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Maximal unramified extensions of imaginary quadratic number fields of small conductors, II

par KEN YAMAMURA

RÉSUMÉ. Dans l'article [15], nous donnions dans une table la structure des groupes de Galois $\mathrm{Gal}(K_{ur}/K)$ des extensions maximales non ramifiées K_{ur} des corps de nombres quadratiques imaginaires K de conducteur $\leqq 1000$ sous l'Hypothèse de Riemann Généralisée, sauf pour 23 d'entre eux (tous de conducteur $\geqq 723$). Ici nous mettons à jour cette table, en précisant, pour 19 de ces corps exceptionnels, la structure de $\mathrm{Gal}(K_{ur}/K)$. En particulier pour $K = \mathbf{Q}(\sqrt{-856})$, nous obtenons $\mathrm{Gal}(K_{ur}/K) \cong \widetilde{S_4} \times C_5$ et $K_{ur} = K_4$, le quatrième corps de classes de Hilbert de K. C'est le premier exemple d'un corps de nombres dont la tour de corps de classes est de longueur 4.

ABSTRACT. In the previous paper [15], we determined the structure of the Galois groups $\mathrm{Gal}(K_{ur}/K)$ of the maximal unramified extensions K_{ur} of imaginary quadratic number fields K of conductors $\leqq 1000$ under the Generalized Riemann Hypothesis (GRH) except for 23 fields (these are of conductors $\geqq 723$) and give a table of $\mathrm{Gal}(K_{ur}/K)$. We update the table (under GRH). For 19 exceptional fields K of them, we determine $\mathrm{Gal}(K_{ur}/K)$. In particular, for $K = \mathbf{Q}(\sqrt{-856})$, we obtain $\mathrm{Gal}(K_{ur}/K) \cong \widetilde{S_4} \times C_5$ and $K_{ur} = K_4$, the fourth Hilbert class field of K. This is the first example of a number field whose class field tower has length four.

1. Introduction

In the previous paper [15], we determined the structure of the Galois groups $\mathrm{Gal}(K_{ur}/K)$ of the maximal unramified extensions K_{ur} of imaginary quadratic number fields K of conductors $\leqq 1000$ under the Generalized Riemann Hypothesis (GRH) except for 23 fields (these are of conductors $\geqq 723$) and give a table of $\mathrm{Gal}(K_{ur}/K)$. (The results are unconditional for conductors $\leqq 420$.)

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We update the table (under GRH). For 19 exceptional fields K of them, we determine $\mathrm{Gal}(K_{ur}/K)$. In [15], we verified $l=1$ for 15 fields of 23 exceptional fields and $l \geqq 2$ and $K_{ur}=K_l$ for the other 8 fields, where l is the length of the class field tower of K: $K=K_0 \subsetneqq K_1 \subsetneqq \cdots \subsetneqq K_l = K_{l+1} = \cdots$ (K_{i+1} is the Hilbert class field of K_i). For 12 of 15 fields with $l=1$, we show $K_{ur}=K_1$, the Hilbert class field of K, that is, we show that K_1 has no

TABLE 1. Table of $\mathrm{Gal}(K_{ur}/K)$ for 23 exceptional fields (not complete)

$-d$	$\mathrm{Cl}(K)$	K_1	K_2	l	G
723	C_4	$K(\sqrt{(-7+3\sqrt{241})/2})$		1	C_4
731	C_{12}	$K(\sqrt{-5+2\sqrt{17}}, \alpha_{87})$		3	$\widetilde{A_4} \curlyvee C_8$
763	C_4	$K(\sqrt{(-9+\sqrt{109})/2})$		1	C_4
771	C_6	$K(\sqrt[3]{16+\sqrt{257}})$		3	$S_4 \times C_3$
772	C_4	$K(\sqrt{(-7+\sqrt{193})/2})$		1	C_4
787	C_5	$K(\gamma_{42})$		1	C_5
808	C_6	$K(\sqrt{-2}, \alpha_{98})$		1	C_6
843	C_6	$K(\sqrt[3]{(5+\sqrt{281})/2})$		1	C_6
856	C_6	$K(\sqrt{2}, \alpha_{105})$	$K_1(\alpha_9)$	4	$\widetilde{S_4} \times C_3$
883	C_3	$K(\alpha_{109})$		1	?
904	C_8			1	C_8
907	C_3	$K(\alpha_{112})$		1	?
916	C_{10}	$K(\sqrt{-1}, \gamma_{52})$		3	$S_4 \times C_5$
932	C_{12}	$K(\sqrt{(13+\sqrt{233})/2}, \alpha_{115})$		1	C_{12}
939	C_8			1	C_8
947	C_5	$K(\gamma_{55})$		1	?
952	$C_4 \times C_2$	$K(\sqrt{-7}, \sqrt{-9+10\sqrt{2}})$		2	$(C_4 \wr C_2) \curlywedge D_5$
964	C_{12}	$K(\sqrt{(-15+\sqrt{241})/2}, \alpha_{119})$		1	C_{12}
971	C_{15}	$K(\alpha_{120}, \gamma_{57})$		1	C_{15}
979	C_8			1	C_8
984	$C_6 \times C_2$	$K(\sqrt{2}, \sqrt[3]{9+\sqrt{82}})$		3	$(Q_{16} \ltimes C_3^2) \times C_3$
987	$C_4 \times C_2$	$K(\sqrt{-3}, \sqrt{(-1+3\sqrt{21})/2})$	$K_1(\gamma_1)$	2	$Q_{20} \times C_2$
996	$C_6 \times C_2$	$K(\sqrt{-1}, \sqrt[3]{82+9\sqrt{83}})$	$K_1(\alpha_2)$	$\geqq 3$	?

Supplements. For $d=-883, -907, -947$. If $K_{ur} \neq K_1$, $\mathrm{Gal}(K_{ur}/K) \cong \mathrm{PSL}(2,7) \times \mathrm{Cl}(K)$.
For $K=\mathbf{Q}(\sqrt{-996})$. $\mathrm{Gal}(K_2/K) \cong D_3 \times C_6$. K_2 has an unramified V_4-extension L which is an $S_4 \times C_6$-extension of K. If $K_{ur} \neq L$, $[K_{ur}:L]=2,4$, or 8. (Supplemental data in the previous paper [15] are wrong.)
As in the previous paper, we use KANT (KASH) for class number calculations with the aid of pari-gp.

unramified nonsolvable Galois extension. Since we have $[K_{ur} : K_1] < 168 = |\mathrm{PSL}(2,7)|$ for these 12 fields by (conditional) discriminant bounds [10], this is to show that K_1 does not have an unramified A_5-extension (which is normal over $\mathbf{Q}$). By using a general fact on the structure of group extensions of A_5 and S_5 by finite abelian groups, we reduce it to the nonexistence of some quintic number fields and available data suffice for this. This idea also enables us to check that the results in [15] are unconditional for $|d| \leqq 463$ except for $d = -427$, where d is the discriminant of K. On the other hand, we show $K_{ur} = K_2$ for $d = -952, -987$, $K_{ur} = K_3$ for $d = -731, -771, -916, -984$, and $K_{ur} = K_4$ for $d = -856$. Among updated results, the following are especially remarkable. The field $\mathbf{Q}(\sqrt{-856})$ is the first imaginary quadratic number field whose class field tower has length (at least) four. Even though we assume GRH, this is the first example of a number field whose class field tower has length four. The field $\mathbf{Q}(\sqrt{-984})$ is a field of a new type: As we remarked in [15], for many $K = \mathbf{Q}(\sqrt{d})$ with $|d| \leqq 1000$, K_{ur} coincides with $(K_g)_1$, the Hilbert class field of the genus field K_g of K, and for each field K for which we verified $K_{ur} \supsetneqq (K_g)_1$ in [15], there exists an S_4-extension M of $\mathbf{Q}$ such that the compositum KM is an unramified extension of K not contained in $(K_g)_1$. For $K = \mathbf{Q}(\sqrt{-984})$, no such S_4-extension of $\mathbf{Q}$ exists, however, there exists a dihedral octic CM-field F such that the compositum KF_1 is an unramified extension of K not contained in $(K_g)_1$. From this fact we can easily deduce a new way to construct imaginary quadratic number fields whose class field towers have length at least three. From each real quadratic number field satisfying some conditions we obtain a family of infinite such imaginary quadratic number fields. We also note that the degree of $\mathbf{Q}(\sqrt{-984})_{ur}$ is 864. Thus, we also update the largest known degree of number fields with class number one (under GRH).

Thus, we obtain Table 1 for 23 exceptional fields. (Notations are as in [15]. Note that $\widetilde{S_4}$ is the double cover of S_4 with $\widetilde{S_4} \cong \mathrm{GL}(2,3)$. This is used in §7 in [15].)

2. Group extensions of groups with trivial center

As described in the Introduction, we use a fact on the structure of group extensions of A_5 and S_5 by finite abelian groups to reduce the nonexistence of unramified A_5-extension of K_1 to that of unramified A_5 and S_5-extensions of its subfields. For this, we consider group extensions of general groups with trivial center. First, we quote some basic result, and then apply it to the groups S_n, A_n, $\mathrm{PGL}(2,p)$ and $\mathrm{PSL}(2,p)$.

Let H and F be groups and G a group extension of H by F:

$$1 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 1 \quad \text{(exact)}.$$

Then as is well known, F acts on H by conjugation and this action induces a group homomorphism $\psi_G : F \to \operatorname{Out} H$, which depends only on G.

Lemma 1 ([12], (7.11)). *Let the situation be as above. Suppose that H has trivial center ($Z(H) = \{1\}$). Then the structure of G is uniquely determined by the homomorphism ψ_G. For any group homomorphism ψ from F into $\operatorname{Out} H$, there exists an extension G of H by F such that $\psi_G = \psi$. Moreover, the isomorphism class of G is uniquely determined by ψ. (In particular, the class of $F \times H$ is determined by ψ with $\psi(F) = 1$.) All the extensions are realized as a subgroup U of the direct product $F \times \operatorname{Aut} H$ satisfying the two conditions $U \cap \operatorname{Aut} H = \operatorname{Inn} H$ and $\pi(U) = F$, where π is the projection from $F \times \operatorname{Aut} H$ to F.*

From this, we immediately obtain the following.

Proposition 1. *Let H be a group with trivial center.*
(i) *If H has trivial outer automorphism group ($\operatorname{Out} H = \{1\}$), then for any group F, any extension of H by F is (isomorphic to) the direct product $F \times H$.*
(ii) *If $\operatorname{Out} H \cong C_2$, then for any group F without quotient group of order two, any extension of H by F is (isomorphic to) the direct product $F \times H$. In particular, for any finite group F with odd order, any group extension of H by F is $F \times H$.*

As is well known, the symmetric group S_n of degree $n \geqq 4$ has trivial center and its outer automorphism group is trivial if $n \neq 6$ [12, (2.18)]. The alternating group A_n of degree $n \geqq 4$ has trivial center and $\operatorname{Out} A_n \cong C_2$ if $n \neq 6$ [12, (2.17)]. Therefore, if we apply Proposition 1 to these groups, then we obtain the following.

Proposition 2. *Let n be a natural number with $n \geqq 4$ and $n \neq 6$.*
(i) *For any group F, any extension of S_n by F is $F \times S_n$.*
(ii) *For any group F without quotient group of order two, any extension of A_n by F is $F \times A_n$. In particular, for any finite group F with odd order, any extension of A_n by F is $F \times A_n$. Moreover, an extension of A_n by C_2 is isomorphic to $C_2 \times A_n$ or S_n. Furthermore, an extension of A_n by C_{2^m}, $m \geqq 2$ is isomorphic to $C_{2^m} \times A_n$ or $C_{2^m} \curlywedge S_n$, where $C_{2^m} \curlywedge S_n$ is the pull back of the epimorphisms $C_{2^m} \to C_2$ and $S_n \to C_2$.*

By using this proposition (repeatedly), we can reduce the nonexistence of an unramified A_5-extension of K_1 which is normal over $\mathbf{Q}$ to that of unramified A_5 and S_5-extensions of K, and then this is reduced to that of some quintic number fields.

Let p be an prime number $\geqq 5$. Then $Z(\mathrm{PGL}(2,p)) = Z(\mathrm{PSL}(2,p)) = \{1\}$ and $\operatorname{Out}\mathrm{PGL}(2,p) = \{1\}$, $\operatorname{Out}\mathrm{PSL}(2,p) \cong C_2$. Therefore we have the following similar assertions.

Proposition 3. *Let p be an prime number $\geqq 5$.*
(i) *For any group F, any extension of* $\mathrm{PGL}(2,p)$ *by F is $F \times \mathrm{PGL}(2,p)$.*
(ii) *For any group F without quotient group of order two, any extension of* $\mathrm{PSL}(2,p)$ *by F is $F \times \mathrm{PSL}(2,p)$. In particular, for any finite group F with odd order, any extension of* $\mathrm{PSL}(2,p)$ *by F is $F \times \mathrm{PSL}(2,p)$. Moreover, an extension of* $\mathrm{PSL}(2,p)$ *by C_2 is isomorphic to $C_2 \times \mathrm{PSL}(2,p)$ or* $\mathrm{PGL}(2,p)$. *Furthermore, a group extension of* $\mathrm{PSL}(2,p)$ *by C_{2^m}, $m \geqq 2$ is isomorphic to $C_{2^m} \times \mathrm{PSL}(2,p)$ or $C_{2^m} \curlywedge \mathrm{PGL}(2,p)$.*

This will be useful for the study of unramified $\mathrm{PSL}(2,7)$-extensions in future (see the next section).

3. Determination

Fields with $l = 1$. We first treat the fields with $-d = 723, 763, 772, 787, 808, 843, 904, 932, 939, 964, 971, 979$. For these K, we have $[K_{ur} : K_1] < 168$ by discriminant bounds. Thus, our task is to show that K_1 does not have an unramified A_5-extension which is normal over $\mathbf{Q}$. For this we use the following proved in [15].

Proposition 4 ([15], Proposition 8). *The field $\mathbf{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified A_5-extension which is normal over $\mathbf{Q}$ in the sense that none of $\mathbf{Q}(\sqrt{d})$ of discriminant d with $0 > d > -1507$ has such an extension.*

By Propositions 2 and 4, we easily see $K_{ur} = K_1$ for K with odd class number. In fact, suppose K_1 has an unramified A_5-extension L. Then L is normal over $\mathbf{Q}$ and $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(K_1/K) \times A_5$ by Proposition 2. Therefore K has an unramified A_5-extension, which is normal over $\mathbf{Q}$. This contradicts Proposition 4.

For the fields with even class number, we use also data for quintic number fields of type S_5 (fields whose normal closure have Galois group isomorphic to S_5). We show $K_{ur} = K_1$ only for $d = -964 = (-4) \cdot 241$. (The other fields are treated similarly.) For this K, $\mathrm{Cl}(K) \cong C_{12}$. Suppose K_1 has an unramified A_5-extension L which is normal over $\mathbf{Q}$. Then by Proposition 2, we have $\mathrm{Gal}(L/K) \cong C_{12} \curlywedge S_5 \cong C_3 \times (C_4 \curlywedge S_5)$. Therefore, the genus field K_g of K, which is the unique unramified quadratic extension of K, has an unramified A_5-extension M, and this is also normal over $\mathbf{Q}$. Then by Propositions 2 and 4, we have $\mathrm{Gal}(M/K) \cong S_5$ and therefore by Proposition 2, $\mathrm{Gal}(M/\mathbf{Q}) \cong S_5 \times C_2$. Consequently by Proposition 4, M is the compositum of K and an S_5-extension N of $\mathbf{Q}$. For the unramifiedness of M/K, a quintic subfield of N must have discriminant -4, 241, $(-4) \cdot 241^2 = -232324$, or $(-4)^2 \cdot 241 = 3856$. Such a quintic number field does not exist. This is a contradiction. Thus, K_1 does not have an unramified A_5-extension which is normal over $\mathbf{Q}$ and $K_{ur} = K_1$.

The same argument works also for other fields K and even if we cannot obtain $[K_{ur} : K_1] < 168$ but $[K_{ur} : K_1] < 3600$, we may conclude that K_1 does not have an unramified A_5-extension which is normal over $\mathbf{Q}$. By applying it to $-d = 424, 436, 443, 451, 456$, we can make the result in [15] unconditional for $|d| \leqq 463$ except for $d = -427$. Similarly we can show that for $d = -883, -907, -947$, K_1 does not have an unramified A_5-extension which is normal over $\mathbf{Q}$. However, for these fields, we obtain only $[K_{ur} : K_1] < 2 \cdot 168 < 360 = |A_6|$ by (conditional) discriminant bounds. Therefore, in order to conclude $K_{ur} = K_1$, we must show that K_1 does not have an unramified $\mathrm{PSL}(2,7)$-extension (which is normal over $\mathbf{Q}$). At present, we do not have sufficient data for number fields for this. Probably, in future we will be able to show $K_{ur} = K_1$ by Proposition 3 and data for septic and octic number fields. Still we do not have the result for unramified $\mathrm{PSL}(2,7)$-extension corresponding to Proposition 4. Recently we found the following example.

Example 1. The imaginary quadratic number field $\mathbf{Q}(\sqrt{-3983})$ have an unramified $\mathrm{PSL}(2,7)$-extension. Such an extension of $K = \mathbf{Q}(\sqrt{-3983})$ is given by the composite field of K with the splitting field M of the septic polynomial $X^7 - X^6 - 3X^5 - X^4 + 2X^3 + 4X^2 + 4X + 1$, which is a $\mathrm{PSL}(2,7)$-extension of $\mathbf{Q}$. We verify the unramifiedness of KM/K as follows. Let E be a septic field defined by the above polynomial. Then the discriminant of E is $3983^2 = 7^2 \cdot 569^2$ and the factorizations of the rational primes 7 and 569 in E are $7 = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^2$ and $569 = \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3^2\mathfrak{q}_4$, where $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$ are primes of degree one and $\mathfrak{p}_4, \mathfrak{q}_4$ are of degree two. These are checked by using pari-gp version 1.39 (the functions 'galois', 'discf' and 'primedec'). Therefore KE/K is unramified and so is KM/K. In view of the size of the discriminant, the author expects that $\mathbf{Q}(\sqrt{-3983})$ is the first imaginary quadratic number field having an unramified $\mathrm{PSL}(2,7)$-extension.

For search for $\mathrm{PSL}(2,7)$-extensions of $\mathbf{Q}$ yielding unramified $\mathrm{PSL}(2,7)$-extensions of quadratic number fields, we used the septic polynomial with parameters a and A whose Galois group over $\mathbf{Q}(a, A)$ is $\mathrm{PSL}(2,7)$ constructed by LaMacchia [4]. We found many other such extensions. The field $\mathbf{Q}(\sqrt{-3983})$ has minimal conductor among imaginary quadratic number fields having an unramified $\mathrm{PSL}(2,7)$-extension we found.

Now we treat the other fields, for which $l \geqq 2$, except for $d = -984$ and $d = -996$, that is, we treat the fields with $-d = 731, 771, 856, 916, 952, 987$. ($\mathbf{Q}(\sqrt{-984})$ is treated in Example 3 in the next section and we have not succeeded in determining the degree $[K_{ur} : K]$ for $K = \mathbf{Q}(\sqrt{-996})$.) For these fields most techniques are as in [15]. We apply computer calculation of class numbers to more octic number fields than in the previous paper. We use class number relations and results for class number divisibility to obtain class numbers of fields of higher degrees. For simplicity, we often omit to describe calculations of class numbers of fields of low degrees below. As in the previous paper, we denote by $B(n)$ the lower bounds for root discriminants of totally imaginary number fields of degrees $\geqq n$ and use Odlyzko's (conditional, i.e., GRH) bounds for $B(n)$ [10].

$d = -731 = 17 \cdot (-43)$. We have $\mathrm{Cl}(K) \cong C_{12}$. Let E be a quartic number field defined by $X^4 - X^3 + 2X^2 - 1$. Then the discriminant d_E equals d and therefore by [15, Proposition 6] the normal closure M of E is an unramified A_4-extension of K. The compositum K_1M is an unramified $A_4 \times C_4$-extension of K and of degree 96. We will show that K_{ur} is its quadratic extension.

First, we show $[K_{ur} : K_1M] = 2$. For this we consider an octic subfield KE of M. Put $N = KE$. By computer calculation we have $\mathrm{Cl}(N) \cong C_{24}$. We easily see that the Hilbert class field N_1 of N is an unramified quadratic extension of K_1M. Since $rd_K = \sqrt{731} = 27.0370\cdots < B(96 \cdot 2 \cdot 3)$ ([10]), we have $[K_{ur} : N_1] \leqq 2$. By [15, Lemma 9] the Hilbert 2-class field $N_1^{(2)}$ of N has odd class number. Since $N_1/N_1^{(2)}$ is cyclic cubic, by [15, Lemma 5] we have $2 \nmid h(N_1)$. Hence $K_{ur} = N_1$.

Next, we show $K_{ur} = K_3$, that is, K_{ur}/K_1 is nonabelian. Since $[K_{ur} : K_1] = 8$, this is equal to showing $\mathrm{Cl}(K_1) \cong V_4$ by [15, Lemma 9]. For this we calculate the class number of $K_1^{(2)}$. By calculations of class numbers of subfields of $K_1^{(2)}$ and by using [15, Lemma 4] we have $h(K_1^{(2)}) = 3$. Since $K_1/K_1^{(2)}$ is cyclic cubic, by [15, Lemma 5] we have $\mathrm{Cl}(K_1) \cong V_4$. Therefore $K_2 = K_1M$ and $K_{ur} = N_1 = K_3$.

Now we determine the structure of $G = \mathrm{Gal}(K_{ur}/K)$. Since $\mathrm{Gal}(K_1M/K_1^{(2)}) \cong \mathrm{Gal}(M/K) \cong A_4$ and K_{ur} is also the third Hilbert class field of $K_1^{(2)}$, we have $\mathrm{Gal}(K_{ur}/K_1^{(2)}) \cong \widetilde{A_4}$. Therefore $G \cong \widetilde{A_4} \rtimes C_4$. This is not the direct product: Since $M = N_1^{(3)}$, we have $\mathrm{Cl}(M) \cong C_8$. Obviously the unique unramified quadratic extension of M is K_gM and $\mathrm{Gal}(K_gM/K) \cong A_4 \times C_2$. Hence K does not have an unramified $\widetilde{A_4}$-extension. More precisely, $\mathrm{Gal}(K_{ur}/K) \cong \widetilde{A_4} \curlyvee C_8$ (central product). In fact, $\mathrm{Gal}(K_{ur}/K_1^{(2)}) \cong \widetilde{A_4} \cong \mathrm{SL}(2,3)$ and its center is $\mathrm{Gal}(K_{ur}/K_1M)$. Moreover, $\mathrm{Gal}(K_{ur}/M) \cong C_8$ and $\mathrm{Gal}(K_{ur}/K_1^{(2)}) \cap \mathrm{Gal}(K_{ur}/M) = \mathrm{Gal}(K_{ur}/K_1M)$. (Note that $K_1^{(2)}M = K_1M$).

$d = -771 = (-3) \cdot 257$. We have $\mathrm{Cl}(K) \cong C_6$. Let E be a quartic number field defined by $X^4 + X^2 - X + 1$ and $F = \mathbf{Q}(\sqrt{257})$. Then $d_E = 257$ and therefore by [15, Proposition 6] the normal closure M of E is an A_4-extension of F unramified at all finite primes. Therefore the compositum K_1M is an unramified $S_4 \times C_3$-extension of K and of degree 144. We will show $K_{ur} = K_1M$. Put $L = K_1M$. Since $rd_K = \sqrt{771} = 27.7668\cdots <$

$B(144 \cdot 5)$ ([10]), we have $[K_{ur} : L] \leqq 4$. Therefore K_{ur} is the Hilbert class field of L and our task is to show $2, 3 \nmid h(L)$.

First, we show $h(L) \neq 3$. For this we first calculate $h(K_1)$. We have $K_1 = K(\sqrt{-3}, \alpha_{95})$ and $K_1/\mathbf{Q}(\sqrt{-3})$ is a D_3-extension. By [15, Lemma 3] we obtain $h(K_1) = 3$ from the class numbers of the intermediate fields of $K_1/\mathbf{Q}(\sqrt{-3})$. Therefore $K_2 = K_1F_1$ (note that $h(F) = 3$) and $3 \nmid h(K_2)$ by [15, Lemma 9]. Assume $h(L) = 3$. Then the action of $\mathrm{Gal}(L/K_1) \cong A_4$ on $\mathrm{Cl}(L) \cong C_3$ induces a group homomorphism $A_4 \to \mathrm{Aut}\,(C_3) \cong C_2$. This is trivial. Since $\mathrm{Gal}(L/K_2) \cong V_4$, the same argument as in the proof of [15, Proposition 2] shows $3 \mid h(K_2)$. This is a contradiction. Hence $h(L) \neq 3$.

Next, we show that the 2-class group $\mathrm{Cl}^{(2)}(KM)$ of KM is trivial. This implies $h(KM) = 3$, which implies $\mathrm{Cl}(L)$ is trivial or $\mathrm{Cl}(L) \cong V_4$ by [15, Lemma 5]. First, in order to show that $\mathrm{Cl}^{(2)}(KM)$ is cyclic, we calculate $h(KF_1)$. Since KF_1/K is a D_3-extension, we obtain $h(KF_1) = 12$ by using [15, Lemma 3]. Since KM/KF_1 is an unramified V_4-extension, consequently we have $\mathrm{Cl}(KF_1) \cong V_4 \times C_3$. Therefore KM is the Hilbert 2-class field of KF_1. Hence we conclude that $\mathrm{Cl}^{(2)}(KM)$ is cyclic by [15, Lemma 9]. Next, to show that the 2-rank of $\mathrm{Cl}^{(2)}(KM)$ is even, we calculate $h(KE)$. By computer calculation we have $h(KE) = 6$. Obviously the Hilbert 2-class field of KE is K_gE. Then by [15, Lemma 9] K_gE has odd class number. Since KM/K_gE is cyclic cubic, the 2-rank of $\mathrm{Cl}^{(2)}(KM)$ is even by [15, Lemma 5] Hence $\mathrm{Cl}^{(2)}(KM)$ is trivial.

Assume $\mathrm{Cl}(L) \cong V_4$. Then L is the Hilbert class field of KM and K_{ur} is the Hilbert class field of L. Therefore $\mathrm{Gal}(K_{ur}/KM) \cong A_4$. Since $\mathrm{Gal}(L/M) \cong \mathrm{Gal}(K_1^{(3)}/\mathbf{Q}) \cong D_3$, we have $\mathrm{Gal}(K_{ur}/M) \cong S_4$. Hence by Proposition 2 we have $\mathrm{Gal}(K_{ur}/\mathbf{Q}) \cong S_4 \times S_4$. This implies that there exists an S_4-extension N of $\mathbf{Q}$ such that $NM = L$ and $N \cap M = \mathbf{Q}$. We can easily check that such an S_4-extension of $\mathbf{Q}$ does not exist by using available data for quartic number fields: Assume that such an S_4-extension of $\mathbf{Q}$ exists. If N contains K, then N is an unramified A_4-extension of K, and therefore N must be an unramified V_4-extension of $K_1^{(3)}$. However, by computer calculation we have $h(K_1^{(3)}) = 2$. This is a contradiction. Hence the unique quadratic subfield of N is $\mathbf{Q}(\sqrt{-3})$. Therefore the discriminant of a quartic subfield N must be $(-3) \cdot 257^2$. Such a quartic number fields does not exist. Hence $\mathrm{Cl}(L)$ is trivial and $K_{ur} = L$.

$d = -856 = 8 \cdot (-107)$. We have $\mathrm{Cl}(K) \cong C_6$. Let E be a quartic number field defined by $X^4 - 2X^3 + 5X^2 - 2X - 1$ and M its normal closure. As explained in the previous paper [15, §7], KM/K is an unramified S_4-extension. The compositum K_1M is an unramified $S_4 \times C_3$-extension of K and of degree 144. We will show that K_{ur} is its quadratic extension.

First, we show $2 \mid h(K_1M)$. For this we consider an octic subfield KE of KM. Put $N = KE$. By computer calculation we have $\mathrm{Cl}(N) \cong C_{12}$. Since KM/N is an unramified S_3-extension, $N_1^{(2)}M$ is an unramified quadratic extension of KM. Therefore K_1N_1 is an unramified quadratic extension of K_1M. Hence $2 \mid h(K_1M)$.

Next, we show $K_{ur} = K_1N_1$. Put $L = K_1N_1$. Since $rd_K = \sqrt{856} = 29.2574\cdots < B(144 \cdot 2 \cdot 4)$ ([10]), we have $[K_{ur} : L] \leqq 3$. Since $L/N_1^{(2)}$ is a C_3^2-extension and $2 \nmid h(N_1^{(2)})$ by [15, Lemma 9], we have $2 \nmid h(L)$ by [15, Lemma 5]. Before showing $3 \nmid h(L)$, we show $L = K_4$. By computer calculation we have $h(K_1) = 3$. Therefore the unique cyclic cubic subextension of K_1M/K_1 is K_2 and $3 \nmid h(K_2)$ by [15, Lemma 9]. Next to see $K_1M = K_3$, we show $3 \nmid h(K_1M)$. Assume $3 \mid h(K_1M)$. Then the action of $\mathrm{Gal}(K_1M/K_1) \cong A_4$ on $\mathrm{Cl}^{(3)}(K_1M) \cong C_3$ induces a group homomorphism $A_4 \to \mathrm{Aut}\,(C_3) \cong C_2$. This is trivial. Since $\mathrm{Gal}(K_1M/K_2) \cong V_4$, the same argument as in the proof of [15, Proposition 2] shows $3 \mid h(K_2)$. This is a contradiction. Thus, $3 \nmid h(K_1M)$. Since $h(K_1) = 3$ and $2 \nmid h(L)$, we conclude $\mathrm{Cl}(K_2) \cong V_4$ by [15, Lemma 5], and therefore $K_1M = K_3$, $h(K_3) = 2$, and $L = K_4$. Then we have $\mathrm{Gal}(K_4/K_1) \cong \widetilde{A_4} \cong \mathrm{SL}(2,3)$. Now we show $3 \nmid h(K_4)$. Assume $3 \mid h(K_4)$. Then the action of $\mathrm{Gal}(K_4/K_1)$ on $\mathrm{Cl}(K_4) \cong C_3$ induces a group homomorphism $\mathrm{SL}(2,3) \to \mathrm{Aut}\,(C_3) \cong C_2$. This is trivial. Since $\mathrm{Gal}(K_4/K_2) \cong Q_8$, the same argument as in the proof of [15, Proposition 2] shows $3 \mid h(K_2)$. This is a contradiction. Hence $K_{ur} = L = K_4$.

Finally, we determine $G = \mathrm{Gal}(K_{ur}/K)$. Put $T = N_1^{(2)}M$. We see that T is an unramified quadratic extension of KM. Since $K_{ur} = K_1^{(3)}T$ is a cyclic sextic extension of KM, we have $\mathrm{Cl}(KM) \cong C_6$ and T is the Hilbert 2-class field of KM. Therefore T is normal over K. Since, as is easily seen, $K_{ur} = K_1^{(3)}T$ and $K_1^{(3)} \cap T = K$, we have $G \cong \mathrm{Gal}(K_{ur}/K_1^{(3)}) \times \mathrm{Gal}(K_{ur}/T)$. Since $\mathrm{Gal}(K_{ur}/T) \cong \mathrm{Gal}(K_3/K_1^{(3)}) \cong C_3$, it remains to determine the structure of $H = \mathrm{Gal}(K_{ur}/K_1^{(3)})$. Since $\mathrm{Gal}(K_3/K_1^{(3)}) \cong \mathrm{Gal}(M/\mathbf{Q}) \cong S_4$, H is a central extension of S_4:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(K_{ur}/K_3) & \longrightarrow & H & \longrightarrow & \mathrm{Gal}(K_3/K_1^{(3)}) & \longrightarrow & 1 \\
 & & \| & & & & \| & & \\
 & & C_2 & & & & S_4 & &
\end{array}$$

We can determine the structure of H from the one of its Sylow 2-subgroups. For a central extension of S_4 by C_2 is isomorphic to $S_4 \times C_2$, $\mathrm{GL}(2,3)$,

$S_4 \curlywedge C_4$, or the group $\widehat{S_4}$ given by

$$\begin{aligned}\langle\, a,b,c,d \mid\, & a^2 = d^2, \\ & a^{-1}ba = b^2,\ b^3 = 1, \\ & a^{-1}ca = d^3,\ b^{-1}cb = d,\ c^2 = d^2, \\ & a^{-1}da = cd^2,\ b^{-1}db = cd,\ c^{-1}dc = d^3,\ d^4 = 1\,\rangle,\end{aligned}$$

[11] and the Sylow 2-subgroups of each group are isomorphic to $D_4 \times C_2$, SD_{16}, $D_4 \curlywedge C_4$, and Q_{16}, respectively. Since $\mathrm{Gal}(K_4/K_2) \cong Q_8$ and $\mathrm{Gal}(K_3/K_1^{(3)}) \cong S_4$, any Sylow 2-subgroup of H has a maximal subgroup isomorphic to Q_8 and a maximal quotient subgroup isomorphic to D_4, and therefore is isomorphic to SD_{16}. Hence we conclude $H \cong \mathrm{GL}(2,3)$. Thus, $G \cong \widetilde{S_4} \times C_3$.

$d = -916 = (-4)\cdot 229$. We have $\mathrm{Cl}(K) \cong C_{10}$. Let E be a quartic number field defined by $X^4 - X + 1$ and $F = \mathbf{Q}(\sqrt{229})$. Then $d_E = 229$ and therefore by [15, Proposition 6] the normal closure M of E is an A_4-extension of F unramified at all finite primes. Therefore the compositum K_1M is an unramified $S_4 \times C_5$-extension of K and of degree 240. We will show $K_{ur} = K_1M$. Put $L = K_1M$. Since $rd_K = \sqrt{916} = 30.2654\cdots < B(240\cdot 7)$ ([10]), we have $[K_{ur} : L] \leqq 6$. Hence our task is to show $2, 3, 5 \nmid h(L)$.

First we show $p \nmid h(L)$ for $p = 3$ and 5. By computer calculation we have $h(K_1) = 3$. Therefore $K_1F_1 = K_2$ (note that $h(F) = 3$) and $3 \nmid h(K_2)$ by [15, Lemma 9]. We have also $5 \nmid h(K_2)$, because by [15, Lemma 5] the 5-rank of $\mathrm{Cl}(K_2)$ is even. Assume $p \mid h(L)$. Then the action of $\mathrm{Gal}(L/K_1) \cong A_4$ on $\mathrm{Cl}^{(p)}(L)$ induces a group homomorphism $A_4 \to \mathrm{Aut}\,(C_p) \cong C_{p-1}$. This is trivial. Since $\mathrm{Gal}(L/K_2) \cong V_4$, the same argument as in the proof of [15, Proposition 2] shows $p \mid h(K_2)$. This is a contradiction. Hence $p \nmid h(L)$.

It remains to show $2 \nmid h(L)$. First we show $\mathrm{Cl}(L)$ is cyclic and isomorphic to $\mathrm{Cl}^{(2)}(KM)$. Since $h(K_1) = 3$, by [15, lemma 5] $\mathrm{Cl}(K_2) \cong V_4$, C_2^4, or C_4^2. Assume $\mathrm{Cl}(K_2) \ncong V_4$. Then $\mathrm{Cl}(L) \cong V_4$. Then by considering the action of $\mathrm{Gal}(L/KM) \cong C_5$ on $\mathrm{Cl}(L)$, we have $\mathrm{Cl}^{(2)}(KM) \cong V_4$. On the other hand, by computer calculation $\mathrm{Cl}(KF_1) \cong V_4 \times C_5$. Therefore by [15, Lemma 9] the 2-class group of KM is cyclic. This is a contradiction. Hence $\mathrm{Cl}(K_2) \cong V_4$ and therefore again by [15, Lemma 9] $\mathrm{Cl}(L)$ is cyclic and isomorphic to $\mathrm{Cl}^{(2)}(KM)$. Next to conclude $\mathrm{Cl}^{(2)}(KM)$ is trivial, we calculate $h(K_gE)$. K_gE/E is a V_4-extension and its intermediate fields are EF, $E(\sqrt{-1})$, and KE. By computer calculation $h(E) = 1$, $h(EF) = 3$, $h(E(\sqrt{-1})) = 1$, and $h(KE) = 10$. Since K_gE/KE is unramified, K_gE is the Hilbert 2-class field of KE and therefore $2 \nmid h(K_gE)$ by [15, Lemma 9]. Hence by class number relation for V_4-extensions we have $h(K_gE) = 15$. Since KM/K_gE is an unramified cyclic cubic extension, KM is the Hilbert

3-class field of K_gE. Therefore by [15, Lemma 5] the 2-rank of $\mathrm{Cl}(KM)$ is even. Hence $\mathrm{Cl}^{(2)}(KM)$ is trivial. Thus, $h(L)=1$ and $L=K_{ur}$.

$d=-952=8\cdot(-7)\cdot 17$. We have $\mathrm{Cl}(K)\cong C_4\times C_2$, $K_g=\mathbf{Q}(\sqrt{2},\sqrt{-7},\sqrt{17})$, and $\mathrm{Cl}(K_g)\cong C_{20}\times C_2$. Put $L=(K_g)_1$. We will show $K_{ur}=L$.

Since $rd_K=\sqrt{952}=30.8544\ldots$ and $rd_K<B(2\cdot 4\cdot 40\cdot 6)$ ([10]), we have $[K_{ur}:L]\leqq 5$. By the result in [1], $\mathrm{Cl}^{(2)}(K_1)$ is cyclic (and therefore the 2-class field tower of K terminates with $K_2^{(2)}$), and $\mathrm{Gal}(K_2^{(2)}/K)\cong\Gamma_{2,2}^2\cong 32\Gamma_3 e$. (This implies $K_2^{(2)}=(K_g)_1^{(2)}$, because $[K_2^{(2)}:K]=32=|\mathrm{Cl}^{(2)}(K_g)|[K_g:K]$.) Therefore, $2\nmid h(K_2^{(2)})$ and since $L/K_2^{(2)}$ is cyclic quintic, $2\nmid h(L)$ by [15, Lemma 5]. Since $3\nmid h(L)$ by [15, Proposition 2], it remains to show $h(L)\neq 5$.

Assume $h(L)=5$. Since by the result in [1] $\mathrm{Gal}(K_2^{(2)}/K(\sqrt{2}))\cong D_4\curlyvee C_4$, and $\mathrm{Cl}(K(\sqrt{2}))\cong V_4\times C_{10}$, we have $\mathrm{Gal}(L/K(\sqrt{2})_1^{(5)})\cong D_4\curlyvee C_4$. We put this Galois group H. The action of H on $\mathrm{Cl}(L)$ induces a group homomorphism $\rho: H\to\mathrm{Aut}(\mathrm{Cl}(L))\cong\mathrm{Aut}(C_5)\cong C_4$. Since $(D_4\curlyvee C_4)_{ab}\cong C_2^3$, we have $|\mathrm{Im}(\rho)|\leqq 2$ and if we let F be the field corresponding to $\mathrm{Ker}(\rho)$, then $5\mid h(F)$. Since $5\nmid h(K(\sqrt{2})_1^{(5)})$ by [15, Lemma 9], $|\mathrm{Im}(\rho)|=2$ and F is an unramified quadratic extension of $K(\sqrt{2})_1^{(5)}$. Hence F is obtained by adjoining γ_4 to an unramified quadratic extension of $K(\sqrt{2})$. There exist exactly seven unramified quadratic extensions of $K(\sqrt{2})$: Three are contained in K_1 and the other four are not. Three are $K_g=K(\sqrt{2},\sqrt{-7})$, $K(\sqrt{-9+10\sqrt{2}})$, and $K(\sqrt{(-7)(-9+10\sqrt{2})})$. Since $h(\mathbf{Q}(\sqrt{-9+10\sqrt{2}}))=1$ and $h(\mathbf{Q}(\sqrt{(-9+\sqrt{119})/2}))=10$, we have $5\parallel h(K(\sqrt{-9+10\sqrt{2}}))$ by [15, Lemma 4]. Since $h(\mathbf{Q}(\sqrt{(-7)(-9+10\sqrt{2})}))=1$ and $h(\mathbf{Q}(\sqrt{(-7)(-9+\sqrt{119})/2}))=10$, we obtain from [15, Lemma 4] that $5\parallel h(K(\sqrt{(-7)(-9+10\sqrt{2})}))$. Hence for the three unramified quadratic extensions of $K(\sqrt{2})$ contained in K_1, the class numbers of the fields obtained by adjoining γ_4 are not divisible by five. Since the Hilbert class field of the field $\mathbf{Q}(\sqrt{-14})$ is $\mathbf{Q}(\sqrt{-1-2\sqrt{2}},\sqrt{-7})$, $K(\sqrt{-1-2\sqrt{2}})$ is an unramified quadratic extension of $K(\sqrt{2})$ not contained in K_1. Since $-9+10\sqrt{2}=(-7+4\sqrt{2})(-1-2\sqrt{2})$, the other three are $K(\sqrt{7(1+2\sqrt{2})})$, $K(\sqrt{-7+4\sqrt{2}})$, and $K(\sqrt{(-7)(-7+4\sqrt{2})})$. We see that $K(\sqrt{-1-2\sqrt{2}})$ and $K(\sqrt{7(1+2\sqrt{2})})$, and $K(\sqrt{-7+4\sqrt{2}})$, and $K(\sqrt{(-7)(-7+4\sqrt{2})})$ are pairwise conjugate and their class group are isomorphic to $C_{20}\times C_2$ and C_{10}, respectively. Hence for any unramified quadratic extension E of $K(\sqrt{2})$, $5\nmid h(E(\gamma_4))$. This is a contradiction. Hence $h(L)\neq 5$ and therefore $h(L)=1$. Thus, we have $K_{ur}=L=(K_g)_1=K_2$ and $G\cong\Gamma_{2,2}^2\ltimes C_5\cong(C_4\wr C_2)\curlywedge D_5$.

$d = -987 = (-3)\cdot(-7)\cdot(-47)$. We have $\mathrm{Cl}(K) \cong C_4 \times C_2$, $K_g = \mathbf{Q}(\sqrt{-3}, \sqrt{-7}, \sqrt{-47})$, and $h(K_g) = 10$. Since $rd_K = \sqrt{987} = 31.4165\ldots$ and $rd_K < B(2\cdot 4\cdot 10\cdot 30)$ ([10]), $[K_{ur} : (K_g)_1] \leqq 29$. We will show $K_{ur} = (K_g)_1$.

First we show that $(K_g)_1 = K_2$. We have $K_1 = K_g(\sqrt{-1+2\sqrt{-47}})$. Then $K_1/K(\sqrt{-47})$ is a V_4-extension, and its intermediate fields are $K(\sqrt{-1+2\sqrt{-47}})$, K_g and $K(\sqrt{(-3)(-1+2\sqrt{-47})})$. By computer calculation, all these fields have class number ten and therefore $h(K_1) = 5$ by [15, Lemmas 2 and 9]. Hence $K_2 = (K_g)_1$.

Now we show $h(K_2) = 1$ and conclude $K_{ur} = K_2$. Since $h(K_1) = 5$ and $K_2 = (K_g)_1$, $5, 11 \nmid h(K_2)$ by [15, Lemma 9 and Proposition 2]. Therefore by [15, Lemma 5] $\mathrm{Cl}(K_2)$ is trivial, or $\mathrm{Cl}(K_2) \cong C_2^4$. For simplicity, we put $F = \mathbf{Q}(\sqrt{-47})$. Then $h(F) = 5$, $h(F_1) = 1$, and $K_2 = K_1F_1$. To eliminate $\mathrm{Cl}(K_2) \cong C_2^4$, we show $\mathrm{Cl}^{(2)}(KF_1) \cong V_4$. If so, K_2 is the Hilbert 2-class field of KF_1 and therefore $\mathrm{Cl}^{(2)}(K_2)$ is cyclic by [15, Lemma 9]. Since $KF_1/KF = K(\sqrt{-47})$ is cyclic quintic, we can conclude the 2-rank of $\mathrm{Cl}^{(2)}(KF_1)$ is 2 or 6 by a generalization of [15, Lemma 5]:

Lemma 2. *Let L/K be a finite cyclic extension of degree n of algebraic number fields. Let p be a prime number with $p \nmid n$ and assume $\mathrm{Cl}^{(p)}(E) \cong \mathrm{Cl}^{(p)}(K)$ for any intermediate field E of L/K. Then for each positive integer a, $r_{p^a}(\mathrm{Cl}^{(p)}(L)) - r_{p^a}(\mathrm{Cl}^{(p)}(K)) \equiv 0 \pmod{f}$, where r_{p^a} denotes the p^a-rank and f is the order of p modulo n.*

In the situation of the lemma, the natural map $\mathrm{Cl}^{(p)}(K) \to \mathrm{Cl}^{(p)}(L)$ is injective, the norm map $N_{L/K} : \mathrm{Cl}^{(p)}(L) \to \mathrm{Cl}^{(p)}(K)$ is surjective, and its restriction on (the image of) $\mathrm{Cl}^{(p)}(K)$ is a bijection. Therefore we obtain the direct decomposition $\mathrm{Cl}^{(p)}(L) \cong \mathrm{Ker}(N_{L/K}) \oplus \mathrm{Cl}^{(p)}(K)$. Thus, by considering the p^a-rank of $\mathrm{Ker}(N_{L/K})$ instead of that of $\mathrm{Cl}^{(p)}(L)$, we get the desired result.

Now it remains to show the 2-rank of $\mathrm{Cl}^{(2)}(KF_1)$ is 2. Since F_1 has class number one, any ideal class of order 2 of its quadratic extension KF_1 is ambiguous. Therefore we consider the ramification of the quadratic extension KF_1/F_1. The primes which are ramified in this extension are the prime divisors of 3 and 7. Both the rational primes 3 and 7 split in F and remain prime in $\mathbf{Q}(\gamma_1)$ (note that $F_1 = F(\gamma_1)$). Therefore they have two prime divisors in F_1 and the number of the primes which are ramified in KF_1/F_1 are four. Hence the 2-rank of $\mathrm{Cl}^{(2)}(KF_1)$ cannot be 6.

4. Families of imaginary quadratic number fields whose class field towers are of length at least three

We explain here that from each real quadratic number field satisfying some conditions we can construct a family of infinite imaginary quadratic number fields whose class field towers are of length at least three.

Let F be a real real quadratic number field with $\mathrm{Cl}^{(2)}(F) \cong C_{2^m}$ and $\mathrm{Cl}_+^{(2)}(F) \cong C_{2^{m+1}}$ for some $m \geqq 1$, where $\mathrm{Cl}_+^{(2)}(F)$ denotes the 2-part of the narrow class group $\mathrm{Cl}_+(F)$ of F. Then the narrow Hilbert 2-class field $F_{1,nar}^{(2)}$ of F is a dihedral CM-field: $\mathrm{Gal}(F_{1,nar}^{(2)}/\mathbf{Q}) \cong D_{2^{m+1}}$. For simplicity, we put $M = F_{1,nar}^{(2)}$ and $M_+ = F_1^{(2)}$. Now we assume that

$(*)$ the relative class number $h^-(M)$ of M is greater than one.

Then from F we can construct a family of infinite imaginary quadratic number fields whose class field towers are of length at least three. In fact, let d' be any negative fundamental discriminant prime to $d(F)$ and put $d = d' \cdot d(F)$. Then $K = \mathbf{Q}(\sqrt{d})$ is an imaginary quadratic number field and the length of the class field tower of K is at least three: Now the Hilbert class field M_1 of M is normal over $\mathbf{Q}$ and if we put $H = \mathrm{Gal}(M_1/\mathbf{Q})$, then H is a solvable group with $H'' \neq \{1\}$. This implies $K_3 \neq K_2$, because the compositum KM_1 is an unramified Galois extension of K with $\mathrm{Gal}(KM_1/K) \cong H$. For since H' corresponds (by Galois theory) to the maximal abelian subfield of M_1 and since M_1/F is unramified at all finite primes, this field is the genus field F_g of F (in the narrow sense). By the assumption on F, we have $F \subsetneqq F_g \subseteqq M_+$. On the other hand, since $2 \nmid h(M)$ by [15, Lemma 9], we have $\mathrm{Cl}_+(M_+) \cong C_2 \times \mathrm{Cl}(M_+)$. Hence the condition $(*)$ $h^-(M) > 1$ implies that M_1/M_+ is not abelian, neither is M_1/F_g. Thus, $H'' \neq \{1\}$ and $K_3 \neq K_2$.

Now we describe characterization of such a real real quadratic number field F in detail. For $\mathrm{Cl}^{(2)}(F) \cong C_{2^m}$ and $\mathrm{Cl}_+^{(2)}(F) \cong C_{2^{m+1}}$ for some $m \geqq 1$, it is necessary and sufficient that $d(F)$ is of the form $8p$ or pq, where p and q are distinct prime numbers with $p \equiv q \equiv 1 \pmod 4$, and the norm of the fundamental unit ϵ of F is 1: $N_{F/\mathbf{Q}}(\epsilon) = 1$. For $m = 1$, by Rédei-Reichardt theory $N_{F/\mathbf{Q}}(\epsilon) = 1$ can be rewritten arithmetically as $(2/p) = 1$ and $(p/2)_4(2/p)_4 = -1$, or $(q/p) = 1$ and $(p/q)_4(q/p)_4 = -1$.

We know that there exist only finitely many normal CM-fields with relative class number one [9] and all dihedral CM-fields with relative class number one have already determined by S. Louboutin et al. [5, 7, 8].

For $m = 1$, we can summarize the above as follows.

Proposition 5. *Let p and q are distinct prime numbers satisfying the following conditions*:

(i) $p \equiv 1$, $q \not\equiv 3 \pmod 4$ *and* $(q/p) = 1$. (*Note that if* $q = 2$, *these are equivalent to* $p \equiv 1 \pmod 8$.)
(ii) $(p/q)_4(q/p)_4 = -1$.
(iii) *If* $q = 2$, $p \neq 17, 73, 89, 233, 281$. *Otherwise,* $\{p, q\} \neq \{5, 41\}$, $\{5, 61\}$, $\{5, 109\}$, $\{5, 149\}$, $\{5, 269\}$, $\{5, 389\}$, $\{13, 17\}$, $\{13, 29\}$, $\{13, 157\}$, $\{13, 181\}$, $\{17, 137\}$, $\{17, 257\}$, $\{29, 53\}$, $\{73, 97\}$.

Then for any negative discriminant d' *prime to* pq, *the imaginary quadratic number field* $\mathbf{Q}(\sqrt{pqd'})$ *has class field tower of length at least three.*

Remark 1. For the pairs $\{p, q\}$ excluded in (iii), $F_{1,nar}$ is a dihedral octic CM-field with relative class number one. (See [8].) The condition $\mathrm{Cl}_+^{(2)}(F) \cong C_{2^{m+1}}$ implies $2 \nmid h(M)$. In particular, if $m = 1$ and $h(F_1^{(2)}) = 1$, then $\mathrm{Cl}(M) \cong \mathrm{Cl}(E)^2 = \mathrm{Cl}(E) \times \mathrm{Cl}(E)$, where E is any nonnormal quartic subfield of M, and therefore $\mathrm{Gal}(M_1/\mathbf{Q}) \cong D_4 \ltimes \mathrm{Cl}(E)^2$.

By taking $d' = -r$, (r a prime number $\equiv 3 \pmod 4$) and imposing some arithmetic conditions on r, we obtain $K = \mathbf{Q}(\sqrt{pqd'})$ whose 2-class field tower is of length two. For example, by [3, (iv), (a)], such conditions are $(r/p) = (r/q) = -1$. (See also [1].) Thus, we can conclude that there exist infinitely many imaginary quadratic number fields with $l^{(2)} = 2$ and $l \geqq 3$, where l (resp. $l^{(2)}$) is the length of the class field tower (resp. 2-class field tower). We note that by using [15, Proposition 6] we can show the infiniteness of K with $l^{(2)} = 1$ and $l \geqq 3$.

Example 2. Let $F = \mathbf{Q}(\sqrt{2 \cdot 97})$. Then $\mathrm{Cl}(F) \cong C_2, \mathrm{Cl}_+(F) \cong C_4$, and $\mathrm{Cl}(M) \cong C_3^2$. We take $d' = -3$, that is, let $K = \mathbf{Q}(\sqrt{(-3) \cdot 2 \cdot 97})$. Then $d(K) = -2388$ and $\mathrm{Cl}(K) \cong C_8 \times C_2$. By the results of Benjamin-Lemmermeyer-Snyder in [1], we have $K_\infty^{(2)} = K_2^{(2)}$ and $\mathrm{Gal}(K_2^{(2)}/K) \cong \Gamma_{4,2}^2$, where $\Gamma_{4,2}^2$ is presented by

$$\langle\, a, b \mid a^4 = 1,\ c = [a, b],\ a^2 = b^{2^4} = c^2,\ [a, c] = c^2,\ [ab, c] = 1 \,\rangle.$$

and $|\Gamma_{4,2}^2| = 128$. Thus, the length of the 2-class field tower of K is two, the length of the l-class field tower of K is zero for any odd l, but the length of the class field tower of K is at least three. We have $\mathrm{Gal}(M_1 K_2^{(2)}/K) \cong \Gamma_{4,2}^2 \ltimes C_3^2$.

For $KM_1 \subsetneqq K_2$, the conditions $\mathrm{Cl}^{(2)}(F) \cong C_{2^m}$ and $\mathrm{Cl}_+^{(2)}(F) \cong C_{2^{m+1}}$ are not essential. For F such that $M = F_{1,nar}^{(2)}$ is a nonabelian CM-field with $h^-(M) > 1$, we may expect the same. For almost all cases, we have $h^-(M) > 1$. Therefore $N_{F/\mathbf{Q}}(\epsilon) = 1$ and that $\mathrm{Cl}^{(2)}(F)$ is not elementary are essential.

Also from some real quadratic number fields whose fundamental units have norm -1, we can obtain families of infinite imaginary quadratic number fields whose class field towers have length at least three.

Let F be a real real quadratic number field with $\mathrm{Cl}^{(2)}(F) \cong C_{2^m}$ $(m \geqq 2)$ and $N_{F/\mathbf{Q}}(\epsilon) = -1$. Then the Hilbert 2-class field $F_1^{(2)}$ of F is a real dihedral number field: $\mathrm{Gal}(F_1^{(2)}/\mathbf{Q}) \cong D_{2^m}$. Let d' be any negative fundamental discriminant prime to $d(F)$ and put $d = d' \cdot d(F)$ and $K = \mathbf{Q}(\sqrt{d})$. Then the compositum $KF_1^{(2)}$ is a normal CM-field with $\mathrm{Gal}(KF_1^{(2)}/\mathbf{Q}) \cong D_{2^m} \times C_2$. Let E be the $C_{2^{m-1}}$-subextension of $F_1^{(2)}/F$. Then $KF_1^{(2)}/E$ is a V_4-extension and $F_1^{(2)}$ and KE are its two intermediate fields. Let M be the other intermediate field. Then M is also a dihedral CM-field. Now we assume that

$$\text{the odd part } h^-_{\mathrm{odd}}(M) \text{ of } h^-(M) \text{ is greater than one.}$$

Then by the same reason as in the case $N_{F/\mathbf{Q}}(\epsilon) = 1$ the class field tower of K has length at least three. (If we do not assume $h^-_{\mathrm{odd}}(M) > 1$, then it is possible that $M_1 = K_2^{(2)}$: $F = \mathbf{Q}(\sqrt{5 \cdot 29})$ and $d' = -3$ is the case.)

For $\mathrm{Cl}^{(2)}(F) \cong C_{2^m}$ for some $m \geqq 2$ and $N_{F/\mathbf{Q}}(\epsilon) = -1$, it is necessary that $d(F)$ is of the form pq, where p and q are distinct prime numbers with $p \equiv 1$, $q \not\equiv 3 \pmod 4$ and $(q/p) = 1$. For general m, it seems not easy to see that there exist infinitely many d' with $h^-_{\mathrm{odd}}(M) > 1$, from now on we assume $m = 2$. Then by Rédei-Reichardt theory $(p/q)_4 = (q/p)_4 = -1$. Now we assume moreover that d' is an odd prime discriminant: $d' = -r$, $r \equiv 3 \pmod 4$ a prime, and that the rational prime r remains prime in the field $\mathbf{Q}(\sqrt{p})$: $(p/r) = -1$. Let E be a nonnormal quartic subfield of M containing $\mathbf{Q}(\sqrt{p})$. Then E is a CM-field and the finite primes ramified in $E/\mathbf{Q}(\sqrt{p})$ are r and $\mathfrak{q}$, where $\mathfrak{q}$ is one of the prime divisors of q in $\mathbf{Q}(\sqrt{p})$. (Note that q splits in $\mathbf{Q}(\sqrt{p})$.) Therefore the 2-rank of $\mathrm{Cl}^{(2)}(E)$ is one. The 4-rank of $\mathrm{Cl}^{(2)}(E)$ is zero or one, according as the Hilbert symbol $(r, \alpha)_{\mathfrak{q}} = (r/q) = -1$, or 1, where α is a totally positive generator of a principal ideal $\mathfrak{q}^{h(\mathbf{Q}(\sqrt{p}))}$. (See [14].) Thus, if $(r/q) = -1$, then $2 \parallel h^-(E)$. We know that there exist only finitely many nonnormal quartic CM-fields with relative class number two [12] and all such fields has been determined by H.-S. Yang and S.-H. Kwon [17]. We also note that in this case by [3, (iv), (a)] the 2-class field tower of K has length two and $\mathrm{Gal}(K_2^{(2)}/K) \cong Q_{2^n}$ $(n \geqq 4)$.

We can summarize the above as follows.

Proposition 6. *Let p and q are distinct prime numbers satisfying the following conditions*:

(i) $p \equiv 1$, $q \not\equiv 3 \pmod 4$ *and* $(q/p) = 1$. (*Note that if $q = 2$, these are equivalent to* $p \equiv 1 \pmod 8$.)
(ii) $(p/q)_4 = (q/p)_4 = -1$.

Then for any prime number r with $r \equiv 3 \pmod 4$, $(p/r) = (q/r) = -1$ and $\{p, q, r\} \neq \{5, 29, 3\}$, $\{5, 101, 3\}$, the imaginary quadratic number field $K = \mathbf{Q}(\sqrt{-pqr})$ has class field tower of length at least three. Moreover, the 2-class field tower of K has length two and $\mathrm{Gal}(K_2^{(2)}/K) \cong Q_{2^n}$ $(n \geqq 4)$.

Remark 2. In the excluded cases where $\{p, q, r\} = \{5, 29, 3\}$, $\{5, 101, 3\}$, we have $h^-(M) = 2$. In the former case, $K = \mathbf{Q}(\sqrt{(-3) \cdot 5 \cdot 29})$ has class field tower of length two (this is unconditional). The relative class numbers of M and E are related by $h^-(M) = Q_M h^-(E)^2/2$, where Q_M is Hasse's unit index of M. If (the norm of) the relative discriminant $d(M/M_+)$ has odd prime divisor, then $Q_M = 1$, where M_+ is the maximal real subfield of M [5, Theorem 1, (i), 1]. Now $M_+ = F_g = \mathbf{Q}(\sqrt{p}, \sqrt{q})$ and $r^2 \mid N_{M_+/\mathbf{Q}}(d(M/M_+))$. Therefore $h^-(M) = h^-(E)^2/2$, hence $\mathrm{Cl}^-(M) \cong C_2 \times \mathrm{Cl}^-(E)^2$.

Example 3. Let $F = \mathbf{Q}(\sqrt{2 \cdot 41})$. Then $\mathrm{Cl}(F) \cong C_4$, $N_{F/\mathbf{Q}}(\epsilon) = -1$, and $\mathrm{Cl}(E) \cong C_6$. We take $d' = -3$, that is, let $K = \mathbf{Q}(\sqrt{(-3) \cdot 2 \cdot 41})$. Then $d(K) = -984$ and $\mathrm{Cl}(K) \cong C_6 \times C_2$. By the result of Kisilevsky in [3], we have $K_\infty^{(2)} = K_2^{(2)}$ and $\mathrm{Gal}(K_2^{(2)}/K) \cong Q_{16}$. But the length of the class field tower of K is at least three.

We have $K_{ur} = K_3 = K_1 M_1$ (under GRH). Since $rd_K = \sqrt{984} = 31.1126\ldots < B(2 \cdot 16 \cdot 3 \cdot 9 \cdot 3)$, $[K_{ur} : K_1 M_1] \leqq 2$. Since $K_1 M_1 / K_2^{(2)}$ is a C_3^3-extension, the 2-rank of $\mathrm{Cl}(K_1 M_1)$ is even and therefore $K_{ur} = K_1 M_1 = K_3$ and $\mathrm{Gal}(K_{ur}/K) \cong (Q_{16} \ltimes C_3^2) \times C_3$. This is conditional, however, we can verify $K_2 = (K_g)_1$ by computer calculation unconditionally.

Correction to the previous paper

There are some errors in the previous paper [15].

On page 407 line 14, $\mathbf{Q}(\sqrt{-423})$ should read $\mathbf{Q}(\sqrt{-424})$.

In the table on page 414, for $d = -943$, $K_2 = K_1(\alpha_1)$ (not $K_1(\alpha_4)$), and for $d = -952$, $K_1 = K(\sqrt{-7}, \sqrt{-9 + 10\sqrt{2}})$ ("$-$" is missing in the second squareroot.)

On page 430 we write that there had been a gap in the proof of R. Schoof communicated to J. Martinet. This is author's misunderstanding. There was not a gap. The author appologizes Prof. Schoof.

On page 434, line 9, $B(2 \cdot 8 \cdot 4 \cdot 11)$ should read $B(2 \cdot 8 \cdot 4 \cdot 16)$.

On page 442. The constant term of $u(X)$ is misprinted. The second term in parenthesis "$a_1^2 a_4^2$" should read "$a_1^4 a_4^2$".

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Ken Yamamura
Department of Mathematics
National Defense Academy
Hashirimizu Yokosuka 239-8686
Japan
E-mail : `yamamura@cc.nda.ac.jp`