^{-ir-s} = \zeta(s-ir)\zeta(s+ir) \qquad (r\in\mathbb{R}, \Re\mathrm{e}\, s > 1).$$

Consequently by the Perron inversion formula (see e.g., [4, p. 486])

$$\sum_{f\le 3K} \sigma_{2ir}(f)f^{-\frac12-ir} \ll_\varepsilon K^{\frac13+\varepsilon} \qquad (K \ll |r| \ll K).$$

Since the relevant range of r in $\mathcal{H}_7(f;h_0)$ is $|r\pm K| \le G\log K$, it follows that the total contribution of $\mathcal{H}_7(f;h_0)$ to (2.1) is $\ll_\varepsilon GK^{3+\varepsilon}$ if G satisfies (1.10). Thus it transpires that what is non-trivial is the contribution to (2.1) of

$$\mathcal{H}_2(f;h_0) = \pi^{-3}\sum_{m=1}^\infty m^{-\frac12}d(m)d(m+f)\Psi^+\Big(\frac{m}{f};h_0\Big), \tag{2.6}$$

with $m \le 2f$ (the terms with $m > 2f$ are negligible by [13, (3.4.21)]) and

$$\mathcal{H}_4(f; h_0) \;=\; \pi^{-3}\sum_{m=1}^{f-1} m^{-\frac12} d(m)d(f-m)\Psi^-\Bigl(\frac{m}{f}; h_0\Bigr). \tag{2.7}$$

We begin with the contribution of (2.6) for $m \le 2f$, noting that by [13, (3.4.20)] we have, for $m \le 2f$ and suitable $c > 0$,

$$\Psi^+\Bigl(\frac{m}{f}; h_\nu\Bigr) \ll K^3 G\left(\frac{G}{K}\right)^\nu \exp\left(-cG^2\frac{m}{f}\right) + \frac{f}{m}\exp(-\tfrac14\log^2 K), \tag{2.8}$$

which clearly shows that the contribution of the portion of (2.6) with $m \le 2f$ is negligibly small if (1.5) holds. Our idea is to evaluate the relevant integrals arising from $\Psi^\pm(m/f; h_0)$ explicitly and then to estimate the ensuing exponential sums, which will permit us to obtain (2.5) with G lying outside of the range given by (1.5). From (2.8) it follows that the nontrivial contribution of (2.6) with $m \le 2f$ will consist of the subsum

$$\pi^{-3}\sum_{G^2/\log^2 K \le f \le 3K} f^{-\frac12}\sum_{m \le fG^{-2}\log^2 K} m^{-\frac12}\ldots\,, \tag{2.9}$$

where the sum over m is non-empty for $G \le \sqrt{3K}\log K$. Henceforth we suppose that

$$K^\varepsilon \le G \le K^{\frac12-\varepsilon}, \tag{2.10}$$

which is actually sufficient for the proof of the Theorem. Namely for the range $K^{\frac12-\varepsilon} \le G \le K^{1-\varepsilon}$ the bound (1.9) follows from (1.4)–(1.5), and for $K^{1-\varepsilon} \le G \le K$ from $\sum_{\kappa_j \le K}\alpha_j H_j^3(\frac12) \ll K^2\log^C K$, with an appropriate change of ε in (1.9). Now we shall use the formula after [13, (3.3.39)] with $x = m/f = o(1)$ (as $K \to \infty$), namely

$$\begin{aligned}\Psi^+(x; h) &= 2\pi\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\\ &\times \Re\mathrm{e}\left\{\frac{\Gamma^2(\frac12+ir)}{\Gamma(1+2ir)}F(\tfrac12+ir,\tfrac12+ir;1+2ir;-\frac1x)x^{-ir}\right\}\mathrm{d}r,\end{aligned} \tag{2.11}$$

where F is the hypergeometric function. We shall apply a classical quadratic transformation formula (see [9, (9.6.12)]) for the hypergeometric function. This is

$$\begin{aligned}F(\alpha,\beta;2\beta;z) \;&=\; \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha}\\ &\times F\left(\alpha,\alpha-\beta+\tfrac12;\beta+\tfrac12;\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right),\end{aligned} \tag{2.12}$$

so that (2.11) will give

$$
\begin{aligned}
\Psi^+(x;h_0) &= 4\pi\frac{\sqrt{x}}{\sqrt{x}+\sqrt{1+x}}\int_{-\infty}^{\infty} rh_0(r)\tanh(\pi r)\\
&\times\Re\mathrm{e}\left\{\frac{\Gamma^2(\frac12+ir)}{\Gamma(1+2ir)}\Bigl(\frac{\sqrt{x}+\sqrt{1+x}}{2}\Bigr)^{-2ir}\right.\\
&\left.\times F\left(\tfrac12+ir,\tfrac12;1+ir;\Bigl(\frac{\sqrt{x}-\sqrt{1+x}}{\sqrt{x}+\sqrt{1+x}}\Bigr)^2\right)\right\}\mathrm{d}r.
\end{aligned}
\tag{2.13}
$$

From the definition (1.6) it is seen that the integral in (2.13) will make a negligible contribution unless $|r+K|\le G\log K$ and $|r-K|\le G\log K$. Since the contributions of both ranges of r are treated analogously (the presence of two exponentials in (1.6) is necessitated by the fact that Motohashi's approach requires $h_0(r)$ to be an even function of r), we shall treat only the latter, noting that $\tanh(\pi r)=1+O(e^{-K})$ for $|r-K|\le G\log K$. For $|z|<1$ one has, by the defining property of the hypergeometric function,

$$
\begin{aligned}
F(\alpha,\beta;\gamma;z) &= \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k\\
&= \sum_{k=0}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+k-1)\beta(\beta+1)\dots(\beta+k-1)}{\gamma(\gamma+1)\dots(\gamma+k-1)k!}z^k.
\end{aligned}
\tag{2.14}
$$

We insert (2.14) in (2.13) with $\alpha=\frac12+ir$, $\beta=\frac12$, $\gamma=1+ir$,

$$
z=\left(\frac{\sqrt{x}-\sqrt{1+x}}{\sqrt{x}+\sqrt{1+x}}\right)^2=\bigl(\sqrt{x}+\sqrt{1+x}\,\bigr)^{-4}=1-4\sqrt{x}+O(x)<1-5\sqrt{x},
$$

since $m\le fG^{-2}\log^2K$ yields $x=m/f=o(1)$. In view of the absolute convergence of the series in (2.14), the resulting relevant expression in (2.13) will be

$$
\frac{4\pi\sqrt{x}}{\sqrt{x}+\sqrt{1+x}}\sum_{k=0}^{\infty}\frac{(\frac12)_k}{k!}\bigl(\sqrt{x}+\sqrt{1+x}\,\bigr)^{-4k}\Re\mathrm{e}\, I_k,
\tag{2.15}
$$

where

$$
\begin{aligned}
I_k = \int\limits_{K-G\log K}^{K+G\log K} & r(r^2+\tfrac14)e^{-(\frac{r-K}{G})^2}\frac{(\frac12+ir)_k}{(1+ir)_k}\\
&\times\left(\frac{\sqrt{x}+\sqrt{1+x}}{2}\right)^{-2ir}\frac{\Gamma^2(\frac12+ir)}{\Gamma(1+2ir)}\mathrm{d}r
\end{aligned}
\tag{2.16}
$$

with $K^\varepsilon \le G \le K^{\frac12-\varepsilon}$. Note that $(\alpha)_0 \equiv 1$ and for $k \ge 1$

$$\left|\frac{(\frac12+ir)_k}{(1+ir)_k}\right| \le 1, \quad \frac{(\frac12+ir)_k}{(1+ir)_k} = 1 + O\left(\frac1r\right)$$

uniformly in k. The contribution of $k \ge K^{1/2}\log^2 K$ will be clearly negligible, by trivial estimation of the tails of the series in (2.15). The contribution of each I_k will be analogous, hence it will suffice to consider in detail only the case $k=0$. Note that

$$\frac{(\frac12)_k}{k!} = \frac{(2k)!}{2^{2k}(k!)^2} \ll \frac{1}{\sqrt{k}}$$

if we use the well-known approximation

$$k! = \sqrt{2\pi}k^{k+\frac12}\exp(-k+\frac{\vartheta}{12k}) \qquad (0<\vartheta<1).$$

Therefore we obtain

$$\begin{aligned}
&\frac{4\pi\sqrt{x}}{\sqrt{x}+\sqrt{1+x}}\sum_{k=0}^\infty \frac{(\frac12)_k}{k!}(\sqrt{x}+\sqrt{1+x}\,)^{-4k}\\
&\ll \sqrt{x}\sum_{k=0}^\infty (k+1)^{-1/2}(1-5\sqrt{x})^k\\
&\ll \sqrt{x}\Big(\sum_{k\le x^{-1/2}}(k+1)^{-1/2} + \sum_{k=0}^\infty x^{1/4}(1-5\sqrt{x})^k\Big)\\
&\ll \sqrt{x}\Big(x^{-1/4}+x^{1/4}x^{-1/2}\Big) \ll x^{1/4}.
\end{aligned} \tag{2.17}$$

Then the expression in (2.9) becomes, up to a negligible error,

$$\begin{aligned}
\Re\mathrm{e}\Big\{\frac{4}{\pi^2}&\sum_{0\le k\le K^{\frac12}\log^2 K}\frac{(\frac12)_k}{k!}\sum_{G^2\log^{-2}K\le f\le 3K} f^{-\frac12}\\
&\times\sum_{m\le fG^{-2}\log^2 K} m^{-\frac12}x^{\frac12}(\sqrt{x}+\sqrt{1+x})^{-4k-\frac12}I_k\Big\},
\end{aligned} \tag{2.18}$$

where $x = m/f \ll K^{-\varepsilon}$. Note that the expression containing x in (2.18) can be conveniently removed by partial summation. For each k the double sum over m and f in (2.18) (without the expression containing x) will be $\ll_\varepsilon GK^{3+\varepsilon}$ uniformly in k (the key fact is that the oscillating factor does not depend on k), as will be shown in the next section. Then using (2.17) (with x different from $x=m/f$, but certainly $x \ll K^{-\varepsilon}$) it follows that the total contribution of (2.9) is $\ll_\varepsilon GK^{3+\varepsilon}$, as asserted.

Thus it suffices to estimate the contribution coming from I_0 in (2.18), and to simplify the gamma-factors in (2.16) we use Stirling's formula in the form $(t \ge t_0 > 0)$

$$\Gamma(s) = \sqrt{2\pi}\, t^{\sigma-\frac12} \exp\left(-\tfrac12\pi t + it\log t - it + \tfrac12\pi i(\sigma - \tfrac12)\right) \times \left(1 + O_\sigma\left(t^{-1}\right)\right), \tag{2.19}$$

with the understanding that the O–term in (2.19) admits an asymptotic expansion in terms of negative powers of t. Therefore we may replace the gamma-factors in (2.16) by $Cr^{-1/2}e^{-2ir\log 2}(1 + O(1/r))$, and then make the change of variable $r = K + Gu$ to obtain that the relevant contribution to I_0 will be a multiple of

$$I' \ := \ G\int_{-\log K}^{\log K} (K+Gu)^{\frac12}((K+Gu)^2+\tfrac14)e^{-u^2}(\sqrt{x}+\sqrt{1+x}\,)^{-2iK-2iGu}\mathrm{d}u.$$

We expand the first two expressions in I' in power series, taking sufficiently many terms so that the error term will, by trivial estimation, make a negligible contribution. The integrals with the remaining terms are evaluated by using the formula

$$\int_{-\infty}^{\infty} u^j e^{Au-u^2}\mathrm{d}u \ = \ P_j(A)e^{\frac14 A^2} \quad (j = 0, 1, 2, \dots\,,\ P_0(A) = \sqrt{\pi}\,), \tag{2.20}$$

where $P_j(z)$ is a polynomial in z of degree j, which may be explicitly evaluated by successive differentiation of the formula

$$\int_{-\infty}^{\infty} e^{Au-u^2}\mathrm{d}u \ = \ \sqrt{\pi}e^{\frac14 A^2},$$

considered as a function of A. We note that in each integral over $[-\log K, \log K]$ we may replace the interval of integration with $(-\infty, \infty)$, making a negligible error. Then we use (2.20) with

$$A = -2iG\log(\sqrt{x} + \sqrt{1+x}\,) \ll G\sqrt{\frac{m}{f}} \quad \left(x \ = \ \frac{m}{f} \ = \ o(1)\right),$$

so that in view of the summation condition in (2.9) we have $A \ll \log^2 K$. The main contribution to I' will come from the term $j = 0$ in (2.20). This is

$$\begin{aligned} &GK^{5/2}(\sqrt{x} + \sqrt{1+x}\,)^{-2iK}\int_{-\infty}^{\infty}(\sqrt{x} + \sqrt{1+x}\,)^{-2iGu}e^{-u^2}\mathrm{d}u \\ &\qquad = \sqrt{\pi}GK^{5/2}(\sqrt{x} + \sqrt{1+x}\,)^{-2iK}\exp\left(-G^2\log^2(\sqrt{x} + \sqrt{1+x}\,)\right), \end{aligned}$$

and it is precisely the factor $(\sqrt{x} + \sqrt{1+x}\,)^{-2iK}$ which is taken into consideration in our analysis and is crucial for the proof of the final result.

3. Estimates of exponential sums

Now we shall insert the above expression in (2.18) (omitting summation over k and disregarding the expression containing x, as was just explained), to obtain that the relevant expression which is to be estimated is a multiple of

$$\begin{aligned}&GK^{5/2}\sum_{G^2/\log^2 K\le f\le 3K} f^{-\frac12}\sum_{m\le fG^{-2}\log^2 K} m^{-\frac12}d(m)d(m+f)\times\\ &\times\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)^{-2iK}\exp\left(-G^2\log^2\left(\sqrt{\frac{m}{f}}+\sqrt{1+\frac{m}{f}}\right)\right).\end{aligned}\tag{3.1}$$

Therefore we have reduced the problem to the estimation of the double exponential sum appearing in (3.1). The exponential factor in (3.1), which is

$$\ll \exp\left(-\frac{CG^2m}{f}\right)\qquad (C>0),$$

is harmless, and can be removed by partial summation, being monotonic in m or f. The first idea that might occur in estimating the sum in (3.1) is to treat it as $\sum\sum d(m)d(m+f)\ldots$, namely as the binary additive divisor problem weighted with an exponential factor. For this problem the error term is precisely evaluated and estimated by Y. Motohashi [12], and various averages of the error term by Y. Motohashi and the author [5]. However, summation over the "shift" parameter f in (3.1) is too "long" for such formulas to be successfully applied. Other possibilities are to use estimates involving one- and two-dimensional exponent pairs, coupled with the Voronoi summation formula (see [2]–[4]), to exploit the particular properties of the function $d(n)$. However all these approaches yield values of G in a range not as large as the one in (1.10).

To prove the Theorem we shall proceed in the following, essentially elementary way. First we change the order of summation in (3.1), keeping in mind that $m \le fG^{-2}\log^2 K$. Then with the help of Taylor's formula we replace $f^{-1/2}$ by $(m+f)^{-1/2}$, taking sufficiently many terms so that the contribution made by trivial estimation of the error term is negligibly small. The contribution of the term $(m+f)^{-1/2}$ will be dominant. We replace $m+f$ by n, use partial summation to remove the factor $\exp(-G^2\ldots)$, and let m and n lie in $O(\log^2 K)$ subsums where $M < m \le M_1 \le 2M$, $N < n \le N_1 \le 2N$. Then we are led to the estimation of the expression

$$\begin{aligned}&GK^{5/2}\sum_{M<m\le M_1\le 2M}\ \sum_{N<n\le N_1\le 2N} d(m)m^{-\frac12}d(n)n^{-\frac12}\times\\ &\qquad\times\exp\left(2iK\log\left(\sqrt{\frac{m}{n-m}}+\sqrt{\frac{n}{n-m}}\right)\right),\end{aligned}\tag{3.2}$$

where we may assume that

$$K^{\varepsilon} \le G \le K^{\frac{1}{2}-\varepsilon},\ M \ll KG^{-2}\log^2 K,$$
$$MG^2\log^2 K \ll N \ll K,\ N \ge K^{\frac{1}{2}},\ MN \ge K. \tag{3.3}$$

The first condition in (3.3) is given by (2.10), and the next two are implied by (3.1). Further, for $N \le K^{\frac{1}{2}}$ (keeping in mind that $M \le N$, because $m \le fG^{-2}\log^2 K$) or for $MN \le K$ we have, by trivial estimation, that the contribution of (3.2) is

$$\ll_{\varepsilon} GK^{5/2+\varepsilon}(MN)^{1/2} \ll_{\varepsilon} GK^{3+\varepsilon},$$

as necessary. Next the range of summation over n in (3.2) is divided into $O(N/N_0)$ subintervals $\mathcal{I}$ of length at most N_0, where N_0 is a parameter that will be suitably chosen a little later, and which satisfies

$$1 \le N_0 \le N. \tag{3.4}$$

Hence the sum to be estimated is

$$\sum := \sum_{M<m\le M_1\le 2M}\ \sum_{n\in\mathcal{I}} d(m)m^{-\frac{1}{2}}d(n)n^{-\frac{1}{2}}\exp(iF(m,n)), \tag{3.5}$$

where

$$F(m,n) := 2K\log\left(\sqrt{\frac{m}{n-m}}+\sqrt{\frac{n}{n-m}}\right). \tag{3.6}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\left|\sum\right|^2 &\le \sum_{M<m\le M_1}\frac{d^2(m)}{m}\sum_{M<m\le M_1}\left|\sum_{n\in\mathcal{I}} d(n)n^{-1/2}e^{iF(m,n)}\right|^2\\
&\ll \log^3 M \sum_{M<m\le M_1}\sum_{n_1\in\mathcal{I}}\sum_{n_2\in\mathcal{I}} d(n_1)d(n_2)(n_1n_2)^{-1/2}e^{i(F(m,n_1)-F(m,n_2))}\\
&\ll_{\varepsilon} K^{\varepsilon}\left(N_0MN^{-1}+\sum_{n_1\ne n_2\in\mathcal{I}} N^{-1}\left|\sum_{M<m\le M_1} e^{i(F(m,n_1)-F(m,n_2))}\right|\right).
\end{aligned}$$

If $F_m(m,n)$ denotes the partial derivative of $F(m,n)$ with respect to m, then

$$F_m(m,n) = \frac{Kn}{(n-m)\sqrt{mn}},$$

and we obtain

$$|F_m(m,n_1)-F_m(m,n_2)| \asymp KM^{-1/2}N^{-3/2}|n_1-n_2|.$$

By hypothesis $|n_1-n_2| \le N_0$, thus we have

$$|F_m(m,n_1)-F_m(m,n_2)| \le \tfrac{1}{2}$$

if with suitable $C > 0$ we choose

$$N_0 = CN^{3/2}M^{1/2}K^{-1}. \tag{3.7}$$

Therefore by standard estimates (see e.g., [4, Lemma 1.2 and Lemma 2.1]) we have

$$\sum_{M<m\le M_1} e^{i(F(m,n_1)-F(m,n_2))} \ll \frac{M^{1/2}N^{3/2}}{K|n_1-n_2|} \qquad (n_1 \ne n_2, n_1 \in \mathcal{I}, n_2 \in \mathcal{I}). \tag{3.8}$$

Hence by using (3.8) we obtain

$$\Big|\sum\Big|^2 \ll_\varepsilon K^\varepsilon(MN_0N^{-1} + M^{1/2}N_0N^{1/2}K^{-1}).$$

Consequently the contribution of (3.5) will be, since $M \le N$ by (3.3),

$$\begin{aligned}&\ll_\varepsilon GK^{5/2+\varepsilon}NN_0^{-1}(M^{1/2}N_0^{1/2}N^{-1/2} + N_0^{1/2}M^{1/4}N^{1/4}K^{-1/2})\\ &\ll_\varepsilon GK^{3+\varepsilon}(M/N)^{1/4} + GK^{5/2+\varepsilon}N^{1/2} \ll_\varepsilon GK^{3+\varepsilon}.\end{aligned}$$

It remains to check that N_0, given by (3.7), verifies (3.4). We have

$$N_0 = CN^{3/2}M^{1/2}K^{-1} \le N$$

for $CN^{1/2}M^{1/2} \le K$, which is true in view of $M \le N \le K$. Also $NM \ge K$ may be assumed in view of (3.3), and therefore

$$N_0 = CN^{3/2}M^{1/2}K^{-1} \ge 1$$

holds for $C(NM)^{1/2}NK^{-1} \gg NK^{-1/2} \ge 1$, that is for, $N \ge K^{1/2}$, which is again true by (3.3). Thus the contribution of (3.1) is $\ll_\varepsilon GK^{3+\varepsilon}$, and consequently the total contribution of $\mathcal{H}_2(f;h)$ is also $\ll_\varepsilon GK^{3+\varepsilon}$, as asserted.

4. Completion of proof

To finish the proof we have yet to deal with the sum in (2.7). For $0 < x < 1$ and $-\frac{3}{2} < \beta < -\frac{1}{2}$ we have [13, (3.3.45)]

$$\begin{aligned}\Psi^-(x;h) = \int_0^\infty &\left\{\int_{(\beta)} x^s(y(y+1))^{s-1}\frac{\Gamma^2(\frac{1}{2}-s)}{\Gamma(1-2s)\cos(\pi s)}\mathrm{d}s\right\}\\ &\times\left\{\int_{-\infty}^{\infty} rh(r)\left(\frac{y}{y+1}\right)^{ir}\mathrm{d}r\right\}\mathrm{d}y,\end{aligned} \tag{4.1}$$

with $h(r)$ given by (1.6) and (2.2). Similarly as in the analysis concerning (2.11), for $h = h_0$, we may consider only the ranges $|r+K| \le G\log K$ and $|r-K| \le G\log K$, and we turn our attention to the latter. Namely for

$|r \pm K| \ge G\log K$ we interchange the order of integration and in the y-integral we integrate the subintegral over $(0,1]$ by parts to obtain that the contribution is $\ll x^\beta \exp(-\frac12\log^2 K)$. Therefore the dominant contribution of the r-integral will be

$$\begin{aligned}&\int_{K-G\log K}^{K+G\log K} r(r^2+\tfrac14)e^{-(r-K)^2G^{-2}}\left(\frac{y}{y+1}\right)^{ir}\mathrm{d}r\\ &= G\int_{-\log K}^{\log K}(K+Gu)((K+Gu)^2+\tfrac14))e^{-u^2}\left(\frac{y}{y+1}\right)^{iK}\left(\frac{y}{y+1}\right)^{iuG}\mathrm{d}u.\end{aligned}$$

We simplify the expression in the first two brackets in the last integral and use (2.20) with $A = iG\log y/(y+1)$ and $P_1(A) = \frac12\sqrt{\pi}A$. Then the above expression equals $O(\exp(-\frac12\log^2 K))$ plus

$$\begin{aligned}&e^{iK\log\frac{y}{y+1}}e^{-\frac14 G^2\log^2\frac{y}{y+1}}\left(\sqrt{\pi}GK^3+\frac{3\sqrt{\pi}}{2}iG^2K^2\log\frac{y}{y+1}\right)\\ &\qquad + O\left(KG^3e^{-\frac18 G^2\log^2\frac{y}{y+1}}\right).\end{aligned}\tag{4.2}$$

In view of the exponential factor in (4.2) we may truncate the y–integral in (4.1) at $G/\log K$ with a negligible error. Therefore the contribution of the O–term in (4.2) is, with $\beta = \varepsilon - 3/2$,

$$\ll KG^3\int_{G/\log K}^\infty x^\beta y^{2\beta-2}\mathrm{d}y \ll \left(\frac{m}{f}\right)^{\varepsilon-3/2}KG^{\varepsilon-1}.$$

The total contribution of this expression is $\ll_\varepsilon K^{3+\varepsilon}G^{-1}$. The main terms in (4.2) are treated analogously, and it is the first one which will make a larger contribution, so it will be treated in detail. The relevant part of $\Psi^-(x;h)$ will be

$$\begin{aligned}&\sqrt{\pi}GK^3\int_{(\beta)}\frac{x^s\Gamma^2(\frac12-s)}{\Gamma(1-2s)\cos(\pi s)}\\ &\qquad\times\Bigl(\int_{\frac{G}{\log K}}^\infty (y(y+1))^{s-1}e^{iK\log\frac{y}{y+1}}e^{-\frac14 G^2\log^2\frac{y}{y+1}}\mathrm{d}y\Bigr)\mathrm{d}s.\end{aligned}\tag{4.3}$$

In view of Stirling's formula and

$$|\cos(x+iy)| = \sqrt{\cos^2 x+\sinh^2 y}\qquad (x\in\mathbb{R},\ y\in\mathbb{R}),$$

it follows that the contribution of $|\Im\mathrm{m}\, s| = |t| > \log^2 K$ in (4.3) will be negligibly small. For $s = \beta + it\,(-\frac32 < \beta < -\frac12,\ |t| \le \log^2 K)$ we write the integral over y in (4.3) as

$$I \;:=\; \int_{G/\log K}^{\infty} (y^2+y)^{\beta-1} e^{iF(y)} e^{-\frac14 G^2 \log^2 \frac{y}{y+1}} \mathrm{d}y \tag{4.4}$$

with

$$F(y) \;:=\; t(\log y + \log(y+1)) + K\log y - K\log(y+1) \qquad (|t| \le \log^2 K),$$

so that

$$F'(y) \;=\; \frac{t}{y} + \frac{t}{y+1} + \frac{K}{y(y+1)} \;\gg\; \frac{K}{y^2}$$

for $y \ll K\log^{-2} K$. We further write

$$I = \int_{G/\log K}^{K^{1-\varepsilon}} + \int_{K^{1-\varepsilon}}^{\infty} = I_1 + I_2 = I_1 + O_\varepsilon(K^{2\beta-1+\varepsilon}),$$

by estimating I_2 trivially. In I_1 we write $e^{iF(y)} = (e^{iF(y)})'/(iF'(y))$ and integrate by parts. Note that the integrated term at $y = G/\log K$ will be negligibly small in view of the second exponential factor in (4.4), and at $y = K^{1-\varepsilon}$ it will be $\ll_\varepsilon K^{2\beta-1+\varepsilon}$. We obtain

$$\begin{aligned} I_1 &= O_\varepsilon(K^{2\beta-1+\varepsilon}) - \\ &- \frac{1}{i}\int\limits_{\frac{G}{\log K}}^{K^{1-\varepsilon}} \Bigg\{ -\frac{F''(y)}{(F'(y))^2}(y^2+y)^{\beta-1} + \frac{1}{F'(y)}(\beta-1)(2y+1)(y^2+y)^{\beta-2} \\ &+ \frac{1}{F'(y)}(y^2+y)^{\beta-1}\left(-\frac{G^2}{2y(y+1)}\log\frac{y}{y+1}\right)\Bigg\} \\ &\quad \times e^{iF(y)} e^{-\frac14 G^2\log^2\frac{y}{y+1}} \mathrm{d}y. \end{aligned}$$

The expression in curly brackets is, since in I_1 we have $F'(y) \gg Ky^{-2}$ and $y \gg G/\log K$,

$$\ll \left(\frac{y}{K} + \frac{G^2}{Ky}\right) y^{2\beta-2} \ll \frac{y}{K}\log^2 K \cdot y^{2\beta-2}. \tag{4.5}$$

Thus we obtain the same type of exponential integral again, only in place of the factor $(y^2+y)^{\beta-1}$ we obtain an expression whose order is given by the right-hand side of (4.5). Since $-3 < 2\beta < -1$, this means that if repeat five times integration by parts we shall obtain an integral which, when majorized, will have a nonnegative exponent of y in the integrand. Trivial estimation of this integral will yield then

$$I_1 \;\ll_\varepsilon\; K^{2\beta-1+\varepsilon},$$

and taking $\beta = \varepsilon - \frac{3}{2}$ we obtain that $(x = m/f)$ for I in (4.4) we have

$$K^3 GI \ll_\varepsilon K^3 G x^\beta K^{2\beta-1+\varepsilon} \ll_\varepsilon G\left(\frac{f}{m}\right)^{3/2} K^{\varepsilon-1}.$$

By using (2.7) we see that this makes a total contribution of $\ll_\varepsilon GK^{1+\varepsilon}$ to (2.1), and thus the proof of the Theorem is complete.

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